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Approximation of a generalized additive mapping in multi-Banach modules and isomorphisms in multi-C^*-algebras: a fixed-point approach

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Abstract

Let $\mathcal{X}, \mathcal{Y}$ be vector spaces. It is shown that if an odd mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation

$$rf\left(\frac{\sum_{j=1}^{d} x_j}{r}\right) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^{d} \iota(j)=l}} rf\left(\frac{\sum_{j=1}^{d}(-1)^{\iota(j)} x_j}{r}\right) = ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1)\sum_{j=1}^{d} f(x_j) \tag{0.1}$$

then the odd mapping $f : \mathcal{X} \to \mathcal{Y}$ is additive, and we use a fixed-point method to prove the Hyers-Ulam stability of the functional equation (0.1) in multi-Banach modules over a unital multi-C^*-algebra. As an application, we show that every almost linear bijection $h : \mathcal{A} \to \mathcal{B}$ of a unital multi-C^*-algebra $\mathcal{A}$ onto a unital multi-C^*-algebra $\mathcal{B}$ is a C^*-algebra isomorphism when $h(\frac{2^n}{r^n}uy) = h(\frac{2^n}{r^n}u)h(y)$ for all unitaries $u \in U(\mathcal{A})$, all $y \in \mathcal{A}$, and $n = 0, 1, 2, \ldots$.

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1 Introduction

Throughout this paper we assume that r is a positive rational number and d, l are integers with $1 < l < \frac{d}{2}$.

Let X and Y be Banach spaces. Consider a mapping $f : X \to Y$ such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, and assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ with

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\left(\|x\|^p + \|y\|^p\right), \quad x, y \in X.$$

Rassias [1] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p, \quad x \in X.$$

Găvruta [2] extended the above theorem as follows: let G be an Abelian group, Y be a Banach space and put

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\big(2^j x, 2^j y\big) < \infty, \quad x, y \in G.$$

If $f : G \to Y$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \le \varphi(x, y), \quad x, y \in G,$$

then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, x), \quad x \in G.$$

Park [3] applied Găvruta's result to linear functional equations in Banach modules over a C^*-algebra. Several functional equations have been investigated in [4, 5] and [6]. In 2006 Baak, Boo and Rassias [7] solved the following functional equation:

$$rf\left(\frac{\sum_{j=1}^{d} x_j}{r}\right) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^{d}\iota(j)=l}} rf\left(\frac{\sum_{j=1}^{d}(-1)^{\iota(j)}x_j}{r}\right)$$

$$= ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1)\sum_{j=1}^{d} f(x_j) \tag{1.1}$$

(any solution of (1.1) will be called a generalized additive mapping) and proved its Hyers-Ulam stability in Banach modules over a unital C^*-algebra via the direct method. These results were applied to investigate C^*-algebra isomorphisms in unital C^*-algebras.

In this paper, we prove the Hyers-Ulam stability of the functional equation (1.1) in multi-Banach modules over a unital multi-C^*-algebra via the fixed-point method. These results are applied to investigate C^*-algebra isomorphisms in unital multi-C^*-algebras.

2 Fixed-point theorems

We recall two fundamental results in the fixed-point theory.

Theorem 2.1 [8, 9] *Let (X, d) be a complete metric space and let $J : X \to X$ be strictly contractive, i.e.,*

$$d(Jx, Jy) \le Ld(x, y), \quad x, y \in X$$

for a Lipschitz constant $L < 1$. Then

(1) the mapping J has a unique fixed point $x^* \in X$,

(2) the fixed point x^* is globally attractive, i.e.,

$$\lim_{n\to\infty} J^n x = x^*, \quad x \in X,$$

(3) the following inequalities hold:

$$d(J^n x, x^*) \le L^n d(x, x^*),$$
$$d(J^n x, x^*) \le \frac{1}{1-L} d(J^n x, J^{n+1} x),$$
$$d(x, x^*) \le \frac{1}{1-L} d(x, Jx)$$

for all $x \in X$ and nonnegative integers n.

Let X be a non-empty set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if for any $x, y, z \in X$, we have:
(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, z) \le d(x, y) + d(y, z)$.

Theorem 2.2 [8, 10] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with a Lipschitz constant $L < 1$. Then, for each $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty, \quad n \ge 0$$

or there exists a positive integer n_0 such that:
(1) $d(J^n x, J^{n+1} x) < \infty$, $n \ge n_0$,
(2) the sequence $(J^n x)$ converges to a fixed point y^* of J,
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$,
(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$, $y \in Y$.

3 Multi-normed spaces

The notion of a multi-normed space was introduced by Dales and Polyakov [11]. This concept is somewhat similar to an operator sequence space and has some connections with operator spaces and Banach lattices. Motivations for the study of multi-normed spaces and many examples are given in [11–13].

Let $(\mathcal{E}, \|\cdot\|)$ be a complex normed space and $k \in \mathbb{N}$. We denote by $\mathcal{E}^k$ the linear space $\mathcal{E} \oplus \cdots \oplus \mathcal{E}$ consisting of k-tuples $(x_1, \dots, x_k)$, where $x_1, \dots, x_k \in \mathcal{E}$. The linear operations on $\mathcal{E}^k$ are defined coordinate-wise. The zero element of either $\mathcal{E}$ or $\mathcal{E}^k$ is denoted by 0. Finally, we denote by $\mathbb{N}_k$ the set $\{1, \dots, k\}$ and by Σ_k the group of permutations on k symbols.

