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JORDAN $*$-DERIVATIONS AND QUADRATIC JORDAN $*$-DERIVATIONS ON REAL C^*-ALGEBRAS AND REAL JC^*-ALGEBRAS

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In this work, we introduce quadratic Jordan $*$-derivations on real C^*-algebras and real JC^*-algebras and prove the Hyers–Ulam stability of Jordan $*$-derivations and of quadratic Jordan $*$-derivations on real C^*-algebras and real JC^*-algebras. We also establish the superstability of such derivations on real C^*-algebras and real JC^*-algebras by using a fixed point theorem.

Keywords: Hyers–Ulam stability; Jordan $*$-derivation; quadratic Jordan $*$-derivation; real C^*-algebra; real JC^*-algebra.

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1. Introduction

Suppose that $\mathcal{A}$ is a complex Banach $*$-algebra. A $\mathbb{C}$-linear mapping $\delta : D(\delta) \to \mathcal{A}$ is said to be a *derivation* on $\mathcal{A}$ if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$, where $D(\delta)$ is a domain of δ and $D(\delta)$ is dense in $\mathcal{A}$. If δ satisfies the additional condition $\delta(a^*) = \delta(a)^*$ for all $a \in \mathcal{A}$, then δ is called a $*$-*derivation* on $\mathcal{A}$. It is well-known that if $\mathcal{A}$ is a C^*-algebra and $D(\delta)$ is $\mathcal{A}$, then the derivation δ is bounded.

A C^*-dynamical system is a triple $(\mathcal{A}, \mathcal{G}, \alpha)$ consisting of a C^*-algebra $\mathcal{A}$, a locally compact group $\mathcal{G}$, and a pointwise norm continuous homomorphism α of $\mathcal{G}$ into the group $\mathrm{Aut}(\mathcal{A})$ of $*$-automorphisms of $\mathcal{A}$. Note that every bounded

$*$-derivation δ arises as an infinitesimal generator of a dynamical system for $\mathbb{R}$. In fact, if δ is a bounded $*$-derivation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, then there exists an element h in the enveloping von Neumann algebra $\mathcal{A}''$, the second dual of $\mathcal{A}$, such that $\delta(x) = ad_{ih}(x)$ for all $x \in \mathcal{A}$. If, for each $t \in \mathbb{R}$, α_t is defined by $\alpha_t(x) = e^{ith}xe^{-ith}$ for all $x \in \mathcal{A}$, then α_t is a $*$-automorphism of $\mathcal{A}$ induced by unitaries $U_t = e^{ith}$ for each $t \in \mathbb{R}$. The action $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$, $t \mapsto \alpha_t$, is a strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{A}$. For several reasons the theory of bounded derivations of C^*-algebras is important in the quantum mechanics (see [5, 6, 19]).

Recall that a functional equation is called *stable* if any function satisfying the functional equation "approximately" is near to a true solution of the functional equation. We say that a functional equation is *superstable* if every approximate solution is an exact solution of it (see [2]).

In 1940, Ulam [28] proposed the following question concerning stability of group homomorphisms: *under what condition does there exists an additive mapping near an approximately additive mapping*? Hyers [16] answered the problem of Ulam for the case that groups are Banach spaces. A generalized version of the theorem of Hyers for an approximately linear mapping was given by Rassias [24]. Since then, the stability problems of various functional equations have been extensively investigated by a number of authors (see [10, 12, 15, 17, 23]).

In 2003, Cădariu and Radu applied a fixed point method to the investigation of the Jensen functional equation. They presented a short and a simple proof for the Cauchy functional equation and the quadratic functional equation in [9] and [8], respectively. After that, this method has been employed by many authors to establish various functional equations. For instance, in [22], the authors established the stability and the superstability of $*$-derivations associated with the Cauchy functional equation and the Jensen functional equation by using this method (see also [3, 13, 20]).

The Hyers–Ulam stability of quadratic derivations on Banach algebras was studied in [14]. Then this is generalized to the stability and the superstability of quadratic $*$-derivations on Banach C^*-algebras in [18].

Jordan $*$-derivations were introduced in [25, 26] for the first time and then the structure of such derivations is investigated in [7]. The importance of the study of these mappings was that the fact that the problem of representing quadratic forms by sesquilinear ones is closely connected with the structure of Jordan $*$-derivations. In [1], An, Cui and Park studied Jordan $*$-derivations on C^*-algebras and Jordan $*$-derivations on JC^*-algebras associated with a special functional inequality.

In this paper, we investigate the Hyers–Ulam stability of Jordan $*$-derivations and quadratic Jordan $*$-derivations on real C^*-algebras and real JC^*-algebras. We also show that Jordan $*$-derivations and quadratic Jordan $*$-derivations on real C^*-algebras and real JC^*-algebras under which conditions are superstable.

2. Stability of Jordan ∗-Derivations

In this section, we prove the Hyers–Ulam stability of Jordan ∗-derivations on real C^*-algebras and real JC^*-algebras.

Definition 2.1. Let $\mathcal{A}$ be a real C^*-algebra. An $\mathbb{R}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is called a *Jordan ∗-derivation* if $D(a^2) = a^*D(a) + D(a)a^*$ for all $a \in \mathcal{A}$.

The mapping $D_x : \mathcal{A} \to \mathcal{A}; a \mapsto a^*x - xa^*$, where x is a fixed element in $\mathcal{A}$, is a Jordan ∗-derivation. Also, a real C^*-algebra $\mathcal{A}$, endowed with the Jordan product $a \circ b := \frac{ab+ba}{2}$ on $\mathcal{A}$, is called a real *JC^*-algebra* (see [1, 21]).

