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Strong convergence theorems based on the viscosity approximation method for a countable family of nonexpansive mappings

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Abstract

In a real Hilbert space, an iterative scheme is considered to obtain a common fixed point for a countable family of nonexpansive mappings. In addition, strong convergence to the common fixed point of this sequence is investigated. As an application, an equilibrium problem is solved. We also state more applications of this procedure to obtain a common fixed point of W-mappings.

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1 Introduction

Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and I be an identity mapping on H. The strong (weak) convergence of $\{x_n\}$ to x is written by $x_n \to x$ ($x_n \rightharpoonup x$) as $n \to \infty$.

It is well known that H satisfies Opial's condition [1]; for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

A metric (nearest point) projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows.

For any point $x \in H$, there exists a unique $P_C x \in C$ such that

$$\|x - P_C x\| \le \|x - y\|,$$

for all $y \in C$. It is well known that P_C is a nonexpansive mapping from H onto C and satisfies the following:

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2,$$

for all $x, y \in H$. Furthermore, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{aligned} &\langle x - P_C x, P_C x - y \rangle \geq 0, \\ &\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \end{aligned} \tag{1.1}$$

for all $y \in C$.

Let A be a mapping of C into H. The variational inequality problem is to find an $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

We shall denote the set of solutions of the variational inequality problem (1.2) by $\mathrm{VI}(C, A)$. Then we have

$$x \in \mathrm{VI}(C, A) \quad \Longleftrightarrow \quad x = P_C(x - \lambda Ax), \quad \forall \lambda > 0. \tag{1.3}$$

A mapping S from C into itself is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$, for all $x, y \in C$. $\mathrm{Fix}(S) := \{x \in C : Sx = x\}$ is the set of fixed point of S. Note that $\mathrm{Fix}(S)$ is closed and convex if S is nonexpansive. A mapping f from C into C is said to be contraction, if there exists a constant $k \in [0, 1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, for all $x, y \in C$.

In 2000, Moudafi [2] introduced the following viscosity approximation methods: $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S x_n, \quad n \in \mathbb{N},$$

where f is a contraction on closed convex subset of a real Hilbert space. It was shown in [2] (also see Xu [3]) that such a sequence converges strongly to the unique solution of the variational inequality problem. In 2007, Chen *et al.* [4] suggested the following iterative scheme:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_C(I - \lambda_n A) x_n, \quad n \in \mathbb{N},$$

where $x_1 \in C$, S is a nonexpansive self-mapping and A an α-inverse strongly monotone mapping. They proved that the sequence $\{x_n\}$ converges strongly to a common fixed point of a nonexpansive mapping which solves the corresponding variational inequality. Recently, Kumam and Plubtieng [5] used the following viscosity iterative method for a countable family of nonexpansive mappings: $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S_n P_C(I - \lambda_n A) x_n, \quad n \in \mathbb{N}.$$

They proved the generated sequence $\{x_n\}$ converges strongly to a common element of the set of common fixed points of a countable family of nonexpansive mappings and the set of solutions of the variational inequality.

On the other hand, in 2009, Yao *et al.* [6] considered a new sequence that is generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(1 - \lambda_n) x_n, \quad n \in \mathbb{N},$$

to find a fixed point of a nonexpansive mapping.

It is worth pointing out that many authors have extended the results in Hilbert space to the more general uniformly convex and uniformly smooth Banach space (see, for instance, [3, 7–11]).

In this work, motivated and inspired by the above results, an iterative scheme based on the viscosity approximation method is utilized to find a common element of the set of common fixed points of a countable family of nonexpansive mappings. Moreover, a strong convergence theorem with different conditions on the parameters is studied. As an application, an equilibrium problem is solved. In addition, a common fixed point for W-mappings is obtained.

The following lemmas will be useful in the sequel.

Lemma 1.1 ([12]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 1$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 1.2 ([13]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0,$$

where

(1) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(2) $\limsup_{n\to\infty}\sigma_n \leq 0$;
(3) $\gamma_n \geq 0$, *for all* $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty}\gamma_n < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

Lemma 1.3 ([14]) *Let* C *be a nonempty closed subset of a Banach space and* $\{S_n\}$ *be a sequence of nonexpansive mappings from* C *into itself. Suppose* $\sum_{n=1}^{\infty}\sup\{\|S_{n+1}x - S_n x\| : x \in C\} < \infty$. *Then, for each* $x \in C$, $\{S_n x\}$ *converges strongly to some point of* C. *If* S *is a mapping from* C *into itself which is defined by* $Sx := \lim_{n\to\infty}S_n x$, *for all* $x \in C$, *then* $\lim_{n\to\infty}\sup\{\|S_n x - Sx\| : x \in C\} = 0$.