Definition 3.1 [11, 14] A multi-norm on $\{\mathcal{E}^k : k \in \mathbb{N}\}$ is a sequence

$$(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$$

such that $\|\cdot\|_k$ is a norm on $\mathcal{E}^k$ for $k \in \mathbb{N}$, $\|\cdot\|_1 = \|\cdot\|$, and for any integer $k \ge 2$, we have
(A1) $\|(x_{\sigma(1)}, \dots, x_{\sigma(k)})\|_k = \|(x_1, \dots, x_k)\|_k$, $\sigma \in \Sigma_k$, $x_1, \dots, x_k \in \mathcal{E}$,
(A2) $\|(\alpha_1 x_1, \dots, \alpha_k x_k)\|_k \le (\max_{i\in\mathbb{N}_k} |\alpha_i|)\|(x_1, \dots, x_k)\|_k$, $\alpha_1, \dots, \alpha_k \in \mathbb{C}$, $x_1, \dots, x_k \in \mathcal{E}$,

(A3) $\|(x_1,\ldots,x_{k-1},0)\|_k = \|(x_1,\ldots,x_{k-1})\|_{k-1}$, $x_1,\ldots,x_{k-1} \in \mathcal{E}$,

(A4) $\|(x_1,\ldots,x_{k-1},x_{k-1})\|_k = \|(x_1,\ldots,x_{k-1})\|_{k-1}$, $x_1,\ldots,x_{k-1} \in \mathcal{E}$.

A sequence $((\mathcal{E}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is then said to be a multi-normed space.

Lemma 3.2 [11, 13] *Suppose that $((\mathcal{E}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space. Then for any $k \in \mathbb{N}$, we have*

(a) $\|(x,\ldots,x)\|_k = \|x\|$, $x \in \mathcal{E}$,

(b) $\max_{i\in\mathbb{N}_k} \|x_i\| \le \|(x_1,\ldots,x_k)\|_k \le \sum_{i=1}^k \|x_i\| \le k \max_{i\in\mathbb{N}_k} \|x_i\|$, $x_1,\ldots,x_k \in \mathcal{E}$.

From Lemma 3.2(b), it follows that if $(\mathcal{E}, \|\cdot\|)$ is a Banach space, then $(\mathcal{E}^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}$ (in this case we say that $((\mathcal{E}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space).

Now, we recall two important examples of multi-norms (see [11, 12]).

Example 3.3 The sequence $(\|\cdot\|_k : k \in \mathbb{N})$ on $\{\mathcal{E}^k : k \in \mathbb{N}\}$ defined by

$$\left\|(x_1,\ldots,x_k)\right\|_k := \max_{i\in\mathbb{N}_k} \|x_i\|, \quad x_1,\ldots,x_k \in \mathcal{E}$$

is a multi-norm called the minimum multi-norm. The terminology 'minimum' is justified by property (b) from Lemma 3.2.

Example 3.4 Let $\{(\|\cdot\|_k^\alpha : k \in \mathbb{N}) : \alpha \in A\}$ be a (non-empty) family of all multi-norms on $\{\mathcal{E}^k : k \in \mathbb{N}\}$. For $k \in \mathbb{N}$, set

$$\left|\!\left|\!\left|(x_1,\ldots,x_k)\right|\!\right|\!\right|_k := \sup_{\alpha\in A} \left\|(x_1,\ldots,x_k)\right\|_k^\alpha, \quad x_1,\ldots,x_k \in \mathcal{E}.$$

Then $(|\!|\!|\cdot|\!|\!|_k : k \in \mathbb{N})$ is a multi-norm on $\{\mathcal{E}^k : k \in \mathbb{N}\}$ called the maximum multi-norm.

Lemma 3.5 [14] *Suppose that $k \in \mathbb{N}$ and $(x_1,\ldots,x_k) \in \mathcal{E}^k$. For each $j \in \{1,\ldots,k\}$, let $(x_n^j)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}$ such that $\lim_{n\to\infty} x_n^j = x_j$. Then for each $(y_1,\ldots,y_k) \in \mathcal{E}^k$ we have*

$$\lim_{n\to\infty} \left(x_n^1 - y_1,\ldots,x_n^k - y_k\right) = (x_1 - y_1,\ldots,x_k - y_k).$$

Definition 3.6 [12, 14] Let $((\mathcal{E}^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ is said to be a multi-null sequence if for each $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \left\|(x_n,\ldots,x_{n+k-1})\right\|_k < \varepsilon, \quad n \ge n_0.$$

We say that the sequence $(x_n)_{n\in\mathbb{N}}$ is multi-convergent to $x \in \mathcal{E}$ and write $\lim_{n\to\infty} x_n = x$ if $(x_n - x)_{n\in\mathbb{N}}$ is a multi-null sequence.

Definition 3.7 [11, 14] Let $(\mathcal{A}, \|\cdot\|)$ be a normed algebra such that $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space. Then $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is called a multi-normed algebra if

$$\left\|(a_1b_1,\ldots,a_kb_k)\right\|_k \le \left\|(a_1,\ldots,a_k)\right\|_k \cdot \left\|(b_1,\ldots,b_k)\right\|_k, \quad k \in \mathbb{N}, a_1,\ldots,a_k,b_1,\ldots,b_k \in \mathcal{A}.$$

The multi-normed algebra $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is said to be a multi-Banach algebra if $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

Example 3.8 Let p, q be such that $1 \le p \le q < \infty$ and $\mathcal{A} = \ell^p$. The algebra $\mathcal{A}$ is a Banach sequence algebra with respect to coordinate-wise multiplication of sequences (see Example 4.1.42 of [15]). Let $(\|\cdot\|_k : k \in \mathbb{N})$ be the standard (p,q)-multi-norm on $\{\mathcal{A}^k : k \in \mathbb{N}\}$ (see [11]). Then $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach algebra.

Definition 3.9 Let $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra and assume that $\mathcal{A}$ is a (unital) C^*-algebra. If the involution $*$ satisfies

$$\left\| \left(a_1^* a_1, \ldots, a_k^* a_k\right) \right\|_k = \left\| (a_1, \ldots, a_k) \right\|_k^2, \quad k \in \mathbb{N}, a_1, \ldots, a_k \in \mathcal{A},$$

then $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is called a (unital) multi-C^*-algebra.