Definition 2.2. Let $\mathcal{A}$ be a real JC^*-algebra. An $\mathbb{R}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a *Jordan ∗-derivation* if $\delta(a^2) = a^* \circ \delta(a) + \delta(a) \circ a^*$ for all $a \in \mathcal{A}$.

Throughout this paper, we denote $\overbrace{\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}}^{n\text{-times}}$ by $\mathcal{A}^n$.

Theorem 2.3. *Let $\mathcal{A}$ be a real C^*-algebra. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^3 \to [0, \infty)$ such that*

$$\tilde{\varphi}(a, b, c) := \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \varphi(2^n a, 2^n b, 2^n c) < \infty, \tag{2.1}$$

$$\|f(\lambda a + b + c^2) - \lambda f(a) - f(b) - f(c)c^* - c^* f(c)\| \leq \varphi(a, b, c) \tag{2.2}$$

for all $\lambda \in \mathbb{R}$ and all $a, b, c \in \mathcal{A}$. Then there exists a unique Jordan ∗-derivation δ on $\mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \leq \tilde{\varphi}(a, a, 0) \tag{2.3}$$

for all $a \in \mathcal{A}$.

Proof. Setting $a = b, c = 0$ and $\lambda = 1$ in (2.2), we have

$$\|f(2a) - 2f(a)\| \leq \varphi(a, a, 0)$$

for all $a \in \mathcal{A}$. One can use induction to show that

$$\left\| \frac{f(2^n a)}{2^n} - \frac{f(2^m a)}{2^m} \right\| \leq \sum_{k=m}^{n-1} \frac{1}{2^{k+1}} \varphi(2^k a, 2^k a, 0) \tag{2.4}$$

for all $n > m \geq 0$ and all $a \in \mathcal{A}$. It follows from (2.1) and (2.4) that the sequence $\{\frac{f(2^n a)}{2^n}\}$ is Cauchy. Due to the completeness of $\mathcal{A}$, this sequence is convergent. Define

$$\delta(a) := \lim_{n \to \infty} \frac{f(2^n a)}{2^n} \tag{2.5}$$

for all $a \in \mathcal{A}$. Then we have

$$\delta\left(\frac{1}{2^k} a\right) = \lim_{n \to \infty} \frac{1}{2^k} \frac{f(2^{n-k} a)}{2^{n-k}} = \frac{1}{2^k} \delta(a)$$

for each $k \in \mathbb{N}$. Putting $c = 0$ and replacing a and b by $2^n a$ and $2^n b$, respectively, in (2.2), we get

$$\left\| \frac{1}{2^n} f(2^n(\lambda a + b)) - \lambda \frac{1}{2^n} f(2^n a) - \frac{1}{2^n} f(2^n b) \right\| \leq \frac{1}{2^n} \varphi(2^n a, 2^n b, 0).$$

Taking the limit as $n \to \infty$, we obtain

$$\delta(\lambda a + b) = \lambda \delta(a) + \delta(b)$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$. So δ is $\mathbb{R}$-linear. Putting $a = b = 0$ and substituting c by $2^n c$ in (2.2), we get

$$\left\| \frac{1}{2^{2n}} f(2^{2n} c^2) - \frac{1}{2^{2n}} f(2^n c)(2^n c^*) - \frac{1}{2^{2n}} (2^n c^*) f(2^n c) \right\|$$
$$\leq \frac{1}{2^{2n}} \varphi(0, 0, 2^n c) \leq \frac{1}{2^n} \varphi(0, 0, 2^n c).$$

Taking the limit as $n \to \infty$, we have

$$\delta(c^2) = \delta(c) c^* + c^* \delta(c)$$

for all $c \in \mathcal{A}$. Moreover, it follows from (2.4) with $m = 0$ and (2.5) that $\|\delta(a) - f(a)\| \leq \tilde{\varphi}(a, a, 0)$ for all $a \in \mathcal{A}$. For the uniqueness of δ, let $\widetilde{\delta} : \mathcal{A} \to \mathcal{A}$ be another Jordan $*$-derivation satisfying (2.3). Then we have

$$\begin{aligned} \|\delta(a) - \widetilde{\delta}(a)\| &= \frac{1}{2^n} \|\delta(2^n a) - \widetilde{\delta}(2^n a)\| \\ &\leq \frac{1}{2^n} (\|\delta(2^n a) - f(2^n a)\| + \|f(2^n a) - \widetilde{\delta}(2^n a)\|) \\ &\leq 2 \sum_{j=1}^{\infty} \frac{1}{2^{n+j}} \varphi(2^{n+j} a, 2^{n+j} a, 0) = 2 \sum_{j=n}^{\infty} \frac{1}{2^j} \varphi(2^j a, 2^j a, 0), \end{aligned}$$

which tends to zero as $n \to \infty$ for all $a \in \mathcal{A}$. So δ is unique. Therefore, δ is a Jordan $*$-derivation on $\mathcal{A}$, as required. □

We have the following theorem which is analogous to Theorem 2.3. Since the proof is similar, it is omitted.

Theorem 2.4. *Let $\mathcal{A}$ be a real C^*-algebra. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^3 \to [0, \infty)$ satisfying* (2.2) *and*

$$\tilde{\varphi}(a, b, c) := \sum_{n=1}^{\infty} 2^{n-1} \varphi\left(\frac{a}{2^n}, \frac{b}{2^n}, \frac{c}{2^n} \right) < \infty$$

for all $a, b, c \in \mathcal{A}$. Then there exists a unique Jordan $*$-derivation δ on $\mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \leq \tilde{\varphi}(a, a, 0)$$

for all $a \in \mathcal{A}$.