2 Strong convergence theorem

In this section, we use the viscosity approximation method to find a common element of the set of common fixed points of a countable family of nonexpansive mappings.

Theorem 2.1 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Assume that* $\{S_n\}$ *is a sequence of nonexpansive mappings from* C *into itself such that* $\bigcap_{n=1}^{\infty}\mathrm{Fix}(S_n) \neq \emptyset$, *and* f *is a contraction from* H *into* C *with constant* $k \leq \frac{1}{2}$. *Suppose that* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\lambda_n\}$ *are real sequences in* $(0,1)$. *Set* $x_1 \in C$ *and let* $\{x_n\}$ *be the iterative sequence defined by*

$$\begin{cases} y_n := P_C(1-\lambda_n)x_n, \\ x_{n+1} := \alpha_n x_n + \beta_n f(x_n) + \gamma_n S_n y_n, \quad n \in \mathbb{N}, \end{cases}$$

satisfying the following conditions:

(1) $\alpha_n + \beta_n + \gamma_n = 1$,
(2) $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$,
(3) $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$,
(4) $\lim_{n\to\infty}\lambda_n = 0$, $\sum_{n=1}^{\infty}\lambda_n = \infty$, $\sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$,
(5) $\sum_{n=1}^{\infty}\sup\{\|S_{n+1}x - S_nx\| : x \in B\} < \infty$, *for any bounded subset* B *of* C.

Let S *be a mapping from* C *into itself defined by* $Sx := \lim_{n\to\infty}S_nx$ *for all* $x \in C$ *and* $\mathrm{Fix}(S) := \bigcap_{n=1}^{\infty}\mathrm{Fix}(S_n)$. *Then* $\{x_n\}$ *converges strongly to an element* $\omega \in \mathrm{Fix}(S)$, *where* $\omega = P_{\mathrm{Fix}(S)}f(\omega)$.

Proof $\mathrm{Fix}(S)$ is a closed convex set, then $P_{\mathrm{Fix}(S)}$ is well defined and $P_{\mathrm{Fix}(S)}$ is nonexpansive. In addition,

$$\left\|P_{\mathrm{Fix}(S)}f(x) - P_{\mathrm{Fix}(S)}f(y)\right\| \le \left\|f(x) - f(y)\right\| \le k\|x - y\|,$$

for all $x, y \in H$. This shows that $P_{\mathrm{Fix}(S)}f$ is a contraction from H into C. Since H is complete, there exists a unique element of $\omega \in \mathrm{Fix}(S) \subset H$ such that $\omega = P_{\mathrm{Fix}(S)}f(\omega)$.

Let $x \in \mathrm{Fix}(S)$, we note that

$$\begin{aligned}
\|x_{n+1} - x\| &\le \alpha_n\|x_n - x\| + \beta_n\left\|f(x_n) - x\right\| + \gamma_n\|S_ny_n - x\| \\
&\le \alpha_n\|x_n - x\| + k\beta_n\|x_n - x\| + \beta_n\left\|f(x) - x\right\| + \gamma_n\|y_n - x\| \\
&\le \alpha_n\|x_n - x\| + k\beta_n\|x_n - x\| + \beta_n\left\|f(x) - x\right\| + \gamma_n\left\|(1 - \lambda_n)x_n - x\right\| \\
&\le \alpha_n\|x_n - x\| + k\beta_n\|x_n - x\| + \beta_n\left\|f(x) - x\right\| \\
&\quad + \gamma_n(1 - \lambda_n)\|x_n - x\| + \gamma_n\lambda_n\|x\| \\
&= (\alpha_n + k\beta_n + \gamma_n - \gamma_n\lambda_n)\|x_n - x\| + \beta_n\left\|f(x) - x\right\| + \gamma_n\lambda_n\|x\| \\
&\le (1 - \beta_n + k\beta_n - \gamma_n\lambda_n)\|x_n - x\| \\
&\quad + (1 - k)\beta_n\frac{\|f(x) - x\|}{1 - k} + \gamma_n\lambda_n\|x\| \\
&\le \max\left\{\|x_n - x\|, \|x\|, \frac{\|f(x) - x\|}{1 - k}\right\} \\
&\;\;\vdots \\
&\le \max\left\{\|x_1 - x\|, \|x\|, \frac{\|f(x) - x\|}{1 - k}\right\}.
\end{aligned}$$