Definition 3.10 Let $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra and $((\mathcal{X}^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach space. Assume also that $\mathcal{X}$ is a Banach left module over $\mathcal{A}$. We say that $((\mathcal{X}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach left module over $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ if there is an $M \ge 0$ such that

$$\left\| (a_1 x_1, \ldots, a_k x_k) \right\|_k \le M \left\| (a_1, \ldots, a_k) \right\|_k \cdot \left\| (x_1, \ldots, x_k) \right\|_k$$

for all $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \mathcal{A}$, $x_1, \ldots, x_k \in \mathcal{X}$.

4 Stability of an odd functional equation in multi-Banach modules over a multi-C^*-algebra

Throughout this section, we assume that $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a unital multi-C^*-algebra, and $((\mathcal{X}^k, \|\cdot\|_k) : k \in \mathbb{N})$ and $((\mathcal{Y}^k, \|\cdot\|_k) : k \in \mathbb{N})$ are multi-Banach left modules over $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$. Moreover, by $U(\mathcal{A})$ we denote the unitary group of $\mathcal{A}$.

Lemma 4.1 [7] *Let X and Y be vector spaces. An odd mapping $f : X \to Y$ satisfies (1.1) for all $x_1, \ldots, x_d \in X$ if and only if f is additive.*

Corollary 4.2 [7] *Let X and Y be vector spaces. An odd mapping $f : X \to Y$ satisfies*

$$rf\left(\frac{x+y}{r}\right) = f(x) + f(y), \quad x, y \in X$$

if and only if f is additive.

Given a mapping $f : \mathcal{X} \to \mathcal{Y}$, we set

$$D_u f(x_1, \ldots, x_d) := rf\left(\frac{\sum_{j=1}^d u x_j}{r}\right) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^d \iota(j)=l}} rf\left(\frac{\sum_{j=1}^d (-1)^{\iota(j)} u x_j}{r}\right) - ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1) \sum_{j=1}^d u f(x_j)$$

for all $u \in U(\mathcal{A})$ and $x_1, \ldots, x_d \in \mathcal{X}$.

Theorem 4.3 *Let $r \neq 2$ and $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that for every $k \in \mathbb{N}$ there is a function $\varphi_k : \mathcal{X}^{kd} \to [0,\infty)$ with*

$$\lim_{j\to\infty} \frac{r^j}{2^j}\varphi_k\left(\frac{2^j}{r^j}x_{11},\ldots,\frac{2^j}{r^j}x_{1d},\ldots,\frac{2^j}{r^j}x_{k1},\ldots,\frac{2^j}{r^j}x_{kd}\right) = 0, \tag{4.1}$$

$$\begin{aligned}&\big\|\big(D_u f(x_{11},\ldots,x_{1d}),\ldots,D_u f(x_{k1},\ldots,x_{kd})\big)\big\|_k \\ &\quad\le \varphi_k(x_{11},\ldots,x_{1d},\ldots,x_{k1},\ldots,x_{kd})\end{aligned} \tag{4.2}$$

for all $u \in U(\mathcal{A})$ and $x_{11},\ldots,x_{1d},\ldots,x_{k1},\ldots,x_{kd} \in \mathcal{X}$. If there exists an $L < 1$ such that

$$\begin{aligned}&\varphi_k\Big(\overbrace{\frac{2}{r}x_{11},\frac{2}{r}x_{11},\ldots,0}^{d},\ldots,\overbrace{\frac{2}{r}x_{k1},\frac{2}{r}x_{k1},\ldots,0}^{d}\Big)\\ &\quad\le \frac{2}{r}L\varphi_k\big(\overbrace{x_{11},x_{11},\ldots,0}^{d},\ldots,\overbrace{x_{k1},x_{k1},\ldots,0}^{d}\big)\end{aligned}$$

for all $k \in \mathbb{N}$ and $x_{11},\ldots,x_{k1} \in \mathcal{X}$, then there is a unique $\mathcal{A}$-linear generalized additive mapping $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ with

$$\begin{aligned}&\big\|\big(\mathcal{L}(x_1)-f(x_1),\ldots,\mathcal{L}(x_k)-f(x_k)\big)\big\|_k \\ &\quad\le \frac{1}{2({}_{d-2}C_l - {}_{d-2}C_{l-2}+1)(1-L)}\varphi_k(x_1,x_1,\underbrace{0,\ldots,0}_{d-2\text{ times}},\ldots,x_k,x_k,\underbrace{0,\ldots,0}_{d-2\text{ times}})\end{aligned} \tag{4.3}$$

for all $k \in \mathbb{N}$ and $x_1,\ldots,x_k \in \mathcal{X}$.

Proof Put

$$X := \{\mathcal{L} : \mathcal{X} \to \mathcal{Y}\}$$

and

$$\begin{aligned}d(\mathcal{L},h) = \inf\big\{&C \in \mathbb{R}_+ : \big\|\big(\mathcal{L}(x_1)-h(x_1),\ldots,\mathcal{L}(x_k)-h(x_k)\big)\big\|_k \\ &\le C\varphi_k\big(\overbrace{x_1,x_1,0,\ldots,0}^{d},\ldots,\overbrace{x_k,x_k,0,\ldots,0}^{d}\big), k \in \mathbb{N}, x_1,\ldots,x_k \in \mathcal{X}\big\}\end{aligned}$$

for all $\mathcal{L},h \in X$. It is easy to show that (X,d) is a complete generalized metric space.

