A FIXED POINT APPROACH TO THE STABILITY OF DOUBLE JORDAN CENTRALIZERS AND JORDAN MULTIPLIERS ON BANACH ALGEBRAS

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We say a functional equation (ξ) *is stable if any function* g *satisfying the equation* (ξ) *approximately is near to true solution of* (ξ)*, moreover, a functional equation* (ξ) *is superstable if any function* g *satisfying the equation* (ξ) *approximately is a true solution of* (ξ)*. In the present paper, we investigate the stability and the superstability of double centralizers and of multipliers on Banach algebras by using the fixed point methods.*

Keywords: Double centralizer; Multiplier; Hyers-Ulam stability.
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1. Introduction

In this paper, we assume that A is a complex Banach algebra. A linear mapping $L : A \to A$ is said to be *left Jordan centralizer* on A if $L(a^2) = L(a)a$ for all $a \in A$. Similarly, a linear mapping $R : A \to A$ satisfying that $R(a^2) = aR(a)$ for all $a \in A$ is called *right Jordan centralizer* on A. A *double Jordan centralizer* on A is a pair (L, R), where L is a left Jordan centralizer, R is a right Jordan centralizer and $aL(b) = R(a)b$ for all $a, b \in A$. For example, (L_c, R_c) is a double Jordan centralizer, where $L_c(a) := ca$ and $R_c(a) := ac$ (see [9] and [11]).

A mapping $T : A \to A$ is said to be a *Jordan multiplier* (see [12]) if $aT(a) = T(a)a$ for all $a \in A$. Clearly, if $A_l(A) = \{0\}$ ($A_r(A) = \{0\}$, respectively) then T is a left (right) Jordan centralizer (see [14]).

In 1940, Ulam [26] proposed the following question concerning stability of group homomorphisms: *under what condition does there is an additive mapping near an approximately additive mapping?* Hyers [10] answered the problem of Ulam for the case where G_1 and G_2 are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mapping was given by Th.

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M. Rassias [25]. After that, several functional equations have been extensively investigated by a number of authors (for instances, [4]–[7], [13] and [15]–[24]).

In 2003, Cădariu and Radu applied the fixed point method to the investigation of the Jensen functional equation [1] (see also [2, 3, 8]). They presented a short and a simple proof (no "*direct method*", initiated by Hyers in 1941) for the Hyers-Ulam stability of the Jensen functional equation [1], for the Cauchy functional equation [3] and for some other functional equations [1, 2, 3].

We need the following known fixed point theorem which is useful for our goals.

Theorem 1.1. *(The alternative of fixed point* [5]*) Suppose* (Ω, d) *be a complete generalized metric space and let* $\mathfrak{J} : \Omega \to \Omega$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$*. Then for each element* $x \in \Omega$*, either* $d(\mathfrak{J}^n x, \mathfrak{J}^{n+1} x) = \infty$ *for all* $n \geq 0$*, or there exists a natural number* n_0 *such that:*
$(*)$ $d(\mathfrak{J}^n x, \mathfrak{J}^{n+1} x) < \infty$ *for all* $n \geq n_0$*;*
$(**)$ *the sequence* $\{\mathfrak{J}^n x\}$ *is convergent to a fixed point* y^* *of* $\mathfrak{J}$*;*
$(***)$ y^* *is the unique fixed point of* $\mathfrak{J}$ *in the set*

$$\Lambda = \{y \in \Omega : d(\mathfrak{J}^{n_0} x, y) < \infty\};$$

$(****)$ $d(y, y^*) \leq \frac{1}{1-L} d(y, \mathfrak{J} y)$ *for all* $y \in \Lambda$.

Moreover, we will use the following known lemma in the proof of main results of our paper. We do not remove the proof of lemma. We suppose that $\mathbb{T}^1 := \{z \in \mathbb{C} : |z| = 1\}$.