Therefore $\{x_n\}$ is bounded. Hence, $\{f(x_n)\}$, $\{y_n\}$, and $\{S_ny_n\}$ are bounded. Also

$$\begin{aligned}
\|S_{n+1}y_{n+1} - S_ny_n\| &\le \|S_{n+1}y_{n+1} - S_{n+1}y_n\| + \|S_{n+1}y_n - S_ny_n\| \\
&\le \|y_{n+1} - y_n\| + \sup\bigl\{\|S_{n+1}x - S_nx\| : x \in \{y_n\}\bigr\} \\
&\le \left\|(1 - \lambda_{n+1})x_{n+1} - (1 - \lambda_n)x_n\right\| \\
&\quad + \sup\bigl\{\|S_{n+1}x - S_nx\| : x \in \{y_n\}\bigr\} \\
&\le \|x_{n+1} - x_n\| + \lambda_{n+1}\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|x_n\| \\
&\quad + \sup\bigl\{\|S_{n+1}x - S_nx\| : x \in \{y_n\}\bigr\}.
\end{aligned} \tag{2.1}$$

Now, we define $x_{n+1} = \alpha_n x_n + (1-\alpha_n)w_n$, for all $n \in \mathbb{N}$. One can observe that

$$\begin{aligned} w_{n+1} - w_n &= \frac{\beta_{n+1} f(x_{n+1}) + \gamma_{n+1} S_{n+1} y_{n+1}}{1-\alpha_{n+1}} - \frac{\beta_n f(x_n) + \gamma_n S_n y_n}{1-\alpha_n} \\ &= \frac{\beta_{n+1}}{1-\alpha_{n+1}} f(x_{n+1}) + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\alpha_{n+1}} S_{n+1} y_{n+1} \\ &\quad - \frac{\beta_n}{1-\alpha_n} f(x_n) - \frac{1-\alpha_n-\beta_n}{1-\alpha_n} S_n y_n \\ &= \frac{\beta_{n+1}}{1-\alpha_{n+1}} \bigl(f(x_{n+1}) - S_{n+1} y_{n+1}\bigr) \\ &\quad + \frac{\beta_n}{1-\alpha_n} \bigl(S_n y_n - f(x_n)\bigr) + S_{n+1} y_{n+1} - S_n y_n. \end{aligned} \tag{2.2}$$

Substituting (2.1) into (2.2), it follows that

$$\begin{aligned} &\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \\ &\quad \le \frac{\beta_{n+1}}{1-\alpha_{n+1}} \bigl\|f(x_{n+1}) - S_{n+1} y_{n+1}\bigr\| + \frac{\beta_n}{1-\alpha_n} \bigl\|S_n y_n - f(x_n)\bigr\| \\ &\quad\quad + \lambda_{n+1} \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \|x_n\| \\ &\quad\quad + \sup\bigl\{\|S_{n+1} x - S_n x\| : x \in \{y_n\}\bigr\}. \end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \le 0.$$

In view of Lemma 1.1, we obtain $\lim_{n\to\infty} \|w_n - x_n\| = 0$, which implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\alpha_n) \|w_n - x_n\| = 0.$$

On the other hand, one has

$$x_{n+1} - x_n = \beta_n \bigl(f(x_n) - S_n y_n\bigr) + (1-\alpha_n)(S_n y_n - x_n).$$

It follows that

$$(1-\alpha_n) \|S_n y_n - x_n\| \le \|x_{n+1} - x_n\| + \beta_n \bigl\|S_n y_n - f(x_n)\bigr\|.$$