Define a mapping $J : X \to X$ by

$$J\mathcal{L}(x) := \frac{r}{2}\mathcal{L}\left(\frac{2}{r}x\right), \quad \mathcal{L} \in X, x \in \mathcal{X}.$$

Analysis similar to that in the proof of Theorem 3.1 in [8] (see also the proof of Lemma 3.2 in [12]) shows that

$$d(J\mathcal{L},Jh) \le Ld(\mathcal{L},h), \quad \mathcal{L},h \in X.$$

Fix $k \in \mathbb{N}$. Putting $u = 1 \in U(\mathcal{A})$, $x_{i1} = x_{i2} = x_1$, and $x_{i3} = \cdots = x_{id} = 0$ for $i \in \{1, \ldots, k\}$ in (4.2), we have

$$\begin{aligned}&\left\| \left(rf\left(\frac{2}{r}x_1\right) - 2f(x_1), \ldots, rf\left(\frac{2}{r}x_k\right) - 2f(x_k) \right) \right\|_k \\ &\quad \le \frac{1}{{}_{d-2}C_l - {}_{d-2}C_{l-2} + 1} \varphi_k(x_1, x_1, \underbrace{0, \ldots, 0}_{d-2 \text{ times}}, \ldots, x_k, x_k, \underbrace{0, \ldots, 0}_{d-2 \text{ times}}),\end{aligned}$$

because f is odd and $t :={}_{d-2} C_l -{}_{d-2} C_{l-2} + 1 ={}_{d-1} C_l -{}_{d-1} C_{l-1} + 1$. We thus get

$$\begin{aligned}&\left\| \left(f(x_1) - \frac{r}{2}f\left(\frac{2}{r}x_1\right), \ldots, f(x_k) - \frac{r}{2}f\left(\frac{2}{r}x_k\right) \right) \right\|_k \\ &\quad \le \frac{1}{2t} \varphi_k(x_1, x_1, \underbrace{0, \ldots, 0}_{d-2 \text{ times}}, \ldots, x_k, x_k, \underbrace{0, \ldots, 0}_{d-2 \text{ times}}), \quad x_1, \ldots, x_k \in \mathcal{X},\end{aligned}$$

and therefore,

$$d(f, Jf) \le \frac{1}{2t}. \tag{4.4}$$

Consequently, by Theorem 2.2, there exists a mapping $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ such that

(1) $\mathcal{L}$ is a fixed point of J, *i.e.*,

$$\mathcal{L}\left(\frac{2}{r}x\right) = \frac{2}{r}\mathcal{L}(x), \quad x \in \mathcal{X}, \tag{4.5}$$

and $\mathcal{L}$ is unique in the set

$$Y = \{\mathcal{L} \in X : d(f, \mathcal{L}) < \infty\}.$$

This means that $\mathcal{L}$ is a unique mapping satisfying (4.5) such that there exists a $C \in (0, \infty)$ with

$$\left\| \left(\mathcal{L}(x_1) - f(x_1), \ldots, \mathcal{L}(x_k) - f(x_k) \right) \right\|_k \le C\varphi_k(x_1, x_1, \underbrace{0, \ldots, 0}_{d-2 \text{ times}}, \ldots, x_k, x_k, \underbrace{0, \ldots, 0}_{d-2 \text{ times}})$$

for all $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in \mathcal{X}$.

(2) $d(J^n f, \mathcal{L}) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{r^n}{2^n} f\left(\frac{2^n}{r^n}x\right) = \mathcal{L}(x) \quad x \in \mathcal{X}. \tag{4.6}$$

(3) $d(f, \mathcal{L}) \le \frac{1}{1-L} d(f, Jf)$, which together with (4.4) gives

$$d(f, \mathcal{L}) \le \frac{1}{2t - 2tL},$$

and therefore, inequality (4.3) holds for all $x_1, \ldots, x_k \in \mathcal{X}$.

Next, note that the fact that the mapping f is odd and (4.6) imply that $\mathcal{L}$ is odd. Moreover, by (4.1) and (4.2), we get

$$\begin{aligned}
&\big\| \big(D_1\mathcal{L}(x_{11},\ldots,x_{1d}),\ldots,D_1\mathcal{L}(x_{k1},\ldots,x_{kd})\big)\big\|_k \\
&\quad = \lim_{n\to\infty}\frac{r^n}{2^n}\left\|\left(D_1f\left(\frac{2^n}{r^n}x_{11},\ldots,\frac{2^n}{r^n}x_{1d}\right),\ldots,D_1f\left(\frac{2^n}{r^n}x_{k1},\ldots,\frac{2^n}{r^n}x_{kd}\right)\right)\right\|_k \\
&\quad \le \lim_{n\to\infty}\frac{r^n}{2^n}\varphi_k\left(\frac{2^n}{r^n}x_{11},\ldots,\frac{2^n}{r^n}x_{1d},\ldots,\frac{2^n}{r^n}x_{k1},\ldots,\frac{2^n}{r^n}x_{kd}\right) = 0
\end{aligned}$$

for all $k\in\mathbb{N}$ and $x_{11},\ldots,x_{1d},\ldots,x_{k1},\ldots,x_{kd}\in\mathcal{X}$, and therefore, $\mathcal{L}$ is a generalized additive mapping.

Fix $u\in U(\mathcal{A})$ and $x\in\mathcal{X}$. Using (4.1) and (4.2), we have

$$\begin{aligned}
&\big\|(D_u\mathcal{L}(x,\underbrace{0,\ldots,0}_{d-1\text{ times}}),\ldots,D_u\mathcal{L}(x,\underbrace{0,\ldots,0}_{d-1\text{ times}})\big\|_k \\
&\quad = \lim_{n\to\infty}\frac{r^n}{2^n}\left\|\left(D_uf\left(\frac{2^n}{r^n}x,\underbrace{0,\ldots,0}_{d-1\text{ times}}\right),\ldots,D_uf\left(\frac{2^n}{r^n}x,\underbrace{0,\ldots,0}_{d-1\text{ times}}\right)\right)\right\|_k \\
&\quad \le \lim_{n\to\infty}\frac{r^n}{2^n}\varphi_k\left(\frac{2^n}{r^n}x,\underbrace{0,\ldots,0}_{d-1\text{ times}},\ldots,\frac{2^n}{r^n}x,\underbrace{0,\ldots,0}_{d-1\text{ times}}\right) = 0,
\end{aligned}$$

and consequently,

$$({}_{d-1}C_l - {}_{d-1}C_{l-1}+1)r\mathcal{L}\left(\frac{ux}{r}\right) = ({}_{d-1}C_l - {}_{d-1}C_{l-1}+1)u\mathcal{L}(x).$$