Corollary 2.5. *Let $\mathcal{A}$ be a real C^*-algebra and ε, p be positive real numbers with $p \neq 1$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying*

$$\|f(\lambda a + b + c^2) - \lambda f(a) - f(b) - cf(c) - f(c)c^*\| \leq \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p) \quad (2.6)$$

for all $\lambda \in \mathbb{R}$ and all $a, b, c \in \mathcal{A}$. Then there exists a unique Jordan ∗-derivation δ on $\mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|a\|^p \quad (2.7)$$

for all $a \in \mathcal{A}$.

Proof. Letting $a = b = c = 0$ and $\lambda = 1$ in (2.6), we get $f(0) = 0$. Now, by considering $\varphi(a, b, c) = \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p)$ in Theorems 2.3 and 2.4, we get the desired result. □

We now investigate the Hyers–Ulam stability of Jordan ∗-derivations on a real JC^*-algebra $\mathcal{A}$. Since the proofs are similar to the above results, we omit them.

Theorem 2.6. *Let $\mathcal{A}$ be a real JC^*-algebra. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^3 \to [0, \infty)$ such that*

$$\tilde{\varphi}(a, b, c) := \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}\varphi(2^n a, 2^n b, 2^n c) < \infty,$$

$$\|f(\lambda a + b + c^2) - \lambda f(a) - f(b) - f(c) \circ c^* - c^* \circ f(c)\| \leq \varphi(a, b, c) \quad (2.8)$$

for all $\lambda \in \mathbb{R}$ and all $a, b, c \in \mathcal{A}$. Then there exists a unique Jordan ∗-derivation δ on $\mathcal{A}$ satisfying $\|f(a) - \delta(a)\| \leq \tilde{\varphi}(a, a, 0)$ for all $a \in \mathcal{A}$.

Proof. See the proof of Theorem 2.3. □

The next theorem is in analogy with Theorem 2.4 for real JC^*-algebras.

Theorem 2.7. *Let $\mathcal{A}$ be a real JC^*-algebra. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^3 \to [0, \infty)$ satisfying* (2.8) *and*

$$\tilde{\varphi}(a, b, c) := \sum_{n=1}^{\infty} 2^{n-1}\varphi\left(\frac{a}{2^n}, \frac{b}{2^n}, \frac{c}{2^n}\right) < \infty$$

for all $a, b, c \in \mathcal{A}$. Then there exists a unique Jordan ∗-derivation δ on $\mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \leq \tilde{\varphi}(a, a, 0)$$

for all $a \in \mathcal{A}$.

Corollary 2.8. *Let $\mathcal{A}$ be a real JC^*-algebra and ε, p be positive real numbers with $p \neq 1$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying*

$$\|f(\lambda a + b + c^2) - \lambda f(a) - f(b) - c^* \circ f(c) - f(c) \circ c^*\| \leq \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p)$$

for all $\lambda \in \mathbb{R}$ and all $a, b, c \in \mathcal{A}$. Then there exists a unique Jordan $*$-derivation δ on $\mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|a\|^p$$

for all $a \in \mathcal{A}$.

Proof. The result follows from Theorems 2.6 and 2.7 by putting $\varphi(a, b, c) = \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p)$. □

3. Stability of Quadratic Jordan $*$-Derivations

In this section, we prove the Hyers–Ulam stability of quadratic Jordan $*$-derivations on real C^*-algebras and real JC^*-algebras. Recall that the functional equation

$$f(a + b) + f(a - b) = 2f(a) + 2f(b)$$

is called *quadratic* functional equation. In addition, every solution of the above equation is said to be a *quadratic mapping*. First, we introduce quadratic Jordan $*$-derivations on real C^*-algebras and real JC^*-algebras as follows.

Definition 3.1. Let $\mathcal{A}$ be a real C^*-algebra. A mapping $D : \mathcal{A} \to \mathcal{A}$ is called a *quadratic Jordan $*$-derivation* if D is a quadratic $\mathbb{R}$-homogeneous mapping, that is, D is quadratic and $D(\lambda a) = \lambda^2 D(a)$ for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{R}$ and

$$D(a^2) = (a^*)^2 D(a) + D(a)(a^*)^2$$

for all $a \in \mathcal{A}$.

The mapping $D_x : \mathcal{A} \to \mathcal{A}; a \mapsto (a^*)^2 x - x(a^*)^2$, where x is a fixed element in $\mathcal{A}$, is a quadratic Jordan $*$-derivation.

Definition 3.2. Let $\mathcal{A}$ be a real JC^*-algebra. A mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a *quadratic Jordan $*$-derivation* if δ is a quadratic $\mathbb{R}$-homogeneous mapping and

$$\delta(a^2) = (a^*)^2 \circ \delta(a) + \delta(a) \circ (a^*)^2$$

for all $a \in \mathcal{A}$.

Theorem 3.3. *Let $\mathcal{A}$ be a real C^*-algebra. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^2 \to [0, \infty)$ such that*

$$\tilde{\varphi}(a, b) := \sum_{k=0}^{\infty} \frac{1}{4^k} \varphi(2^k a, 2^k b) < \infty,$$

$$\|f(\lambda a + \lambda b) + f(\lambda a - \lambda b) - 2\lambda^2 f(a) - 2\lambda^2 f(b)\| \leq \varphi(a, b), \tag{3.1}$$

$$\|f(a^2) - f(a)(a^*)^2 - (a^*)^2 f(a)\| \leq \varphi(a, a) \tag{3.2}$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$. Then there exists a unique quadratic Jordan $*$-derivation δ on $\mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \leq \frac{1}{4} \tilde{\varphi}(a, a) \tag{3.3}$$

for all $a \in \mathcal{A}$.