Hence, $\lim_{n\to\infty} \|x_n - S_n y_n\| = 0$. Also, from $\|y_n - S_n y_n\| \le \|x_n - S_n y_n\| + \lambda_n \|x_n\|$, we obtain

$$\lim_{n\to\infty} \|y_n - S_n y_n\| = 0.$$

Now, we prove

$$\limsup_{n\to\infty} \bigl\langle f(\omega) - \omega, S_n y_n - \omega\bigr\rangle \le 0,$$

where $\omega = P_{\mathrm{Fix}(S)}f(\omega)$. Indeed, since $\{S_n y_n\}$ is bounded, one can find a subsequence $\{S_{n_i} y_{n_i}\}$ of $\{S_n y_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\omega)-\omega, S_n y_n-\omega\rangle = \lim_{i\to\infty}\langle f(\omega)-\omega, S_{n_i} y_{n_i}-\omega\rangle.$$

$\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to z. Without loss of generality, assume that $y_{n_i} \rightharpoonup z$. y_{n_i} is a sequence in C and C is closed and convex, so $z \in C$. Now, using the fact that $\|S_n y_n - y_n\| \to 0$, we obtain $S_{n_i} y_{n_i} \rightharpoonup z$. Next we show $z \in \mathrm{Fix}(S)$.

Assume that $z \notin \mathrm{Fix}(S)$. From Opial's condition and Lemma 1.3, we have

$$\begin{aligned}
\liminf_{i\to\infty}\|y_{n_i}-z\| &< \liminf_{i\to\infty}\|y_{n_i}-Sz\| \\
&= \liminf_{i\to\infty}\|y_{n_i}-S_{n_i}y_{n_i}+S_{n_i}y_{n_i}-Sy_{n_i}+Sy_{n_i}-Sz\| \\
&\le \liminf_{i\to\infty}\|Sy_{n_i}-Sz\| \\
&\le \liminf_{i\to\infty}\|y_{n_i}-z\|.
\end{aligned}$$

This is a contradiction. Thus, $z \in \mathrm{Fix}(S)$.

Also, we note that $\omega = P_{\mathrm{Fix}(S)}f(\omega)$ and so, by (1.1), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(\omega)-\omega, S_n y_n-\omega\rangle &= \lim_{i\to\infty}\langle f(\omega)-\omega, S_{n_i} y_{n_i}-\omega\rangle \\
&= \langle f(\omega)-\omega, z-\omega\rangle \le 0.
\end{aligned}$$

To complete the proof, we show $\{x_n\}$ converges strongly to $\omega \in F(S)$. For this, by convexity of $\|\cdot\|^2$, we have

$$\|S_n y_n-\omega\|^2 \le \|y_n-\omega\|^2 \le \big\|(1-\lambda_n)x_n-\omega\big\|^2 \le (1-\lambda_n)\|x_n-\omega\|^2+\lambda_n\|\omega\|^2.$$