Since $\mathcal{L}$ is a generalized additive mapping, from Lemma 4.1 it follows that $\mathcal{L}$ is additive, and therefore,

$$\mathcal{L}(ux) = r\mathcal{L}\left(\frac{ux}{r}\right) = u\mathcal{L}(x),\quad u\in U(\mathcal{A}), x\in\mathcal{X}.$$

As in the proof of Theorem 3.1, in [7] one can now show that $\mathcal{L}$ is an $\mathcal{A}$-linear mapping. □

Corollary 4.4 *Let $r\neq 2$ and $\theta,p\in(0,\infty)$. Assume also that $p>1$ for $r>2$, and $p<1$ for $r<2$. If $f:\mathcal{X}\to\mathcal{Y}$ is an odd mapping such that*

$$\big\|\big(D_uf(x_{11},\ldots,x_{1d}),\ldots,D_uf(x_{k1},\ldots,x_{kd})\big)\big\|_k \le \theta\left(\sum_{j=1}^{d}\|x_{1j}\|^p+\cdots+\sum_{j=1}^{d}\|x_{kj}\|^p\right)$$

for all $u\in U(\mathcal{A})$, $k\in\mathbb{N}$, and $x_{11},\ldots,x_{1d},\ldots,x_{k1},\ldots,x_{kd}\in\mathcal{X}$, then there exists a unique $\mathcal{A}$-linear generalized additive mapping $\mathcal{L}:\mathcal{X}\to\mathcal{Y}$ with

$$\begin{aligned}
&\big\|\big(\mathcal{L}(x_1)-f(x_1),\ldots,\mathcal{L}(x_k)-f(x_k)\big)\big\|_k \\
&\quad \le \frac{r^{p-1}\theta}{(r^{p-1}-2^{p-1})({}_{d-2}C_l - {}_{d-2}C_{l-2}+1)}\big(\|x_1\|^p+\cdots+\|x_k\|^p\big)
\end{aligned}$$

for all $k\in\mathbb{N}$ and $x_1,\ldots,x_k\in\mathcal{X}$.

Proof Putting $L = \frac{2^{p-1}}{r^{p-1}}$ and

$$\begin{aligned}&\varphi_k(x_{11},\ldots,x_{1d},\ldots,x_{k1},\ldots,x_{kd})\\&\quad=\theta\left(\sum_{j=1}^{d}\|x_{1j}\|^p+\cdots+\sum_{j=1}^{d}\|x_{kj}\|^p\right),\end{aligned}$$

for all $k \in \mathbb{N}$ and $x_{11},\ldots,x_{1d},\ldots,x_{k1},\ldots,x_{kd} \in \mathcal{X}$, in Theorem 4.3, we get the desired assertion. □

Theorem 4.5 *Let $r \neq 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping for which there is a function $\varphi : \mathcal{X}^{kd} \to [0,\infty)$ such that*

$$\begin{aligned}&\lim_{j\to\infty}\frac{2^j}{r^j}\varphi\left(\frac{r^j}{2^j}x_{11},\ldots,\frac{r^j}{2^j}x_{1d},\ldots,\frac{r^j}{2^j}x_{k1},\ldots,\frac{r^j}{2^j}x_{kd}\right)=0,\\&\left\|\left(D_uf(x_{11},\ldots,x_{1d}),\ldots,D_uf(x_{k1},\ldots,x_{kd})\right)\right\|_k\\&\quad\le\varphi(x_{11},\ldots,x_{1d},\ldots,x_{k1},\ldots,x_{kd})\end{aligned}\tag{4.7}$$

for all $u \in U(\mathcal{A})$ and all $x_{11},\ldots,x_{1d},\ldots,x_{k1},\ldots,x_{kd} \in \mathcal{X}$. If there exists an $L < 1$ such that

$$\begin{aligned}&\varphi\Bigl(\overbrace{\frac{r}{2}x_{11},\frac{r}{2}x_{11},\ldots,0}^{d},\overbrace{\frac{r}{2}x_{21},\frac{r}{2}x_{21},\ldots,0}^{d},\ldots,\overbrace{\frac{r}{2}x_{k1},\frac{r}{2}x_{k1},\ldots,0}^{d}\Bigr)\\&\quad\le\frac{r}{2}L\varphi\bigl(\overbrace{x_{11},x_{11},\ldots,0}^{d},\overbrace{x_{21},x_{21},\ldots,0}^{d},\ldots,\overbrace{x_{k1},x_{k1},\ldots,0}^{d}\bigr)\end{aligned}$$

for all $x_{11},x_{21},\ldots,x_{k1} \in \mathcal{X}$. Then there exists a unique $\mathcal{A}$-linear generalized additive mapping $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned}&\sup_{k\in\mathbb{N}}\left\|\left(\mathcal{L}(x_1)-f(x_1),\ldots,\mathcal{L}(x_k)-f(x_k)\right)\right\|_k\\&\quad\le\sup_{k\in\mathbb{N}}\frac{L}{2({}_{d-2}C_l-{}_{d-2}C_{l-2}+1)(1-L)}\varphi(x_1,x_1,\underbrace{0,\ldots,0}_{d-2\text{ times}},\ldots,x_k,x_k,\underbrace{0,\ldots,0}_{d-2\text{ times}})\end{aligned}$$

for all $x_1,\ldots,x_k \in \mathcal{X}$.

