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Automorphism Group of a Non-Associative Algebra I

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Abstract. Automorphisms of a Weyl-type non-associative subalgebra of $\overline{WN_{n,m,s}}$ were studied in [2], [3], [4], [12], [13]. There are various papers on the automorphism groups of an associative algebra, a Lie algebra, and a non-associative algebra [4], [5], [11]. It seems that there is no paper on automorphisms of a semi-Lie algebra in the literature. A degree on an algebra is used to find the derivation group of an algebra in the paper [13]. We find the automorphism groups $Auto_{non}(\overline{WN_{0,0,1_2}})$ and $Auto_{semi-Lie}(\overline{WN_{0,0,1_{2[,]}}})$ of the non-associative algebra $\overline{WN_{0,0,1_2}}$ and the semi-Lie algebra $\overline{WN_{0,0,1_{2[,]}}}$ respectively in this paper. The results of an algebra in this paper do not depend on its standard basis.

Keywords: Simple; Non-associative algebra; Semi-Lie algebra; Right identity; Annihilator; m-abelian; Automorphism.

1. Preliminaries

Let $\mathbb{N}$ be the set of all non-negative integers and $\mathbb{Z}$ be the set of all integers. Let $\mathbb{F}$ be a field of characteristic zero. The non-associative algebra $\overline{WN_{0,1,0_n}}$ is spanned

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by the standard basis $\{x^i\partial^n | i \in \mathbb{Z}\}$ with the usual addition and the multiplication is defined as follows: for any basis elements $x^i\partial^n, x^j\partial^n \in \overline{WN_{0,1,0}}_n$,

$$x^i\partial^n * x^j\partial^n = x^i(\partial^n(x^j))\partial^n \tag{1}$$

by extending linearly on $\overline{WN_{0,1,0}}_n$ [1], [6], [13], [14]. $\overline{WN_{0,0,1}}_n$ is a subalgebra of $\overline{WN_{0,1,0}}_n$ spanned by $\{x^i\partial^n | i \in \mathbb{N}\}$. Similarly, we define the non-associative algebra $\overline{WN_{0,0,1}}_{1,2}$ spanned by $\{x^i\partial, x^j\partial^2 | i, j \in \mathbb{N}\}$ with the similar multiplication (1) (please refer to the papers [9], [15] for these kinds of non-associative algebras.) For any element $x \in \overline{WN_{0,0,1}}_n$, $l \in \overline{WN_{0,0,1}}_n$ is a right (multiplicative) identity of x, if $x*l = x$ holds. The semi-Lie algebra $\overline{WN_{0,0,1}}_{n_{[,]}}$ is spanned by the standard basis $\{x^i\partial^n | i \in \mathbb{N}\}$ with the commutator of $\overline{WN_{0,0,1}}_n$ and the semi-Lie algebra $\overline{WN_{0,0,1}}_{1,2_{[,]}}$ can be defined as $\overline{WN_{0,0,1}}_{n_{[,]}}$. We shall define the degree of $x^i\partial^n$ as $deg(x^i\partial^n) = i$ for $x^i\partial^n \in \overline{WN_{0,0,1}}_{n_{[,]}}$. Thus for any element l of $\overline{WN_{0,0,1}}_{n_{[,]}}$, we can define $deg(l)$ as the highest degree of the non-zero basis term of l [13]. We can define the non-associative algebra $\overline{WN_{0,0,2}}_{n_1,n_2}$ which contains $\overline{WN_{0,0,1}}_n$ with the standard basis $\{x_1^{i_1}x_2^{i_2}\partial^{n_r} | i_1, i_2 \in \mathbb{N}, r = 1, 2\}$ with the usual addition and the multiplication as $\overline{WN_{0,0,1}}_n$. So we can define the semi-Lie algebra $\overline{WN_{0,0,2}}_{n_1,n_{2_{[,]}}}$ as $\overline{WN_{0,0,1}}_{n_{[,]}}$. Throughout this paper, $Aut_{semi-Lie}(\overline{WN_{0,0,1}}_{n_{[,]}})$ denotes the set of all semi-Lie algebra automorphisms of $\overline{WN_{0,0,1}}_{n_{[,]}}$. A Lie (resp. semi-Lie) algebra is m-abelian if the dimension of its maximal finite dimensional abelian subalgebra is m [13]. A Lie algebra is 1-abelian if and only if it is self-centralizing [13], [7], [8]. Note that m-abelian is auto-invariant.