Lemma 1.1. *Let* $n_0 \in \mathbb{N}$ *be a positive integer number and let* X, Y *be linear vector spaces on* $\mathbb{C}$*. Suppose that* $f : X \to Y$ *is an additive mapping. Then* f *is* $\mathbb{C}-$*linear if and only if* $f(\lambda x) = \lambda f(x)$ *for all* x *in* X *and* λ *in* $\mathbb{T}^1_{\frac{1}{n_o}} := \{e^{i\theta} \; ; \; 0 \leq \theta \leq \frac{2\pi}{n_o}\}$.

Proof. Suppose that f is additive and $f(\lambda x) = \lambda f(x)$ for all x in X and all λ in $T^1_{\frac{1}{n_o}}$. Now, let $\mu \in T^1$. Then we have $\mu = e^{i\theta}$ such that $0 \leq \theta \leq 2\pi$. We set

$$\mu_1 = e^{\frac{i\theta}{n_o}},$$

thus μ_1 belongs to $T^1_{\frac{1}{n_o}}$ and

$$f(\mu x) = f(\mu_1^{n_o} x) = \mu_1^{n_o} f(x) = \mu f(x)$$

for all x in X. If μ belongs to $nT^1 = \{nz \; ; \; z \in T^1\}$ then by additivity of f, $f(\mu x) = \mu f(x)$ for all x in X and μ in nT^1. If $t \in (0, \infty)$ then by archimedean property there exists a natural number n such that the point $(t, 0)$ lies in the interior of circle with center at origin and radius n in $\mathbb{C}$. Put

$$t_1 := t + \sqrt{n^2 - t^2}\; i \in nT^1$$

and

$$t_2 := t - \sqrt{n^2 - t^2}\ i \in nT^1.$$

We have $t = \frac{t_1+t_2}{2}$ and

$$f(tx) = f(\frac{t_1+t_2}{2}x) = \frac{t_1+t_2}{2}f(x) = tf(x)$$

for all x in X.

If $\mu \in \mathbb{C}$, then

$$\mu = |\mu|e^{i\mu_1},$$

therefore

$$f(\mu x) = f(|\mu|e^{i\mu_1}x) = |\mu|e^{i\mu_1}f(x) = \mu f(x)$$

for all x in X. In the other words f is $\mathbb{C}-$linear. The converse is clear. □

From now on, we suppose that $n_0 \in \mathbb{N}$ is a positive integer number, and that

$$\mathbb{T}^1_{\frac{1}{n_o}} := \{e^{i\theta}\ ;\ 0 \leq \theta \leq \frac{2\pi}{n_o}\}.$$

2. Stability of double centralizers

For every $x, y \in A$, we put $x^0 - y^0 = 0, x^0 y = y$ and also denote $\overbrace{A \times A \times ... \times A}^{n-times}$ by A^n. We establish the Hyers-Ulam stability of double Jordan centralizers as follows.

Theorem 2.1. *Let $f_j : A \to A$ be mappings with $f_j(0) = 0$ $(j = 0, 1)$, and let $\varphi : A^4 \to [0, \infty)$ be a function such that*

$$\|f_j(\lambda x + \lambda y + z^2) - \lambda f_j(x) - \lambda f_j(y) - [j(zf_j(z))^j + (1-j)(f_j(z)z)^{1-j}]$$
$$+ f_{1-j}(a)a - af_j(a)\| \leq \varphi(x, y, z, a) \tag{1}$$

for all $\lambda \in \mathbb{T}^1_{\frac{1}{n_o}}$. If

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z, 2^n a)}{2^n} = 0, \tag{2}$$

and there exists a constant K in which $0 < K < 1$ such that

$$\psi(2x) \leq 2K\psi(x) \tag{3}$$

for all $x \in A$, then there exists a unique double Jordan centralizer (L, R) on A satisfying

$$\|f_0(x) - L(x)\| \leq \frac{\psi(x)}{2(1-K)}, \tag{4}$$

$$\|f_1(x) - R(x)\| \leq \frac{\psi(x)}{2(1-K)} \tag{5}$$

for all $x \in A$*, where* $\psi(x) = \varphi(x, x, 0, 0)$.