Proof Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in \mathcal{X}$ since f is an odd mapping. Let $u = 1 \in U(\mathcal{A})$. Putting $x_{i1} = x_{i2} = x_1$ and $x_{i3} = \cdots = x_{im} = 0$, $1 \le i \le k$ in (4.7), we have

$$\begin{aligned}&\left\|\left(rf\left(\frac{2}{r}x_1\right)-2f(x_1),\ldots,rf\left(\frac{2}{r}x_k\right)-2f(x_k)\right)\right\|_k\\&\quad\le\frac{1}{{}_{d-2}C_l-{}_{d-2}C_{l-2}+1}\varphi(x_1,x_1,\underbrace{0,\ldots,0}_{d-2\text{ times}},\ldots,x_k,x_k,\underbrace{0,\ldots,0}_{d-2\text{ times}}).\end{aligned}$$

Letting $t :=_{d-2} C_l -_{d-2} C_{l-2} + 1$, we get

$$\begin{aligned}
&\left\| \left(f(x_1) - \frac{2}{r} f\left(\frac{r}{2}x_1\right), \ldots, f(x_k) - \frac{2}{r} f\left(\frac{r}{2}x_k\right) \right) \right\|_k \\
&\quad \le \frac{1}{rt} \varphi\left(\frac{r}{2}x_1, \frac{r}{2}x_1, \underbrace{0, \ldots, 0}_{d-2 \text{ times}}, \ldots, \frac{r}{2}x_k, \frac{r}{2}x_k, \underbrace{0, \ldots, 0}_{d-2 \text{ times}} \right) \\
&\quad \le \frac{L}{2t} \varphi(x_1, x_1, \underbrace{0, \ldots, 0}_{d-2 \text{ times}}, \ldots, x_k, x_k, \underbrace{0, \ldots, 0}_{d-2 \text{ times}})
\end{aligned}$$

for all $x_1, \ldots, x_k \in \mathcal{X}$.

The rest of the proof is similar to the proof of Theorem 4.3. □

Corollary 4.6 *Let $r < 2$, and let θ and $p > 1$ be positive real numbers, or let $r > 2$, and let θ and $p < 1$ be positive real numbers. Let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that*

$$\left\| \left(D_u f(x_{11}, \ldots, x_{1d}), \ldots, D_u f(x_{k1}, \ldots, x_{kd}) \right) \right\|_k \le \theta \left(\sum_{j=1}^{d} \|x_{1j}\|^p + \cdots + \sum_{j=1}^{d} \|x_{kj}\|^p \right)$$

for all $u \in U(\mathcal{A})$ and all $x_{11}, \ldots, x_{1d}, \ldots, x_{k1}, \ldots, x_{kd} \in \mathcal{X}$. Then there exists a unique $\mathcal{A}$-linear generalized additive mapping $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned}
&\sup_{k \in \mathbb{N}} \left\| \left(\mathcal{L}(x_1) - f(x_1), \ldots, \mathcal{L}(x_k) - f(x_k) \right) \right\|_k \\
&\quad \le \sup_{k \in \mathbb{N}} \frac{r^{p-1}\theta}{(2^{p-1} - r^{p-1})(_{d-2}C_l -_{d-2} C_{l-2} + 1)} \left(\|x_1\|^p + \cdots + \|x_k\|^p \right)
\end{aligned}$$

for all $x \in X$.

Proof Define

$$\varphi(x_{11}, \ldots, x_{1d}, \ldots, x_{k1}, \ldots, x_{kd}) = \theta \left(\sum_{j=1}^{d} \|x_{1j}\|^p + \cdots + \sum_{j=1}^{d} \|x_{kj}\|^p \right).$$

Putting $L = \frac{r^{p-1}}{2^{p-1}}$ in Theorem 4.5, we get the desired result. □

Now we investigate the Hyers-Ulam stability of linear mappings for the case $d = 2$.

Theorem 4.7 *Let $r \neq 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping for which there is a function $\varphi : \mathcal{X}^{2k} \to [0, \infty)$ such that*

$$\begin{aligned}
&\lim_{j \to \infty} \frac{r^j}{2^j} \varphi\left(\frac{2^j}{r^j}x_1, \frac{2^j}{r^j}y_1, \ldots, \frac{2^j}{r^j}x_k, \frac{2^j}{r^j}y_k \right) = 0, \\
&\left\| \left(rf\left(\frac{ux_1 + uy_1}{r}\right) - uf(x_1) - uf(y_1), \ldots, rf\left(\frac{ux_k + uy_k}{r}\right) - uf(x_k) - uf(y_k) \right) \right\|_k \\
&\quad \le \varphi(x_1, y_1, \ldots, x_k, y_k) \qquad (4.8)
\end{aligned}$$

for all $u \in U(\mathcal{A})$ *and all* $x_1, \dots x_k, y_1 \dots, y_k \in \mathcal{X}$. *If there exists an* $L < 1$ *such that*

$$\varphi\left(\frac{2}{r}x_1, \frac{2}{r}x_1, \frac{2}{r}x_2, \frac{2}{r}x_2, \dots, \frac{2}{r}x_k, \frac{2}{r}x_k\right) \le \frac{2}{r}L\varphi(x_1, x_1, x_2, x_2, \dots, x_k, x_k)$$

for all $x_1, \dots, x_k \in \mathcal{X}$. *Then there exists a unique* $\mathcal{A}$*-linear generalized additive mapping* $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \left\| \left(\mathcal{L}(x_1) - f(x_1), \dots, \mathcal{L}(x_k) - f(x_k)\right) \right\|_k \\
&\quad \le \sup_{k\in\mathbb{N}} \frac{L}{2(1-L)} \varphi(x_1, x_1, \dots, x_k, x_k)
\end{aligned}$$

for all $x_1, \dots, x_k \in \mathcal{X}$.

Proof Let $u = 1 \in U(\mathcal{A})$. Putting $x = y$ in (4.8), we have

$$\left\| \left(rf\left(\frac{2}{r}x_1\right) - 2f(x_1), \dots, rf\left(\frac{2}{r}x_k\right) - 2f(x_k) \right) \right\|_k \le \varphi(x_1, x_1, \dots, x_k, x_k)$$

for all $x \in X$. So

$$\left\| \left(f(x_1) - \frac{r}{2}f\left(\frac{2}{r}x_1\right), \dots, f(x_k) - \frac{r}{2}f\left(\frac{2}{r}x_k\right) \right) \right\|_k \le \frac{1}{2}\varphi(x_1, x_1, \dots, x_k, x_k)$$

for all $x \in X$.

