GENERALIZED CONDITIONAL YEH-WIENER INTEGRALS FOR THE SAMPLE PATH-VALUED CONDITIONING FUNCTION

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Abstract. The purpose of this paper is to treat the generalized conditional Yeh-Wiener integral for the sample path-valued conditioning function. As a special case of our results, we obtain the results in [6].

1. Introduction

Let \( t = g(s) \) be a monotonically decreasing and continuous function on \([0, S]\) with \( g(S) > 0 \) and let \( \Omega = \{(s, t) \mid 0 \leq s \leq S, 0 \leq t \leq g(s)\} \). Let \( C(\Omega) \) be a space of all real continuous functions \( x \) on \( \Omega \) such that \( x(s, t) = 0 \) for all \((s, t)\) in \( \Omega \) satisfying \( st = 0 \).

In [3], the authors treated the generalized conditional Yeh-Wiener integral which includes the conditional Yeh-Wiener integral in [5] and the modified conditional Yeh-Wiener integral in [1]. In [5–8], Park and Skoug treated the conditional Yeh-Wiener integral for various kinds of conditioning functions including the sample path-valued conditioning function.

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The purpose of this paper is to treat the generalized conditional Yeh-Wiener integral for the sample path-valued conditioning function. We obtain the formula for the generalized conditional Yeh-Wiener integral and then evaluate it for two kinds of functionals. As a special case of our results, we obtain the results in [6].

2. Generalized Conditional Yeh-Wiener integrals for sample path-valued conditioning function

For a functional \( F \) of \( x \) in \( C(\Omega) \), \( E(F) = \int_{C(\Omega)} F(x)dm(x) \) is called a generalized Yeh-Wiener integral of \( F \) if it exists ([3]). As a stochastic process, \( \{(x(s,t))|(s,t) \in \Omega\} \) has a mean \( E(x(s,t)) = 0 \) and \( E(x(s,t)x(u,v)) = \min\{s,u\}\min\{t,v\} \). Let \( C[0,g(S)] \) denote the standard Wiener space with the Wiener measure and assume that \( \psi \) is in \( C[0,g(S)] \).

For a generalized Yeh-Wiener integrable function \( F \) of \( x \) in \( C(\Omega) \), consider the generalized conditional Yeh-Wiener integral of the form

\[
(2.1) \quad E(F(x)|x(S,(\cdot) \wedge T) = \psi((\cdot) \wedge T))
\]

with \( g(S) = T \) and \( a \wedge b = \min\{a,b\} \). Here, \((\cdot)\) belongs to \([0,g(s)]\) for \( 0 \leq s \leq S \). Since two processes \( x(S,t \wedge T) \) and \( \{x(s,t) - (s/S)x(S,t \wedge T)|(s,t) \in \Omega\} \) are (stochastically) independent, we have

\[
(2.2) \quad E(F(x)|x(S,(\cdot) \wedge T) = \psi((\cdot) \wedge T)) = E(F(x(\cdot,\cdot)) - \frac{S}{S}x(S,(\cdot) \wedge T) + \frac{S}{S}\psi((\cdot) \wedge T)))
\]

for almost all \( \psi \) in \( C[0,T] \). Here, for the notational convenience, we denote \( \cdot = (\cdot) \).

Especially, if \( g(s) = T \) for all \( 0 \leq s \leq S \), then \((\cdot) \wedge T = (\cdot) \), which agrees with (2.2) in [6]. This means that our result (2.2) is a slight generalization of the result in [6].

Let \( y(\cdot) \) be a tied-down Brownian motion, that is,

\[ \{y(t) \mid 0 \leq t \leq T\} = \{w \in C[0,T] \mid w(T) = \xi\} \]

Then, as is well known([6]), \( y(\cdot) \) can be expressed in terms of the standard Wiener process,

\[ y(t) = w(t) - \frac{t}{T}w(T) + \frac{t}{T}\xi. \]
The following theorem is one of our main results, which is slightly different from Theorem 1 in [6].