Hence,

$$\begin{aligned}
\|x_{n+1}-\omega\|^2 &= \big\|\alpha_n(x_n-\omega)+\beta_n\big(f(x_n)-\omega\big)+\gamma_n(S_n y_n-\omega)\big\|^2 \\
&\le \big\|\alpha_n(x_n-\omega)+\beta_n\big(f(x_n)-\omega\big)\big\|^2+\gamma_n^2\|S_n y_n-\omega\|^2 \\
&\quad + 2\gamma_n\big\langle\alpha_n(x_n-\omega)+\beta_n\big(f(x_n)-\omega\big), S_n y_n-\omega\big\rangle \\
&= \big(\alpha_n\|x_n-\omega\|+\beta_n\big\|f(x_n)-\omega\big\|\big)^2+\gamma_n^2\|S_n y_n-\omega\|^2 \\
&\quad + 2\alpha_n\gamma_n\langle x_n-\omega, S_n y_n-\omega\rangle+2\beta_n\gamma_n\big\langle f(x_n)-\omega, S_n y_n-\omega\big\rangle \\
&\le \alpha_n^2\|x_n-\omega\|^2+\beta_n^2\big\|f(x_n)-\omega\big\|^2+\gamma_n^2\|S_n y_n-\omega\|^2 \\
&\quad + 2\alpha_n\beta_n\|x_n-\omega\|\big\|f(x_n)-\omega\big\|+2\alpha_n\gamma_n\|x_n-\omega\|\|S_n y_n-\omega\| \\
&\quad + 2\beta_n\gamma_n\big\langle f(x_n)-f(\omega), S_n y_n-\omega\big\rangle+2\beta_n\gamma_n\big\langle f(\omega)-\omega, S_n y_n-\omega\big\rangle \\
&\le \alpha_n^2\|x_n-\omega\|^2+\beta_n^2\big\|f(x_n)-\omega\big\|^2+\gamma_n^2\|S_n y_n-\omega\|^2 \\
&\quad + \alpha_n\beta_n\big(\|x_n-\omega\|^2+\big\|f(x_n)-\omega\big\|^2\big)+\alpha_n\gamma_n\big(\|x_n-\omega\|^2+\|S_n y_n-\omega\|^2\big) \\
&\quad + 2k\beta_n\gamma_n\|x_n-\omega\|\|S_n y_n-\omega\|+2\beta_n\gamma_n\big\langle f(\omega)-\omega, S_n y_n-\omega\big\rangle
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\alpha_n^2 + \alpha_n\beta_n + \alpha_n\gamma_n\right)\|x_n - \omega\|^2 + \left(\beta_n^2 + \alpha_n\beta_n\right)\left\|f(x_n) - \omega\right\|^2 \\
&\quad + \left(\gamma_n^2 + \alpha_n\gamma_n\right)\|S_n y_n - \omega\|^2 + k\beta_n\gamma_n\left(\|x_n - \omega\|^2 + \|S_n y_n - \omega\|^2\right) \\
&\quad + 2\beta_n\gamma_n\left\langle f(\omega) - \omega, S_n y_n - \omega\right\rangle \\
&\le (\alpha_n + k\beta_n\gamma_n)\|x_n - \omega\|^2 + \beta_n(1 - \gamma_n)\left\|f(x_n) - \omega\right\|^2 \\
&\quad + \left(\gamma_n^2 + \alpha_n\gamma_n + k\beta_n\gamma_n\right)\|S_n y_n - \omega\|^2 + 2\beta_n\gamma_n\left\langle f(\omega) - \omega, S_n y_n - \omega\right\rangle \\
&\le (\alpha_n + k\beta_n\gamma_n)\|x_n - \omega\|^2 + \beta_n(1 - \gamma_n)\left\|f(x_n) - \omega\right\|^2 \\
&\quad + \left(\gamma_n^2 + \alpha_n\gamma_n + k\beta_n\gamma_n\right)\left[(1 - \lambda_n)\|x_n - \omega\|^2 + \lambda_n\|\omega\|^2\right] \\
&\quad + 2\beta_n\gamma_n\left\langle f(\omega) - \omega, S_n y_n - \omega\right\rangle \\
&\le \left(\alpha_n + k\beta_n\gamma_n + \left(\gamma_n^2 + \alpha_n\gamma_n + k\beta_n\gamma_n\right)(1 - \lambda_n)\right)\|x_n - \omega\|^2 \\
&\quad + \beta_n(1 - \gamma_n)\left\|f(x_n) - \omega\right\|^2 + \left(\gamma_n^2 + \alpha_n\gamma_n + \beta_n\gamma_n\right)\lambda_n\|\omega\|^2 \\
&\quad + 2\beta_n\gamma_n\left\langle f(\omega) - \omega, S_n y_n - \omega\right\rangle.
\end{aligned}
$$

Now, suppose $L = \sup\{\|x_n - \omega\|, \|f(x_n) - \omega\|, \|\omega\|\}$. Then