The rest of the proof is the same as in the proof of Theorem 4.3. □

Corollary 4.8 *Let* $r > 2$, *and let* θ *and* $p > 1$ *be positive real numbers, or let* $r < 2$, *and let* θ *and* $p < 1$ *be positive real numbers. Let* $f : X \to Y$ *be an odd mapping such that*

$$\begin{aligned}
&\left\| \left(rf\left(\frac{ux_1 + uy_1}{r}\right) - uf(x_1) - uf(y_1), \dots, rf\left(\frac{ux_k + uy_k}{r}\right) - uf(x_k) - uf(y_k) \right) \right\|_k \\
&\quad \le \theta \sum_{j=1}^{k} \left(\|x_j\|^p + \|y_j\|^p\right)
\end{aligned}$$

for all $u \in U(\mathcal{A})$ *and for all* $x_1, \dots, x_k \in \mathcal{X}$. *Then there exists a unique* $\mathcal{A}$*-linear generalized additive mapping* $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\sup_{k\in\mathbb{N}} \left\| \left(\mathcal{L}(x_1) - f(x_1), \dots, \mathcal{L}(x_k) - f(x_k)\right) \right\|_k \le \sup_{k\in\mathbb{N}} \frac{r^{p-1}\theta}{r^{p-1} - 2^{p-1}} \sum_{j=1}^{k} \|x_j\|^p$$

for all $x_1, \dots, x_k \in \mathcal{X}$.

Proof Define $\varphi(x_1, y_1, \dots, x_k, y_k) = \theta \sum_{j=1}^{k} (\|x_j\|^p + \|y_j\|^p)$, and apply Theorem 4.7. Then we get the desired result. □

Theorem 4.9 *Let $r \neq 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping for which there is a function $\varphi : \mathcal{X}^{2k} \to [0, \infty)$ such that*

$$\lim_{j\to\infty} \frac{2^j}{r^j} \varphi\left(\frac{r^j}{2^j}x_1, \frac{r^j}{2^j}y_1, \dots, \frac{r^j}{2^j}x_k, \frac{r^j}{2^j}y_k\right) = 0,$$
$$\left\| \left(rf\left(\frac{ux_1 + uy_1}{r}\right) - uf(x_1) - uf(y_1), \dots, rf\left(\frac{ux_k + uy_k}{r}\right) - uf(x_k) - uf(y_k) \right) \right\|_k \le \varphi(x_1, y_1, \dots, x_k, y_k), \tag{4.9}$$

for all $u \in U(\mathcal{A})$ and all $x_1, \dots, x_k, y_1, \dots, y_k \in \mathcal{X}$. If there exists an $L < 1$ such that

$$\varphi\left(\frac{r}{2}x_1, \frac{r}{2}x_1, \frac{r}{2}x_2, \frac{r}{2}x_2, \dots, \frac{r}{2}x_k, \frac{r}{2}x_k\right) \le \frac{r}{2} L\varphi(x_1, x_1, x_2, x_2, \dots, x_k, x_k)$$

for all $x_1, \dots, x_k \in \mathcal{X}$. Then there exists a unique $\mathcal{A}$-linear generalized additive mapping $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ such that

$$\sup_{k\in\mathbb{N}} \left\| \left(\mathcal{L}(x_1) - f(x_1), \dots, \mathcal{L}(x_k) - f(x_k)\right) \right\|_k \le \sup_{k\in\mathbb{N}} \frac{1}{2(1-L)} \varphi(x_1, x_1, \dots, x_k, x_k)$$

for all $x_1, \dots, x_k \in \mathcal{X}$.

Proof Let $u = 1 \in U(\mathcal{A})$. Putting $x = y$ in (4.9), we have

$$\left\| \left(rf\left(\frac{2}{r}x_1\right) - 2f(x_1), \dots, rf\left(\frac{2}{r}x_k\right) - 2f(x_k) \right) \right\|_k \le \varphi(x_1, x_1, \dots, x_k, x_k)$$

for all $x_1, \dots, x_k \in \mathcal{X}$. So

$$\begin{aligned} \left\| \left(f(x_1) - \frac{2}{r} f\left(\frac{r}{2}x_1\right), \dots, f(x_k) - \frac{2}{r} f\left(\frac{r}{2}x_k\right) \right) \right\|_k &\le \frac{1}{r} \varphi\left(\frac{r}{2}x_1, \frac{r}{2}x_1, \dots, \frac{r}{2}x_k, \frac{r}{2}x_k\right) \\ &\le \frac{1}{2} L\varphi(x_1, x_1, \dots, x_k, x_k) \end{aligned}$$

for all $x_1, \dots, x_k \in \mathcal{X}$.

The rest of the proof is similar to the proof of Theorem 4.3. □

Corollary 4.10 *Let $r > 2$, and let θ and $p > 1$ be positive real numbers. Or let $r < 2$, and let θ and $p < 1$ be positive real numbers. Let $f : X \to Y$ be an odd mapping such that*

$$\left\| \left(rf\left(\frac{ux_1 + uy_1}{r}\right) - uf(x_1) - uf(y_1), \dots, rf\left(\frac{ux_k + uy_k}{r}\right) - uf(x_k) - uf(y_k) \right) \right\|_k \le \theta \sum_{j=1}^{k} \left(\|x_j\|^p + \|y_j\|^p\right)$$

for all $u \in U(\mathcal{A})$ and all $x_1, \dots, x_k \in \mathcal{X}$. Then there exists a unique $\mathcal{A}$-linear generalized additive mapping $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ such that

$$\sup_{k\in\mathbb{N}} \left\| \left(\mathcal{L}(x_1) - f(x_1), \dots, \mathcal{L}(x_k) - f(x_k)\right) \right\|_k \le \sup_{k\in\mathbb{N}} \frac{r^{p-1}\theta}{2^{p-1} - r^{p-1}} \sum_{j=1}^{k} \|x_j\|^p$$

for all $x_1, \ldots, x_k \in \mathcal{X}$.