**THEOREM 2.1.** If $F \in L_1(C(\Omega), m)$, then we have

\[(2.3) \quad E_w(E(F(x) \mid x(S, (\cdot) \land T)) = \sqrt{S} w((\cdot) \land T)) = E(F(x)),\]

\[(2.4) \quad E(F(x)|x(S, T) = \sqrt{S} \xi) = E_w \left\{ E \left( F(x) \mid x(S, (\cdot) \land T) = \sqrt{S} \left( w((\cdot) \land T) - \frac{(\cdot) \land T}{T} w((\cdot) \land T) \right) + \frac{(\cdot) \land T}{T} \xi \right) \right\}.\]

**Proof.**

(1) Using (2.2), we write

\[(2.5) \quad E_w(E(F(x) \mid x(S, (\cdot) \land T)) = \sqrt{S} w((\cdot) \land T))\]

\[= E_w \left\{ E \left( F(x(s, \cdot)) - \star \frac{s}{S} x(S, (\cdot) \land T) + \frac{\star}{\sqrt{S}} \psi((\cdot) \land T) \right) \right\}.\]

Let $y(s, t) = x(s, t) - (s/S)x(S, t \land T) + (s/\sqrt{S})\psi(t \land T)$ for all $(s, t)$ in $\Omega$. Then we have $E(y(s, t)) = 0$ and $E(y(s, t)y(u, v)) = \min\{s, u\}\min\{t, v\}$. This means that $\{\{y(s, t)\mid(s, t) \in \Omega\}$ is a generalized Yeh-Wiener process, and the right-hand side of (2.5) becomes $\int_{C(\Omega)} F(y)dm(y) = E(F(x))$.

Thus, we obtain the formula (2.3).

(2) We use Theorem 2 in [5] to have

\[(2.6) \quad E(F(x) \mid x(S, T) = \sqrt{S} \xi) = E \left\{ E \left( F(x(s, \cdot)) - \star \frac{s}{S} x(S, T) + \frac{\star}{\sqrt{S}} \frac{(\cdot) \land T}{T} \xi \right) \right\}.\]

We can rewrite the right-hand side of (2.6) as the following form:

\[(2.7) \quad E \{ F(x(s, \cdot)) - \star \frac{s}{S} x(S, (\cdot) \land T) + \frac{\star}{\sqrt{S}} x(S, (\cdot) \land T) - \frac{(\cdot) \land T}{T} x(S, T) + \frac{(\cdot) \land T}{T} \sqrt{S} \xi)\}.\]

We use $E(x(s, t)x(u, v)) = \min\{s, u\}\min\{t, v\}$ to show that two processes $x(s, \cdot) - (s/S)x(S, (\cdot) \land T)$ and $x(S, (\cdot) \land T) - ((\cdot) \land T)(x(S, T)/T)$ are stochastically independent. Furthermore, $\sqrt{S} (w((\cdot) \land T)(w(T)/T))$ and $x(S, (\cdot) \land T) - ((\cdot) \land T)(x(S, T)/T)$ are equivalent processes, where $w(\cdot)$ is the standard Wiener process. Thus, (2.7) becomes
\[(2.8) \quad E_w\{E\{F(x, \cdot) - \frac{x}{S}(S, (\cdot) \land T) + \frac{1}{\sqrt{S}}[(w(\cdot) - ((\cdot) \land T)(w(T)/T)) + ((\cdot) \land T) \frac{\xi}{T}])\}\} = E_w\left\{E\left(F(x) \bigg| x(S, (\cdot) \land T)\right) - \sqrt{S}(w(\cdot) - \frac{(\cdot) \land T}{T}w((\cdot) \land T) + \frac{(\cdot) \land T}{T}\xi)\right\} \right. \]

Therefore, we get the formula (2.4). \hfill \Box

For the special case \(g(s) = T\) for \(0 \leq s \leq S\), we have the same result of Theorem 1 in [6]. In a certain sense, our result is a slight generalization of the result in [6].