Proof. We consider the set Ω as follows:

$$\Omega = \{h : A \longrightarrow A | h(0) = 0\}.$$

We also define the generalized metric on Ω:

$$d(g, h) := inf\{C \in [0, \infty] : \|g(x) - h(x)\| \leq C\psi(x) \quad \text{for all } x \in A\}.$$

One can show that (Ω, d) is a complete metric space. Now, we define a mapping $\mathcal{J} : \Omega \longrightarrow \Omega$ by

$$\mathcal{J}h(x) = \frac{1}{2}h(2x) \tag{6}$$

for all $x \in A$. Given $g, h \in \Omega$, let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq C$, i.e.,

$$\|g(x) - h(x)\| \leq C\psi(x) \tag{7}$$

for all $x \in A$. Substituting x by $2x$ in the inequality (7) and using from (3) and (6), we have

$$\|\mathcal{J}g(x) - \mathcal{J}h(x)\| \leq \frac{1}{2}\|g(2x) - h(2x)\| \leq \frac{1}{2}C\psi(2x) \leq CK\psi(x)$$

for all $x \in A$, and so $d(\mathcal{J}g, \mathcal{J}h) \leq CK$. This shows that $\mathcal{J}$ is strictly contractive on Ω. Hence we can conclude that

$$d(\mathcal{J}g, \mathcal{J}h) \leq Kd(g, h)$$

for all $g, h \in \Omega$. Now, we prove that for all $h \in \Omega$, $d(\mathcal{J}h, h) < \infty$. Letting $j = 0, \lambda = 1, x = y, z = a = 0$ in (1), we obtain $\|f_0(2x) - 2f_0(x)\| \leq \psi(x)$ for all $x \in A$. Thus

$$\|\frac{1}{2}f_0(2x) - f_0(x)\| \leq \frac{1}{2}\psi(x) \tag{8}$$

for all $x \in A$.

It follows from (8) that $d(\mathcal{J}f, f) \leq \frac{1}{2}$. By Theorem 1.1, the sequence $\{\mathcal{J}^n f_0\}$ converges to a unique fixed point $L : A \to A$ in the set $\Omega_1 = \{h \in \Omega, d(f, h) < \infty\}$, i.e.,

$$\lim_{n\to\infty} \frac{f_0(2^n x)}{2^n} = L(x) \tag{9}$$

for all $x \in A$, and so

$$d(f_0, L) \leq \frac{1}{1-K}d(\mathcal{J}f_0, L) \leq \frac{1}{2(1-K)}.$$

Thus the inequality (4) holds for all $x \in A$. Now, replace $2^n x$ and $2^n y$ by x and y, respectively, and put $z = a = 0$ in (1). If we divide both sides of the resulting inequality by 2^n, and letting n tend to infinity, then it follows from (1), (2) and (9) that

$$L(\lambda x) = \lambda L(x),$$

for all $x \in A$ and all $\lambda \in \mathbb{T}^1_{\frac{1}{n_o}}$. It follows from Lemma 1.1 that L is $\mathbb{C}-$linear. It is routine to show that $L(a^2) = L(a)^2$ from (1), and so it is a left Jordan centralizer on A.

According the above argument, one can show that there exists a unique mapping $R : A \to A$ which is a point of T such that

$$\lim_{n\to\infty} \frac{f_1(2^n x)}{2^n} = R(x) \tag{10}$$

for all $x \in A$. Indeed, R belongs to the set Ω_1. If we put $i = 0, x = y = z = 0$ and substitute a by $2^n a$ in (1) and we divide the both sides of the obtained inequality by 2^n, then we get

$$\|a\frac{f_0(2^n a)}{2^n} - \frac{f_1(2^n a)}{2^n}a\| \le \frac{\varphi(0, 0, 0, 2^n a)}{2^n}.$$

Passing to the limit as $n \to \infty$, and from (2) we conclude that $aL(a) = R(a)a$, for all $a \in A$. □