Proof Define $\varphi(x_1, y_1, \ldots, x_k, y_k) = \theta \sum_{j=1}^{k}(\|x_j\|^p + \|y_j\|^p)$, and apply Theorem 4.9. Then we get the desired result. □

5 Isomorphisms in unital multi-C^*-algebras

Throughout this section, assume that $\mathcal{A}$ and $\mathcal{B}$ are unital multi-C^*-algebras with unit e. Let $U(\mathcal{A})$ be the set of unitary elements in $\mathcal{A}$.

We investigate C^*-algebra isomorphisms in unital multi-C^*-algebras.

Theorem 5.1 *Let* $r \neq 2$. *Let* $h : \mathcal{A} \to \mathcal{B}$ *be an odd bijective mapping satisfying* $h(\frac{2^n}{r^n}uy) = h(\frac{2^n}{r^n}u)h(y)$ *for all* $u \in U(\mathcal{A})$, *all* $y \in \mathcal{A}$, *and* $n = 0, 1, 2, \ldots$, *for which there exists a function* $\varphi : \mathcal{A}^{kd} \to [0, \infty)$ *such that*

$$\lim_{j\to\infty} \frac{r^j}{2^j}\varphi\left(\frac{2^j}{r^j}x_{11}, \ldots, \frac{2^j}{r^j}x_{1d}, \ldots, \frac{2^j}{r^j}x_{k1}, \ldots, \frac{2^j}{r^j}x_{kd}\right) = 0,$$

$$\left\|\left(D_\mu h(x_{11}, \ldots, x_{1d}), \ldots, D_\mu h(x_{k1}, \ldots, x_{kd})\right)\right\|_k \le \varphi(x_{11}, \ldots, x_{1d}, \ldots, x_{k1}, \ldots, x_{kd}),$$

$$\left\|\left(h\left(\frac{2^n}{r^n}u_1^*\right) - h\left(\frac{2^n}{r^n}u_1\right)^*, \ldots, h\left(\frac{2^n}{r^n}u_k^*\right) - h\left(\frac{2^n}{r^n}u_k\right)^*\right)\right\|_k \le \varphi\Bigl(\underbrace{\frac{2^n}{r^n}u_1, \ldots, \frac{2^n}{r^n}u_1}_{d \text{ times}}, \ldots, \underbrace{\frac{2^n}{r^n}u_k, \ldots, \frac{2^n}{r^n}u_k}_{d \text{ times}}\Bigr)$$

for all $\mu \in S^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, *all* $u_1, \ldots, u_k \in U(\mathcal{A})$, $n = 0, 1, 2, \ldots$, *and all* $x_{11}, \ldots, x_{kd} \in A$. *Assume that* $\lim_{n\to\infty} \frac{r^n}{2^n}h(\frac{2^n}{r^n}e)$ *is invertible. Then the odd bijective mapping* $h : \mathcal{A} \to \mathcal{B}$ *is a* C^**-algebra isomorphism.*

Proof Consider the multi-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ as left Banach modules over the unital multi-C^*-algebra $\mathbb{C}$. By Theorem 4.3, there exists a unique $\mathbb{C}$-linear generalized additive mapping $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ such that

$$\sup_{k\in\mathbb{N}}\left\|\left(h(x_1) - \mathcal{H}(x_1), \ldots, h(x_k) - \mathcal{H}(x_k)\right)\right\|_k \le \sup_{k\in\mathbb{N}} \frac{1}{2({}_{d-2}C_l - {}_{d-2}C_{l-2} + 1)}\varphi(x_1, x_1, \underbrace{0, \ldots, 0}_{d-2 \text{ times}}, \ldots, x_k, x_k, \underbrace{0, \ldots, 0}_{d-2 \text{ times}})$$

for all $x_1, \ldots, x_k \in \mathcal{A}$ in which $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ is given by

$$\mathcal{H}(x) = \lim_{n\to\infty} \frac{r^n}{2^n}h\left(\frac{2^n}{r^n}x\right)$$

for all $x \in \mathcal{A}$.

The rest of the proof is similar to the proof of Theorem 4.1 of [7]. □

Corollary 5.2 *Let* $r > 2$, *and let* θ *and* $p > 1$ *be positive real numbers. Or let* $r < 2$, *and let* θ *and* $p < 1$ *be positive real numbers. Let* $h : \mathcal{A} \to \mathcal{B}$ *be an odd bijective mapping satisfying*

$h(\frac{2^n}{r^n}uy) = h(\frac{2^n}{r^n}u)h(y)$ for all $u \in U(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, such that

$$\big\| \big(D_\mu h(x_{11}, \ldots, x_{1d}), \ldots, D_\mu h(x_{k1}, \ldots, x_{kd})\big) \big\|_k \le \theta \sum_{j=1}^{d} \big(\|x_{1j}\|^p + \cdots + \|x_{kj}\|^p\big),$$

$$\left\| \left(h\left(\frac{2^n}{r^n}u_1^*\right) - h\left(\frac{2^n}{r^n}u_1\right)^*, \ldots, h\left(\frac{2^n}{r^n}u_k^*\right) - h\left(\frac{2^n}{r^n}u_k\right)^* \right) \right\|_k \le kd\frac{2^{pn}}{r^{pn}}\theta$$

for all $\mu \in S^1$, all $u \in U(\mathcal{A})$, $n = 0, 1, 2, \ldots$, and all $x_{11}, \ldots, x_{kd} \in \mathcal{A}$. Assume that $\lim_{n\to\infty} \frac{r^n}{2^n} \times h(\frac{2^n}{r^n}e)$ is invertible. Then the odd bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof Define $\varphi(x_{11}, \ldots, x_{1d}, \ldots, x_{k1}, \ldots, x_{kd}) = \theta \sum_{j=1}^{d}(\|x_{1j}\|^p + \cdots + \|x_{kj}\|^p)$, and apply Theorem 5.1. Then we get the desired result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the proof. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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