2. Automorphisms of $\overline{WN_{0,0,1}}_n$ and $\overline{WN_{0,0,1}}_{n_{[,]}}$

It is well known that the algebras $\overline{WN_{0,0,1}}_n$ and $\overline{WN_{0,0,1}}_{n_{[,]}}$ are simple [4], [12], [13]. Since every non-zero endomorphism of them is a monomorphism, we can find the following results.

Lemma 2.1. *For any $\theta \in Aut_{non}(\overline{WN_{0,0,1}}_2)$ and any basis element $x^q\partial^2$ of $\overline{WN_{0,0,1}}_2$, $\theta(x^q\partial^2) = c_3^{-q+2}(x + \frac{c_4}{c_3})^q\partial^2$ holds, where $c_3 \in \mathbb{F}^\bullet$ and $c_4 \in \mathbb{F}$.*

Proof. Let θ be an automorphism of $\overline{WN_{0,0,1}}_2$. Since the right annihilator of $\overline{WN_{0,0,1}}_2$ is spanned by ∂^2 and $x\partial^2$, and $\frac{x^2}{2}\partial^2$ is a right identity of $\overline{WN_{0,0,1}}_2$, we have

$$\theta(\partial^2) = c_1x\partial^2 + c_2\partial^2 \tag{2}$$

$$\theta(x\partial^2) = c_3x\partial^2 + c_4\partial^2 \tag{3}$$

$$\theta(x^2\partial^2) = x^2\partial^2 + c_5x\partial^2 + c_6\partial^2 \tag{4}$$

where $c_1, \cdots, c_6 \in \mathbb{F}$. We have the following two cases $c_1 = 0$ and $c_1 \neq 0$.

Case I. We assume that $c_1 \neq 0$ in (1). By $\theta(\partial^2 * x^3\partial^2) = 6\theta(x\partial^2)$, we have $\theta(x^3\partial^2) = c_7x^2\partial^2 + c_8x\partial^2 + c_9\partial^2$, where $c_7, c_8, c_9 \in \mathbb{F}$. By $\theta(x\partial^2 * x^3\partial^2) = 6\theta(x^2\partial^2)$, we have

$$\theta(x\partial^2) * \theta(x^3\partial^2) = 6\theta(x^2\partial^2) \tag{5}$$

Since $deg(\theta(x\partial^2)) = 1$, $deg(\theta(x^3\partial^2)) = 1$, and $deg(\theta(x^2\partial^2)) = 2$, the equality (5) does not hold. This contradiction shows that $c_1 = 0$.

Case II. Now, we assume that $c_1 = 0$ in (1). We put the equalities (2) and (3) hold. It is easy to prove that $c_3 \neq 0$. By $\theta(x\partial^2 * x^3\partial^2) = 6\theta(x^2\partial^2)$, we have

$$\theta(x^3\partial^2) = \frac{x^3}{c_3}\partial^2 + c_{10}x^2\partial^2 + c_{11}x\partial^2 + c_{12}\partial^2,$$

where $c_{10}, c_{11}, c_{12} \in \mathbb{F}$. By $\theta(\partial^2 * x^3\partial^2) = 6\theta(x\partial^2)$, $c_2 = c_3^2$ and $c_{10} = \frac{3c_4}{c_3^2}$, that is,

$$\theta(x^3\partial^2) = \frac{x^3}{c_3}\partial^2 + \frac{3c_4}{c_3^2}x^2\partial^2 + c_{11}x\partial^2 + c_{12}\partial^2$$