Proof. Putting $a = b$ and $\lambda = 1$ in (3.1), we have

$$\|f(2a) - 4f(a)\| \leq \varphi(a, a)$$

for all $a \in \mathcal{A}$. We can deduce by induction that

$$\left\| \frac{f(2^n a)}{4^n} - \frac{f(2^m a)}{4^m} \right\| \leq \frac{1}{4} \sum_{k=m}^{n-1} \frac{\varphi(2^k a, 2^k a)}{4^k} \tag{3.4}$$

for all $n > m \geq 0$ and all $a \in \mathcal{A}$. Since the right-hand side of the inequality (3.4) tends to 0 as m and n tend to infinity, the sequence $\{\frac{f(2^n a)}{4^n}\}$ is Cauchy. Now, since $\mathcal{A}$ is complete, this sequence can be convergent to a mapping, say δ. Indeed,

$$\delta(a) := \lim_{n \to \infty} \frac{f(2^n a)}{4^n}.$$

Since $f(0) = 0$, we have $\delta(0) = 0$. Replacing a and b by $2^n a$ and $2^n b$, respectively, in (3.1), we get

$$\left\| \frac{f(2^n(\lambda a + \lambda b))}{4^n} + \frac{f(2^n(\lambda a - \lambda b))}{4^n} - 2\lambda^2 \frac{f(2^n a)}{4^n} - 2\lambda^2 \frac{f(2^n b)}{4^n} \right\| \leq \frac{\varphi(2^n a, 2^n b)}{4^n}.$$

Taking the limit as $n \to \infty$, we obtain

$$\delta(\lambda a + \lambda b) + \delta(\lambda a - \lambda b) = 2\lambda^2 \delta(a) + 2\lambda^2 \delta(b) \tag{3.5}$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$. Putting $\lambda = 1$ in (3.5), we obtain that δ is a quadratic mapping. It is easy to check that the quadratic mapping δ satisfying (3.3) is unique (see the proof of Theorem 2.3). Setting $b = a$ in (3.5), we get $\delta(2\lambda a) = 4\lambda^2 \delta(a)$ for all $a \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$. Hence $\delta(\lambda a) = \lambda^2 \delta(a)$ for all $a \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$.

Replacing a by $2^n a$, in (3.2), we get

$$\begin{aligned}&\left\|\frac{f(2^n a\cdot 2^n a)}{4^{2n}}-\frac{2^{2n}(a^*)^2 f(2^n a)}{4^{2n}}-\frac{f(2^n a)2^{2n}(a^*)^2}{4^{2n}}\right\|\\ &\quad=\left\|\frac{f(2^{2n}a^2)}{4^{2n}}-\frac{2^{2n}(a^*)^2}{2^{2n}}\frac{f(2^n a)}{4^n}-\frac{f(2^n a)}{4^n}\frac{2^{2n}(a^*)^2}{2^{2n}}\right\|\\ &\quad\le\frac{\varphi(2^n a,2^n a)}{4^{2n}}\le\frac{\varphi(2^n a,2^n a)}{4^n}\end{aligned}$$

for all $a\in\mathcal{A}$. Thus we have

$$\|\delta(a^2)-(a^*)^2\delta(a)-\delta(a)(a^*)^2\|\le\lim_{n\to\infty}\frac{\varphi(2^n a,2^n a)}{4^n}=0.$$

The above statement shows that δ is a quadratic Jordan $*$-derivation on $\mathcal{A}$ which is unique. □

The following parallel result can be proved in a similar way, and so we omit its proof.

Theorem 3.4. *Let $\mathcal{A}$ be a real C^*-algebra. Suppose that $f:\mathcal{A}\to\mathcal{A}$ is a mapping with $f(0)=0$ for which there exists a function $\varphi:\mathcal{A}^2\to[0,\infty)$ satisfying* (3.1), (3.2) *and*

$$\tilde{\varphi}(a,b):=\sum_{k=1}^{\infty}4^k\varphi\left(\frac{a}{2^k},\frac{b}{2^k}\right)<\infty$$

for all $a,b\in\mathcal{A}$. Then there exists a unique quadratic Jordan $*$-derivation δ on $\mathcal{A}$ satisfying

$$\|f(a)-\delta(a)\|\le\frac{1}{4}\tilde{\varphi}(a,a)$$

for all $a\in\mathcal{A}$.

Corollary 3.5. *Let $\mathcal{A}$ be a real C^*-algebra and ε,p be positive real numbers with $p\neq 2$. Suppose that $f:\mathcal{A}\to\mathcal{A}$ is a mapping such that*

$$\|f(\lambda a+\lambda b)+f(\lambda a-\lambda b)-2\lambda^2 f(a)-2\lambda^2 f(b)\|\le\varepsilon(\|a\|^p+\|b\|^p),$$

$$\|f(a^2)-a^2 f(a)-f(a)(a^*)^2\|\le 2\varepsilon\|a\|^p$$

for all $a,b\in\mathcal{A}$ and all $\lambda\in\mathbb{R}$. Then there exists a unique quadratic Jordan $*$-derivation $\delta:\mathcal{A}\to\mathcal{A}$ satisfying

$$\|f(a)-\delta(a)\|\le\frac{2\varepsilon}{|4-2^p|}\|a\|^p \tag{3.6}$$

for all $a\in\mathcal{A}$.