In [6], Park and Skoug treated the rectangle \(Q\), but we treat the more general region \(\Omega\). Let \(\Omega\) be the region given by

\[\Omega = \{(s, t) \mid 0 \leq s \leq S, 0 \leq t \leq g(s)\}\]

where \(t = g(s)\) is a monotonically decreasing and continuous function on \([0, S]\) with \(g(S) = T > 0\). In the following two theorems we evaluate the generalized conditional Yeh-Wiener integral for the sample path-valued conditioning function.

**Theorem 2.2.** Let \(F\) be a functional on \(C(\Omega)\) given by \(F(x) = \int_{\Omega} x(s, t) \, ds \, dt\). Then we have

\[(2.9) \quad E(F(x) \mid x(S, (\cdot) \land T) = \psi((\cdot) \land T)) = \frac{S}{2} \int_0^T \psi(t) \, dt + \frac{\psi(T)}{S} \int_0^S \int_T^{g(s)} \, s \, dt \, ds.\]

**Proof.** By (2.2) and Fubini theorem, we have

\[(2.10) \quad E(F(x) \mid x(S, (\cdot) \land T) = \psi((\cdot) \land T)) = \int_{\Omega} E\left(x(s, t) - \frac{s}{S} x(S, (\cdot) \land T) + \frac{s}{S}\psi((\cdot) \land T)\right) \, ds \, dt.
= \int_{\Omega} \frac{s}{S}\psi((\cdot) \land T) \, ds \, dt.\]
The right hand side of the last equality in (2.10) comes from the fact that $E(x(s, t)) = 0$ and $m(C(\Omega)) = 1$. By the straightforward calculation, we have

$$(2.11) \quad E(F(x) \mid x(S, (\cdot) \wedge T) = \psi((\cdot) \wedge T))$$

$$= \int_0^S \int_0^{g(s)} \frac{s}{S} \psi((\cdot) \wedge T) dtds$$

$$= \frac{S}{2} \int_0^T \psi(t) dt + \frac{\psi(T)}{S} \int_0^S \int_T^{g(s)} s dtds,$$

which is our desired result.

**Theorem 2.3.** Let $F$ be a functional on $C(\Omega)$ given by $F(x) = \int_\Omega x^2(s, t) dsdt$ and $g(S) = T$. Then we have

$$(2.12) \quad E(F(x) \mid x(S, (\cdot) \wedge T) = \psi((\cdot) \wedge T))$$

$$= \frac{S^2 T^2}{12} + \frac{S}{3} \int_0^T \psi^2(t) dt + \int_0^S \int_T^{g(s)} \left( st - \frac{s^2}{S} T + \frac{s^2}{S^2} \psi^2(T) \right) dtds.$$

**Proof.** By (2.2) and Fubini theorem, we have

$$(2.13) \quad E(F(x) \mid x(S, (\cdot) \wedge T) = \psi((\cdot) \wedge T))$$

$$= \int_\Omega E \left( \left\{ x(s, t) - \frac{s}{S} x(S, (\cdot) \wedge T) + \frac{s}{S} \psi((\cdot) \wedge T) \right\}^2 \right) dsdt$$

$$= \int_\Omega \left\{ st - \frac{s^2}{S} (t \wedge T) + \frac{s^2}{S^2} \psi^2(t \wedge T) \right\} dtds.$$

The right hand side of the last equality in (2.13) comes from the fact that $E(x(s, t)) = 0$, $E(x(s, t|x(u, v))) = \min\{s, u\} \min\{t, v\}$ and $m(C(\Omega)) = 1$. By the straightforward calculation, the right hand side of the last equality in (2.13) becomes

$$(2.14) \quad \frac{S^2 T^2}{12} + \frac{S}{3} \int_0^T \psi^2(t) dt$$

$$+ \int_0^S \int_T^{g(s)} \left( st - \frac{s^2}{S} T + \frac{s^2}{S^2} \psi^2(T) \right) dtds,$$

which is our desired result.
Remark 2.4. In Theorem 2.2 and Theorem 2.3, we have the extra terms which does not appear in Example 1 and Example 2 of [6]. This means that Park and Skoug’s examples in [6] are the special case of our results for the rectangle $\Omega$.

References


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