By $\theta(x\partial^2 * x^3\partial^2) = 6\theta(x^2\partial^2)$, we also have $c_5 = \frac{2c_4}{c_3}$ and $c_6 = \frac{c_4^2}{c_3^2}$, i.e.,

$$\theta(x^2\partial^2) = x^2\partial^2 + \frac{2c_4}{c_3}x\partial^2 + \frac{c_4^2}{c_3^2}\partial^2 = (x + \frac{c_4}{c_3})^2\partial^2 \tag{6}$$

By $\theta(x^2\partial^2 * x^3\partial^2) = 6\theta(x^3\partial^2)$, we have $c_{11} = \frac{3c_4^2}{c_3^3}$ and $c_{12} = \frac{c_4^3}{c_3^4}$, i.e.,

$$\theta(x^3\partial^2) = \frac{x^3}{c_3}\partial^2 + \frac{3c_4}{c_3^2}x^2\partial^2 + \frac{3c_4^2}{c_3^3}x\partial^2 + \frac{c_4^3}{c_3^4}\partial^2 = c_3^{-1}(x + \frac{c_4}{c_3})^3\partial^2$$

By (6) and $\theta(\partial^2 * x^4\partial^2) = 12\theta(x^2\partial^2)$, we have

$$\theta(x^4\partial^2) = \frac{x^4}{c_3^2}\partial^2 + \frac{6c_5x^3}{3c_3^2}\partial^2 + \frac{6c_6x^2}{c_3^2}\partial^2 + c_{13}x\partial^2 + c_{14}\partial^2$$

where $c_{13}, c_{14} \in \mathbb{F}$. By $\theta(x^2\partial^2 * x^4\partial^2) = 12\theta(x^4\partial^2)$, we have $c_{13} = \frac{4c_4^3}{c_3^5}$ and $c_{14} = \frac{c_4^4}{c_3^5}$, that is,

$$\theta(x^4\partial^2) = \frac{x^4}{c_3^2}\partial^2 + \frac{6c_5x^3}{3c_3^2}\partial^2 + \frac{6c_6x^2}{c_3^2}\partial^2 + \frac{4c_4^3}{c_3^5}x\partial^2 + \frac{c_4^4}{c_3^6}\partial^2 = c_3^{-2}(x + \frac{c_4}{c_3})^4\partial^2$$

Thus by induction on $p \in \mathbb{N}$ of $x^p\partial^2$, we can assume that $\theta(x^p\partial^2) = c_3^{-p+2}(x + \frac{c_4}{c_3})^p\partial^2$ holds. Since the right annihilator of ∂^2 is spanned by $\{\partial^2, x\partial^2\}$ and the fact that $deg(\theta(x^2\partial^2 * x^{p+1}\partial^2)) = p+1$, $deg(\partial^2 * l_1) = deg(l_1) - 2$ and $deg(x\partial^2 * l_1) = deg(l_1) - 1$ we can prove that $\theta(x^{p+1}\partial^2) = c_3^{-p+1}(x + \frac{c_4}{c_3})^{p+1}\partial^2$ by appropriate inductions where $l_1, l_2 \in \overline{WN_{0,0,1_2}}$. This completes the proof of the Lemma. ∎

Note 2.2. For any basis element $x^p\partial^2$ of $\overline{WN_{0,0,1_2}}$ (resp. $\overline{WN_{0,0,1_2}}_{[,]}$), $c_3 \in \mathbb{F}^{\bullet}$ and $c_4 \in \mathbb{F}$, we can define an $\mathbb{F}$-linear map θ_{c_3,c_4} of $\overline{WN_{0,0,1_2}}$ (resp. $\overline{WN_{0,0,1_2}}_{[,]}$) as follows:

$$\theta_{c_3,c_4}(x^p\partial^2) = c_3^{-p+2}(x + \frac{c_4}{c_3})^p\partial^2$$

Then θ can be linearly extended to a non-associative (resp. semi-Lie) algebra automorphism of $\overline{WN_{0,0,1_2}}$ (resp. $\overline{WN_{0,0,1_2}}_{[,]}$).