Proof. Defining $\varphi(a,b)=\varepsilon(\|a\|^p+\|b\|^p)$ and applying Theorems 3.3 and 3.4, we obtain the result. □

From here to the end of this section, we assume that $\mathcal{A}$ is a real JC^*-algebra. In the upcoming theorems, we indicate the Hyers–Ulam stability of quadratic Jordan $*$-derivations on $\mathcal{A}$. Since the proofs are similar to the case that $\mathcal{A}$ is a real C^*-algebra, we skip them.

Theorem 3.6. *Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^2 \to [0, \infty)$ such that*

$$\tilde{\varphi}(a, b) := \sum_{k=0}^{\infty} \frac{1}{4^k} \varphi(2^k a, 2^k b) < \infty,$$

$$\|f(\lambda a + \lambda b) + f(\lambda a - \lambda b) - 2\lambda^2 f(a) - 2\lambda^2 f(b)\| \le \varphi(a, b), \tag{3.7}$$

$$\|f(a^2) - (a^*)^2 \circ f(a) - f(a) \circ (a^*)^2\| \le \varphi(a, a) \tag{3.8}$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$. Then there exists a unique quadratic Jordan $*$-derivation δ on $\mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \le \frac{1}{4} \tilde{\varphi}(a, a)$$

for all $a \in \mathcal{A}$.

Theorem 3.7. *Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^2 \to [0, \infty)$ satisfying* (3.7), (3.8) *and*

$$\tilde{\varphi}(a, b) := \sum_{k=1}^{\infty} 4^k \varphi\left(\frac{a}{2^k}, \frac{b}{2^k}\right) < \infty$$

for all $a, b \in \mathcal{A}$. Then there exists a unique quadratic Jordan $*$-derivation δ on $\mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \le \frac{1}{4} \tilde{\varphi}(a, a)$$

for all $a \in \mathcal{A}$.

Corollary 3.8. *Let ε, p be positive real numbers with $p \neq 2$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping such that*

$$\|f(\lambda a + \lambda b) + f(\lambda a - \lambda b) - 2\lambda^2 f(a) - 2\lambda^2 f(b)\| \le \varepsilon(\|a\|^p + \|b\|^p),$$

$$\|f(a^2) - (a^*)^2 \circ f(a) - f(a) \circ (a^*)^2\| \le 2\varepsilon \|a\|^p$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$. Then there exists a unique quadratic Jordan $*$-derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \le \frac{2\varepsilon}{|4 - 2^p|} \|a\|^p$$

for all $a \in \mathcal{A}$.

Proof. We can obtain the result by letting $\varphi(a, b) = \varepsilon(\|a\|^p + \|b\|^p)$ in Theorems 3.6 and 3.7. □

4. A Fixed Point Approach

In this section, we establish the Hyers–Ulam stability and the superstability of Jordan $*$-derivations and of quadratic Jordan $*$-derivations on real C^*-algebras and real JC^*-algebras by using the fixed point method (Theorem 4.1). To prove the main results, we bring this theorem which has been proved by Diaz and Margolis in [11]. Later, an extension of this result has been given by Turinici in [27].

Theorem 4.1 (The fixed point alternative). *Let (Ω, d) be a complete generalized metric space and $T : \Omega \to \Omega$ be a mapping with Lipschitz constant $L < 1$. Then, for each element $\alpha \in \Omega$, either $d(T^n\alpha, T^{n+1}\alpha) = \infty$ for all $n \geq 0$, or there exists a natural number n_0 such that*:

(i) *$d(T^n\alpha, T^{n+1}\alpha) < \infty$ for all $n \geq n_0$;*
(ii) *the sequence $\{T^n\alpha\}$ is convergent to a fixed point β^* of T;*
(iii) *β^* is the unique fixed point of T in the set $\Lambda = \{\beta \in \Omega : d(T^{n_0}\alpha, \beta) < \infty\}$;*
(iv) *$d(\beta, \beta^*) \leq \frac{1}{1-L} d(\beta, T\beta)$ for all $\beta \in \Lambda$.*

Here and subsequently, we suppose that $\mathcal{A}$ is a real C^*-algebra.

Theorem 4.2. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping with $f(0) = 0$ and let $\varphi : \mathcal{A}^3 \to [0, \infty)$ be a function such that*

$$\|f(\lambda a + b + c^2) - \lambda f(a) - f(b) - f(c)c^* - c^* f(c)\| \leq \varphi(a, b, c) \tag{4.1}$$

for all $\lambda \in \mathbb{R}$ and $a, b, c \in \mathcal{A}$. If there exists a constant $k \in (0, 1)$ such that

$$\varphi(2a, 2b, 2c) \leq 2k\varphi(a, b, c) \tag{4.2}$$

for all $a, b, c \in \mathcal{A}$, then there exists a unique Jordan $*$-derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \leq \frac{1}{2(1-k)}\varphi(a, a, 0) \tag{4.3}$$

for all $a \in \mathcal{A}$.