Corollary 2.1. *Let r and θ be nonnegative real numbers such that $r < 1$, and let $f_j : A \to A$ be mappings with $f_j(0) = 0$ $(j = 0, 1)$ such that*

$$\|f_j(\lambda x + \lambda y + z^2) - \lambda f_j(x) - \lambda f_j(y) - [j(zf_j(z))^j + (1 - j)(f_j(z)z)^{1-j}]$$

$$-af_j(a) + f_{1-j}(a)a\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|a\|^r) \tag{11}$$

for all $\lambda \in \mathbb{T}^1_{\frac{1}{n_o}}$ and all $x, y, z, a \in A$. Then there exists a unique double centralizer (L, R) on A satisfying

$$\|f_0(x) - L(x)\| \le \frac{\theta}{2 - 2^r},$$

$$\|f_1(x) - R(x)\| \le \frac{\theta}{2 - 2^r}$$

for all $x, y \in A$.

Proof. By putting $K = 2^{r-1}$ and

$$\varphi(x, y, z, w, a) = \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|a\|^r)$$

for all $x, y, z, a \in A$ in Theorem 2.1, we obtain the desired result. □

In the following corollary, we prove the superstability of double centralizers under some conditions.

Corollary 2.2. *Let r and θ be nonnegative real numbers such that $r < \frac{1}{6}$, and let $f_j : A \to A$ be mappings with $f_j(0) = 0$ ($j = 0, 1$) such that*

$$\|f_j(\lambda x + \lambda y + z^2) - \lambda f_j(x) - \lambda f_j(y) - [j(zf_j(z))^j + (1-j)(f_j(z)z)^{1-j}]$$
$$-af_j(a) + f_{1-j}(a)a\| \le \theta(\|x\|^r\|y\|^r\|z\|^r\|a\|^r) \quad (12)$$

for all $\lambda \in \mathbb{T}^1_{\frac{1}{n_o}}$ and all $x, y, z, a \in A$. Then (f_0, f_1) is a double centralizer.

Proof. It follows from Theorem 2.1 by taking

$$\varphi(x, y, z, a) = \theta(\|x\|^r\|y\|^r\|z\|^r\|a\|^r)$$

for all $x, y, z, a \in A$. □

3. Stability Of multipliers

In this section, we investigate the Hyers-Ulam stability and the superstability of multipliers on Banach algebras. First we need the next theorem which is the main key to investigation of the stability and superstability.

Theorem 3.1. *let $f : A \to A$ be a mapping with $f(0) = 0$ and let $\phi : A^3 \to [0, \infty)$ be a function such that*

$$\|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y) - f(z)z + zf(z)\| \le \phi(x, y, z) \quad (13)$$

for all $\lambda \in \mathbb{T}^1_{\frac{1}{n_o}}$ and all $x, y, z \in A$. If

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z)}{2^n} = 0, \quad (14)$$

and there exists a constant K, $0 < K < 1$, such that

$$\psi(2x) \le 2K\psi(x) \quad (15)$$

for all $x \in A$, then there exists a unique multiplier T on A satisfying

$$\|f(x) - L(x)\| \le \frac{1}{2(1-K)}\psi(x) \quad (16)$$

for all $x \in A$, where $\psi(x) = \varphi(x, x, 0)$.

Proof. First, similar to the proof of Theorem 2.1, we Consider the set $\Omega := \{h : A \to A \mid h(0) = 0\}$ and introduce the generalized metric d on Ω as follows:

$$d(g, h) := inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \le C\psi(x) \text{ for all } x \in A\}.$$

Again, similar to the proof of Theorem 2.1, the space Ω equipped to the metric d is complete. Now we define a mapping $\mathcal{J} : \Omega \to \Omega$ by

$$\mathcal{J}(h)(x) = \frac{1}{2}h(2x)$$

for all $x \in A$. By the same reasoning as in the proof of Theorem 2.1, $\mathcal{J}$ is strictly contractive on Ω. Putting $\lambda = 1, x = y$ and $z = 0$ in (13), we obtain