Proposition 2.3. *The non-associative algebra automorphism group $Aut_{non}(\overline{WN_{0,0,1_2}})$ of $\overline{WN_{0,0,1_2}}$ is generated by θ_{c_3,c_4} which is defined in Note 2.2 with the appropriate constants in Note 2.2.*

Proof. The proof of this Proposition is straightforward by Lemma 2.1, hence we omit the proof. ∎

Theorem 2.4. *The automorphism group $Aut_{non}(\overline{WN_{0,0,1_2}})$ of $\overline{WN_{0,0,1_2}}$ is generated by θ_{c_3,c_4} which is defined in the above Note with the appropriate scalars in the Note.*

Proof. The right annihilator of $\overline{WN_{0,0,1_2}}$ is spanned by $\{\partial^2, x^2\partial^2\}$ which is auto-invariant. For any $l \in \overline{WN_{0,0,1_2}}$, by the facts that $deg(x^i\partial^2 * l) = deg(l) + i - 2$, $\frac{x^2}{2!}\partial^2$ is a right identity of $\overline{WN_{0,0,1_2}}$, and $deg(x^2\partial^2 * l) = deg(l)$, we can prove the similar results of Lemma 2.1 for $\overline{WN_{0,0,1_2}}$. This completes the proof of the Theorem by Note 2.2. ∎

Theorem 2.5. *The non-associative algebra automorphism group $Aut_{non}(\overline{WN_{0,1,0_2}})$ of $\overline{WN_{0,1,0_2}}$ is generated by $\theta_{c_3,0}$ which is defined in Note 2.2 with the constants in Note 2.2. For $n_1, n_2 \in \mathbb{N}$, if $n_1 \neq n_2$, then the non-associative algebra automorphism group $Aut_{non}(\overline{WN_{0,1,0_{n_1}}})$ is isomorphic to the non-associative algebra automorphism group $Aut_{non}(\overline{WN_{0,1,0_{n_2}}})$.*

Proof. Since the right annihilator of $\overline{WN_{0,1,0_2}}$ is spanned by $\{x^j\partial^2 | 1 \le j \le 2\}$ and by $\theta(x\partial^2 * \frac{x^2}{2!}\partial^2) = \theta(x\partial^2)$, we can prove the similar results in Lemma 1 with $c_4 = 0$. Thus, $Aut_{non}(\overline{WN_{0,1,0_2}})$ is generated by $\theta_{c_3,0}$ in Note 2.2. The remaining results of the Theorem is obvious. This completes the proof of the Theorem. ■

We note that the automorphism group $Aut_{non}(\overline{WN_{0,1,0_n}})$ is a subgroup of the automorphism group $Aut_{non}(\overline{WN_{0,0,1_n}})$.

Lemma 2.6. *The semi-Lie algebra $(\overline{WN_{0,0,1_{2[,]}}})$ is 2-abelian and its finite dimensional maximal subalgebra $< \partial^2, x\partial^2 >$ spanned by ∂^2 and $x\partial^2$ is auto-invariant.*

Proof. The proof of Lemma is straightforward by the fact that the algebra has the well defined order, and hence the proof is omitted. ■

Lemma 2.7. *For any $\theta \in Aut_{non}(\overline{WN_{0,0,1_{2[,]}}})$ and any basis element $x^q\partial^2$ of $\overline{WN_{0,0,1_{2[,]}}}$, $\theta(x^q\partial^2) = c_3^{-q+2}(x + \frac{c_4}{c_3})^q\partial^2$ holds, where $c_3 \in \mathbb{F}^\bullet$ and $c_4 \in \mathbb{F}$.*

Proof. Since the semi-Lie algebra $\overline{WN_{0,0,1_{2[,]}}}$ is 2-abelian, $\frac{x^2}{2}\partial^2$ is ad-diagonal with respect to its standard basis, and every non-associative algebra automorphism of the non-associative algebra $\overline{WN_{0,0,1_2}}$ is a semi-Lie algebra automorphism of $\overline{WN_{0,0,1_{2[,]}}}$, the similar results of Lemma 2.1 holds for the semi-Lie algebra $\overline{WN_{0,0,1_2}}$. By Note 2.2, this completes the proof of the Lemma. ■