Proof. It follows from (4.2) that

$$\lim_{j\to\infty} \frac{\varphi(2^j a, 2^j b, 2^j c)}{2^j} = 0$$

for all $a, b, c \in \mathcal{A}$. Putting $\lambda = 1, a = b$, and $c = 0$ in (4.1), we have

$$\|f(2a) - 2f(a)\| \leq \varphi(a, a, 0)$$

for all $a \in \mathcal{A}$, and so

$$\left\|f(a) - \frac{1}{2}f(2a)\right\| \leq \frac{1}{2}\varphi(a, a, 0) \tag{4.4}$$

for all $a \in A$. We consider the set $\Omega := \{h : \mathcal{A} \to \mathcal{A} \,|\, h(0) = 0\}$ and introduce the generalized metric on Ω as follows:

$$d(h_1, h_2) := \inf\{C \in (0, \infty) : \|h_1(a) - h_2(a)\| \leq C\varphi(a, a, 0),\ \forall\, a \in \mathcal{A}\},$$

if there exists such constant C, and $d(h_1, h_2) = \infty$, otherwise. Similar to the proof of [4, Theorem 2.2], one can show that d is a generalized metric on Ω and the metric space (Ω, d) is complete. We now define the linear mapping $T : \Omega \to \Omega$ by

$$Th(a) = \frac{1}{2}h(2a) \tag{4.5}$$

for all $a \in \mathcal{A}$. Given $h_1, h_2 \in \Omega$. Let $C \in \mathbb{R}^+$ be an arbitrary constant with $d(h_1, h_2) \leq C$, that is,

$$\|h_1(a) - h_2(a)\| \leq C\varphi(a, a, 0) \tag{4.6}$$

for all $a \in \mathcal{A}$. Substituting a by $2a$ in the inequality (4.6) and using the equalities (4.2) and (4.5), we have

$$\|Th_1(a) - Th_2(a)\| = \frac{1}{2}\|h_1(2a) - h_2(2a)\| \leq \frac{1}{2}C\varphi(2a, 2a, 0) \leq Ck\varphi(2a, 2a, 0)$$

for all $a \in \mathcal{A}$, and thus $d(Th_1, Th_2) \leq Ck$. Therefore, we conclude that $d(Th_1, Th_2) \leq kd(h_1, h_2)$ for all $h_1, h_2 \in \Omega$. The inequality (4.4) implies that

$$d(Tf, f) \leq \frac{1}{2}. \tag{4.7}$$

It also follows from Theorem 4.1 that $d(T^n h, T^{n+1} h) < \infty$ for all $n \geq 0$, and thus in this theorem we have $n_0 = 0$. Therefore, Parts (iii) and (iv) of Theorem 4.1 hold on the whole Ω. Thus, the sequence $\{T^n f\}$ converges to a unique fixed point $\delta : \mathcal{A} \to \mathcal{A}$ in the set $\Omega_1 = \{h \in \Omega;\ d(f, h) < \infty\}$, that is,

$$\lim_{n\to\infty} \frac{f(2^n a)}{2^n} = \delta(a)$$

for all $a \in \mathcal{A}$. By Theorem 4.1 and (4.7), we have

$$d(f, \delta) \leq \frac{d(Tf, f)}{1 - k} \leq \frac{1}{2(1 - k)}.$$

The above inequalities show that (4.3) holds for all $a \in \mathcal{A}$. Similar to the proof of Theorem 2.3, we can deduce that δ is $\mathbb{R}$-linear by letting $c = 0$ and replacing a and b by $2^n a$ and $2^n b$, respectively, in (4.1). By a similar way we have $\delta(c^2) = \delta(c)c^* + c^*\delta(c)$ for all $c \in \mathcal{A}$. □

The following corollary shows that we can obtain a more accurate approximation of (2.7) in the case $p < 1$.

Corollary 4.3. *Let p, θ be non-negative real numbers with $p < 1$ and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping such that*

$$\begin{aligned} &\|f(\lambda a + b + c^2) - \lambda f(a) - f(b) - f(c)c^* - c^* f(c)\| \\ &\quad \le \theta(\|a\|^p + \|b\|^p + \|c\|^p) \end{aligned} \tag{4.8}$$

for all $\lambda \in \mathbb{R}$ and all $a, b, c \in \mathcal{A}$. Then there exists a unique Jordan $*$-derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \le \frac{\theta}{2 - 2^p}\|a\|^p$$

for all $a \in \mathcal{A}$.

Proof. First, note that the inequality (4.8) implies that $f(0) = 0$. Now, the result follows from Theorem 4.2 by taking $\varphi(a, b, c) = \theta(\|a\|^p + \|b\|^p + \|c\|^p)$. □

In the following corollary, under some conditions, the superstability for Jordan $*$-derivations on real C^*-algebras is shown.

Corollary 4.4. *Let p, q, r, θ be non-negative real numbers such that $p + q + r \in (0, 1)$. Suppose that a mapping $f : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\|f(\lambda a + b + c^2) - \lambda f(a) - f(b) - f(c)c^* - c^* f(c)\| \le \theta(\|a\|^p\|b\|^q\|c\|^r) \tag{4.9}$$

for all $a, b, c \in \mathcal{A}$. Then f is a Jordan $*$-derivation on $\mathcal{A}$.

Proof. Letting $a = b = c = 0$ in (4.9), we have $f(0) = 0$. Once more, if we put $\lambda = 1, c = 0$ and $a = b$ in (4.9), then we get $f(2a) = 2f(a)$ for all $a \in \mathcal{A}$. It is easy to see that by induction, we have $f(2^n a) = 2^n f(a)$, and so $f(a) = \frac{f(2^n a)}{2^n}$ for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$. Now, it follows from Theorem 4.2 that f is a Jordan $*$-derivation. □

Note that in Corollary 4.4, if $p+q+r \in (0, 1)$ and $p > 0$ such that the inequality (4.9) holds, then by applying $\varphi(a, b, c) = \theta(\|a\|^p\|b\|^q\|c\|^r)$ in Theorem 4.2, f is again a Jordan $*$-derivation.