$$\|f(2x) - 2f(x)\| \leq \psi(x)$$

for all $x \in A$. So

$$\|\frac{1}{2}f(2x) - f(x)\| \leq \frac{1}{2}\psi(x) \tag{17}$$

for all $x \in A$. The inequality (17) shows that

$$d(\mathcal{J}f, f) \leq \frac{1}{2}.$$

By Theorem 1.1, $\mathcal{J}$ has a unique fixed point in the set $\Omega_1 := \{h \in \Omega : d(f,h) < \infty\}$. Let T be the fixed point of $\mathcal{J}$. Then T is the unique mapping with

$$T(2x) = 2T(x)$$

for all $x \in A$ such that there exists $C \in (0, \infty)$ such that

$$\|T(x) - f(x)\| \leq K\psi(x)$$

for all $x \in A$. On the other hand, we have

$$lim_{n\to\infty} d(\mathcal{J}^n(f), h) = 0.$$

Thus

$$lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = h(x) \tag{18}$$

for all $x \in A$. Hence

$$d(f,T) \leq \frac{1}{1-K} d(T, \mathcal{J}(f)) \leq \frac{1}{2(1-L)}. \tag{19}$$

This implies the inequality (16). From (13), (14) and (18) we obtain

$$\begin{aligned}\|T(x+y) - T(x) - T(y)\| &= lim_{n\to\infty} \frac{1}{2^n} \|f(2^n(x+y)) + f(2^n(x)) - f(2^n y)\| \\ &\leq lim_{n\to\infty} \frac{1}{2^n} \phi(2^n x, 2^n y, 0) = 0\end{aligned}$$

for all $x, y \in A$. So $T(x+y) = T(x) + T(y)$ for all $x, y \in A$. Thus T is Cauchy additive. If we put $y = x, z = 0$ in (13), we can conclude that

$$\|2\lambda f(x) - f(2\lambda x)\| \leq \psi(x)$$

for all $x \in A$. It follows that

$$\begin{aligned}\|T(2\lambda x) - 2\lambda T(x)\| &= lim_{n\to\infty} \frac{1}{2^n} \|f(2\lambda 2^n x) - 2\lambda f(2^n x)\| \\ &\leq lim_{n\to\infty} \frac{1}{2^n} \phi(2^n x, 2^n x, 0) = 0\end{aligned}$$

for all $\lambda \in \mathbb{T}$ and all $x \in A$. So $T(\lambda x) = \lambda T(x)$ for all $\lambda \in \mathbb{T}^1_{\frac{1}{n_o}}$ and all $x \in A$. Linearity of T follows from the proof of Theorem 2.1. If we substitute z by

$2^n z$ in (13), and put $x = y = 0$ and we divide the both sides of the obtained inequality by 2^n, we get

$$\|z\frac{f(2^n z)}{2^n} - \frac{f(2^n z)}{2^n}z\| \leq \frac{\phi(0,0,2^n z)}{2^n}.$$

Passing to the limit as $n \to \infty$, and from (16) we conclude that $zT(z) = T(z)z$ for all $z \in A$. □

Corollary 3.1. *Let r and θ be nonnegative real numbers such that $r < 1$, and let $f : A \to A$ be a mapping with $f(0) = 0$ such that*

$$\|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y) - f(z)z + zf(z)\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $\lambda \in \mathbb{T}^1_{\frac{1}{n_o}}$ and all $x, y, z \in A$. Then there exists a unique multiplier T on A satisfying

$$\|f(x) - T(x)\| \leq \frac{\theta}{2 - 2^r}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1 by taking

$$\phi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $x, y, z \in A$ and by putting $K = 2^{r-1}$. □

Now, we have the following result for the superstability of multipliers.

Corollary 3.2. *Let r and θ be nonnegative real numbers such that $r < \frac{1}{4}$, and let $f : A \to A$ be a mapping with $f(0) = 0$ such that*

$$\|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y) - f(z)z + zf(z)\| \leq \theta(\|x\|^r \|y\|^r \|z\|^r)$$

for all $\lambda \in \mathbb{T}^1_{\frac{1}{n_o}}$ and all $x, y, z \in A$. Then f is a multiplier on A.

Proof. It follows from Theorem 3 by taking

$$\phi(x, y, z) = \theta(\|x\|^r \|y\|^r \|z\|^r)$$

for all $x, y, z \in A$. □

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