Theorem 2.8. *The automorphism group $Aut_{semi}(\overline{WN_{0,0,1_{2[,]}}})$ of $\overline{WN_{0,0,1_{2[,]}}}$ is generated by θ_{c_3,c_4} which is defined in Notes with the constants in the Notes.*

Proof. Since $\overline{WN_{0,0,1_{2[,]}}}$ is 2-abelian, the proof of the Theorem is similar to the proof of Theorem 2.4, and is hence omitted. ■

Corollary 2.9. *The automorphism group $Aut_{semi}(\overline{WN_{0,1,0_{2[,]}}})$ of $\overline{WN_{0,1,0_{2[,]}}}$ is generated by $\theta_{c_3,0}$ which is defined in Notes with the constants in the Notes.*

Proof. By Theorem 2.5 and Theorem 2.8, the proof of the Corollary is straightforward, and is hence omitted. ■

Proposition 2.10. *If $n_1 \neq n_2$, then the non-associative algebra $\overline{WN_{0,0,1_{n_1}}}$ is not isomorphic to the non-associative algebra $\overline{WN_{0,0,1_{n_2}}}$ as non-associative algebras.*

Proof. Without loss of generality, we can assume that $n_1 > n_2$. If θ is an isomorphism from $\overline{WN_{0,0,1_{n_1}}}$ to $\overline{WN_{0,0,1_{n_2}}}$, then there is no pre-image of ∂^{n_2} in

$\overline{WN_{0,0,1}}_{n_1}$. This contradiction shows that there is no isomorphism between them. ■

Actually, if $n_1 \neq n_2$, then there is no non-zero non-associative algebra homomorphism from $\overline{WN_{0,0,1}}_{n_1}$ to the non-associative algebra $\overline{WN_{0,0,1}}_{n_2}$. Similar proof of Proposition 2.10, if $n_1 \neq n_2$, then there is no semi-Lie algebra isomorphism from $\overline{WN_{0,0,1}}_{n_1[,]}$ to $\overline{WN_{0,0,1}}_{n_2[,]}$. The semi-Lie algebras $\overline{WN_{0,0,1}}_{n_1[,]}$ and $\overline{WN_{0,0,1}}_{n_2[,]}$ are simple. Thus if $n_1 \neq n_2$, then it is easy to prove that there is no non-zero homomorphism from $\overline{WN_{0,0,1}}_{n_1[,]}$ to $\overline{WN_{0,0,1}}_{n_2[,]}$.

Theorem 2.11. *Let L_1 be a m_1-abelian Lie (resp. semi-Lie) algebra and L_2 be a m_2-abelian Lie (resp. semi-Lie) algebra. If $m_1 \neq m_2$, then L_1 is not isomorphic to L_2 as Lie (resp. semi-Lie) algebras.*

Proof. The proof of the Theorem is standard, so we omit the proof. ■

Proposition 2.12. *If $n_1 \neq n_2$, then there is no non-zero non-associative (resp. semi-Lie) endomorphism θ of $\overline{WN_{0,0,2}}_{n_1,n_2}$ (resp. $\overline{WN_{0,0,2}}_{n_1,n_2[,]}$) such that $\theta(\partial_1^{n_1}) = c\partial_2^{n_2}$, where c is a non-zero scalar.*

Proof. If there is a non-zero endomorphism between them, then we can derive a contradiction easily. We omit the proof of the Proposition. ■

Proposition 2.13. *The non-associative algebra $\overline{WN_{0,0,2}}_{n_1,n_2}$ has a subalgebra spanned by $\frac{x_1^{n_1}}{n_1!}\partial_1^{n_1}$, $\frac{x_1^{n_1}}{n_1!}\partial_2^{n_2}$, $\frac{x_2^{n_2}}{n_2!}\partial_1^{n_1}$, and $\frac{x_2^{n_2}}{n_2!}\partial_2^{n_2}$, which is isomorphic to the matrix ring $M_2(\mathbf{F})$. Thus the semi-Lie algebra $\overline{WN_{0,0,2}}_{n_1,n_2[,]}$ has a Lie subalgebra which is isomorphic to $sl_2(\mathbf{F})$.*

Proof. The proof of this Proposition is straightforward, and we hence omit the details. ■

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