The following parallel theorem for the stability of quadratic Jordan $*$-derivations on real C^*-algebras can be proved in a similar method to Theorem 4.2. But, we include the proof.

Theorem 4.5. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping with $f(0) = 0$ and let $\varphi : \mathcal{A}^2 \to [0, \infty)$ be a function such that*

$$\begin{aligned} &\|f(\lambda a + \lambda b) + f(\lambda a - \lambda b) - 2\lambda^2 f(a) - 2\lambda^2 f(b)\| \le \varphi(a, b), \\ &\|f(a^2) - f(a)(a^*)^2 - (a^*)^2 f(a)\| \le \varphi(a, a) \end{aligned} \tag{4.10}$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$. If there exists a constant $k \in (0, 1)$ such that

$$\varphi(2a, 2b) \le 4k\varphi(a, b) \tag{4.11}$$

for all $a, b \in \mathcal{A}$, then there exists a unique quadratic Jordan $*$-derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \leq \frac{1}{4(1-k)}\varphi(a, a) \tag{4.12}$$

for all $a \in \mathcal{A}$.

Proof. Similar to the proof of Theorem 4.2, we consider the set $\Omega = \{g : \mathcal{A} \to \mathcal{A} \,|\, g(0) = 0\}$ and define the following mapping d on $\Omega \times \Omega$:

$$d(g, h) := \inf\{c \in (0, \infty) : \|g(a) - h(a)\| \leq c\phi(a, a), \text{ for all}\, a \in \mathcal{A}\},$$

if there exists such constant c, and $d(g, h) = \infty$, otherwise. One can easily show that (Ω, d) is complete (see the proof of Theorem 4.2). Now, we consider the mapping $T : \Omega \to \Omega$ defined by

$$Tg(a) = \frac{1}{4}g(2a), \quad (a \in \mathcal{A}).$$

Given $g, h \in \Omega$ with $d(g, h) < c$. By definition of d and T, we get

$$\left\|\frac{1}{4}g(2a) - \frac{1}{4}h(2a)\right\| \leq \frac{1}{4}c\varphi(2a, 2a)$$

for all $a \in \mathcal{A}$. Using (4.11), we have

$$\left\|\frac{1}{4}g(2a) - \frac{1}{4}h(2a)\right\| \leq ck\varphi(a, a)$$

for all $a \in \mathcal{A}$. The above inequality shows that $d(Tg, Th) \leq kd(g, h)$ for all $g, h \in \Omega$. Hence, T is a strictly contractive mapping on Ω with a Lipschitz constant k. Now, we prove that $d(Tf, f) < \infty$. Putting $a = b$ and $\lambda = 1$ in (4.10), we obtain $\|f(2a) - 4f(a)\| \leq \varphi(a, a)$ for all $a \in \mathcal{A}$. Hence

$$\left\|\frac{1}{4}f(2a) - f(a)\right\| \leq \frac{1}{4}\varphi(a, a) \tag{4.13}$$

for all $a \in \mathcal{A}$. We deduce from (4.13) that $d(Tf, f) \leq \frac{1}{4}$. It follows from Theorem 4.1 that $d(T^n g, T^{n+1} g) < \infty$ for all $n \geq 0$, and thus the parts (iii) and (iv) of this theorem hold on the whole Ω. Hence there exists a unique mapping $\delta : \mathcal{A} \to \mathcal{A}$ such that δ is a fixed point of T and that $T^n f \to \delta$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} \frac{f(2^n a)}{4^n} = \delta(a)$$

for all $a \in \mathcal{A}$, and so

$$d(f, \delta) \leq \frac{1}{1-k}d(Tf, f) \leq \frac{1}{4(1-k)}.$$

The above equalities show that (4.12) is true for all $a \in \mathcal{A}$. Now, it follows from (4.11) that

$$\lim_{n\to\infty} \frac{\varphi(2^n a, 2^n b)}{4^n} = 0.$$

The rest of the proof is similar to the proof of Theorem 3.3. □

In the following corollary, we find a more accurate approximation relative to Corollary 3.5 with the same conditions on the mapping f when $p < 2$. In fact, we obtain a refinement of the inequality (3.6).

Corollary 4.6. *Let θ, p be positive real numbers with $p < 2$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping such that*

$$\begin{gathered}\|f(\lambda a + \lambda b) + f(\lambda a - \lambda b) - 2\lambda^2 f(a) - 2\lambda^2 f(b)\| \leq \theta(\|a\|^p + \|b\|^p), \\ \|f(a^2) - (a^*)^2 f(a) - f(a)(a^*)^2\| \leq 2\theta\|a\|^p\end{gathered} \tag{4.14}$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$. Then there exists a unique quadratic Jordan $*$-derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfying

$$\|f(a) - \delta(a)\| \leq \frac{\theta}{4 - 2^p}\|a\|^p$$

for all $a \in \mathcal{A}$.

Proof. If we put $a = b = 0$ and $\lambda = 1$ in (4.14), we get $f(0) = 0$. Letting $\varphi(a, b) = \theta(\|a\|^p + \|b\|^p)$ in Theorem 4.5, we obtain the result. □

The next result shows that under what conditions a quadratic Jordan $*$-derivation on a real C^*-algebra is superstable.

Corollary 4.7. *Let θ, p, q be positive real numbers with $p + q \neq 2$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping such that*

$$\|f(\lambda a + \lambda b) + f(\lambda a - \lambda b) - 2\lambda^2 f(a) - 2\lambda^2 f(b)\| \leq \theta(\|a\|^p\|b\|^q), \tag{4.15}$$

$$\|f(a^2) - (a^*)^2 f(a) - f(a)(a^*)^2\| \leq \theta\|a\|^{p+q} \tag{4.16}$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$. Then f is a quadratic Jordan $*$-derivation on $\mathcal{A}$.

Proof. Putting $a = b = 0$ in (4.15), we get $f(0) = 0$. Now, if we put $a = b$, $\lambda = 1$ in (4.15), then we have $f(2a) = 4f(a)$ for all $a \in \mathcal{A}$. It is easy to see by induction that $f(2^n a) = 4^n f(a)$, and so $f(a) = \frac{f(2^n a)}{4^n}$ for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$. It follows from Theorem 4.5 that f is a quadratic homogeneous mapping. Letting $\varphi(a, b) = \theta(\|a\|^p\|b\|^q)$ in Theorem 4.5, we can obtain the desired result. □

One should remember that all of the results in this section hold when we replace a real C^*-algebra by a real JC^*-algebra with its corresponding product.

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References

[1] J. An, J. Cui and C. Park, Jordan $*$-derivations on C^*-algebras and JC^*-algebras, *Abstr. Appl. Anal.* **2008** (2008), Article ID: 410437, 12pp.

[2] J. Baker, The stability of the cosine equation, *Proc. Amer. Math. Soc.* **80** (1979) 242–246.

[3] A. Bodaghi, I. A. Alias and M. E. Gordji, On the stability of quadratic double centralizers and quadratic multipliers: A fixed point approach, *J. Inequal. Appl.* **2011** (2011) 957541, 9 pp.

[4] A. Bodaghi, I. A. Alias and M. H. Ghahramani, Ulam stability of a quartic functional equation, *Abstr. Appl. Anal.* **2012** (2012) 232630, 9 pp., doi: 10.1155/2012/232630.

[5] O. Bratteli, *Derivation, Dissipation and Group Actions on C^*-Algebras*, Lecture Notes in Mathematics, Vol. 1229 (Springer-Verlag, Berlin, 1986).

[6] O. Bratteli, F. M. Goodman and P. E. T. Jørgensen, Unbounded derivations tangential to compact groups of automorphisms II, *J. Funct. Anal.* **61** (1985) 247–289.

[7] M. Brešar and B. Zalar, On the structure of Jordan $*$-derivations, *Colloq. Math.* **63** (1992) 163–171.

[8] L. Cădariu and V. Radu, Fixed points and the stability of quadratic functional equations, *Ann. Univ. Timişoara, Ser. Mat. Inform.* **41** (2003) 25–48.

[9] L. Cădariu and V. Radu, On the stability of the Cauchy functional equation: A fixed point approach, *Grazer Math. Ber.* **346** (2004) 43–52.

[10] S. Czerwik, On the stability of the quadratic mapping in normed spaces, *Abh. Math. Sem. Univ. Hamburg* **62** (1992) 59–64.

[11] J. B. Diaz and B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, *Bull. Amer. Math. Soc.* **74** (1968) 305–309.

[12] M. E. Gordji and A. Bodaghi, On the Hyers–Ulam–Rassias stability problem for quadratic functional equations, *East J. Approx.* **16**(2) (2010) 123–130.

[13] M. E. Gordji, A. Bodaghi and C. Park, A fixed point approach to the stability of double Jordan centralizers and Jordan multipliers on Banach algebras, *Politehn. Univ. Bucharest Sci. Bull. Ser. A* **73**(2) (2011) 65–73.

[14] M. E. Gordji and F. Habibian, Hyers–Ulam–Rassias stability of quadratic derivations on Banach Algebras, *Nonlinear Funct. Anal. Appl.* **14** (2009) 759–766.

[15] Z. Gajda, On stability of additive mappings, *Int. J. Math. Math. Sci.* **14** (1991) 431–434.

[16] D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA* **27** (1941) 222–224.

[17] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables* (Birkhäuser, Basel, 1998).

[18] S. Jang and C. Park, Approximate $*$-derivations and approximate quadratic $*$-derivations on C^*-algebras, *J. Inequal. Appl.* **2011** (2011) 55, 13 pp.

[19] S. Lee and S. Jang, Unbounded derivations on compact actions of C^*-algebras. *Commun. Korean Math. Soc.* **5** (1990) 79–86.

[20] C. Park, Fixed points and Hyers–Ulam–Rassias stability of Cauchy–Jensen functional equations in Banach algebras, *Fixed Point Theory Appl.* **2007** (2007) 50175, 15 pp.

[21] C. Park, Homomorphisms between Poisson JC^*-algebras, *Bull. Brazilian Math. Soc.* **36** (2005) 79–97.

[22] C. Park and A. Bodaghi, On the stability of $*$-derivations on Banach $*$-algebras, *Adv. Difference Equations* **2012** (2012) 138, 10 pp.

[23] C. Park and D. Boo, Isomorphisms and generalized derivations in proper CQ^*-algebras, *J. Nonlinear Sci. Appl.* **4** (2011) 19–36.

[24] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978) 297–300.
[25] P. Šemrl, On Jordan *-derivations and an application, *Colloq. Math.* **59** (1990) 241–251.
[26] P. Šemrl, Quadratic functionals and Jordan ∗-derivations, *Studia Math.* **97** (1991) 157–163.
[27] M. Turinici, Sequentially iterative processes and applications to Volterra functional equations, *Ann. Univ. Mariae Curie-Sklodowska Sect. A* **32** (1978) 127–134.
[28] S. M. Ulam, *Problems in Modern Mathematics*, Chap. VI, Science edn. (Wiley, New York, 1940).