$$
\begin{aligned}
\|x_{n+1} - \omega\|^2 &\le (1 - \beta_n\gamma_n)\|x_n - \omega\|^2 \\
&\quad + \beta_n\gamma_n\left[2\left\langle f(\omega) - \omega, S_n y_n - \omega\right\rangle + \frac{\lambda_n}{\beta_n}\|\omega\|^2 + \frac{1 - \gamma_n}{\gamma_n}\left\|f(x_n) - \omega\right\|^2\right. \\
&\quad \left. + \frac{\alpha_n + k\beta_n\gamma_n + (\gamma_n^2 + \alpha_n\gamma_n + k\beta_n\gamma_n)(1 - \lambda_n) + \beta_n\gamma_n - 1}{\beta_n\gamma_n}\|x_n - \omega\|^2\right] \\
&\le (1 - \beta_n\gamma_n)\|x_n - \omega\|^2 + \beta_n\gamma_n\left[2\left\langle f(\omega) - \omega, S_n y_n - \omega\right\rangle\right. \\
&\quad + \left[\frac{\lambda_n\gamma_n + \beta_n - \beta_n\gamma_n + \alpha_n + k\beta_n\gamma_n + (\gamma_n(1 - \beta_n) + k\beta_n\gamma_n)(1 - \lambda_n)}{\beta_n\gamma_n}\right. \\
&\quad \left.\left. + \frac{\beta_n\gamma_n - 1}{\beta_n\gamma_n}\right]L^2\right] \\
&= (1 - \delta_n)\|x_n - \omega\|^2 + \delta_n\sigma_n,
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_n &= \beta_n\gamma_n, \\
\sigma_n &= 2\left\langle f(\omega) - \omega, S_n y_n - \omega\right\rangle \\
&\quad + \left[\frac{\lambda_n\gamma_n + \beta_n - \beta_n\gamma_n + \alpha_n + k\beta_n\gamma_n + (\gamma_n(1 - \beta_n) + k\beta_n\gamma_n)(1 - \lambda_n)}{\beta_n\gamma_n}\right. \\
&\quad \left. + \frac{\beta_n\gamma_n - 1}{\beta_n\gamma_n}\right]L^2 \\
&= 2\left\langle f(\omega) - \omega, S_n y_n - \omega\right\rangle \\
&\quad + \left[\frac{\lambda_n\gamma_n + k\beta_n\gamma_n - \gamma_n + \gamma_n(1 - \beta_n)(1 - \lambda_n) + k\beta_n\gamma_n(1 - \lambda_n)}{\beta_n\gamma_n}\right]L^2 \\
&= 2\left\langle f(\omega) - \omega, S_n y_n - \omega\right\rangle
\end{aligned}
$$

$$+\left[\frac{\lambda_n + k\beta_n - 1 + (1-\beta_n)(1-\lambda_n) + k\beta_n(1-\lambda_n)}{\beta_n}\right]L^2$$
$$\leq 2\langle f(\omega) - \omega, S_n y_n - \omega\rangle + \left[(2k-1) + (1-k)\lambda_n\right]L^2.$$

It is easy to see that $\{\delta_n\} \subset [0,1]$, $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \leq 0$. Hence, by Lemma 1.2, we find that $\{x_n\}$ strongly converges to $\omega \in \mathrm{Fix}(S)$, where $\omega = P_{\mathrm{Fix}(S)}f(\omega)$. This completes the proof of this theorem. □

The following example shows that this theorem is not a special case of [5, Theorem 3.1].

Example 2.2 Let $C = [-1,1] \subset H = \mathbb{R}$ with $\alpha_n = \frac{n-1}{10n-9}$, $\beta_n = \frac{1}{n}$, and $\lambda_n = \frac{9}{10n}$. Set $f(x) = \frac{x}{10}$ and $S_n(x) = \frac{x}{n}$. Then f is a $\frac{1}{10}$-contraction and S_n is a sequence of nonexpansive mappings. It readily follows that the sequence $\{x_n\}$ generated by

$$\begin{cases} y_n := P_C(1-\lambda_n)x_n = \frac{10n-9}{10n}x_n, \\ x_{n+1} := \alpha_n x_n + \beta_n f(x_n) + \gamma_n S_n y_n = \frac{10n^2-9}{100n^2-90n}x_n + \frac{9n^2-18n+9}{10n^3-9n^2}y_n, \end{cases}$$

with initial value $x_1 \in C$, converges strongly to an element (zero) of $\mathrm{Fix}(S) = \bigcap_{n=1}^{\infty}\mathrm{Fix}(S_n)$ and $P_{\mathrm{Fix}(S)}f(0) = 0$.

3 Applications

In this section, we consider the equilibrium problems and W-mappings.

3.1 Equilibrium problems

Equilibrium theory plays a central role in various applied sciences such as physics, mechanics, chemistry, and biology. In addition, it represents an important area of the mathematical sciences such as optimization, operations research, game theory, and financial mathematics. Equilibrium problems include fixed point problems, optimization problems, variational inequalities, Nash equilibria problems, and complementary problems as special cases.

Let $\varphi : C \to \mathbb{R}$ be a real-valued function and $A : C \to H$ a nonlinear mapping. Also suppose $F : C \times C \to \mathbb{R}$ is a bifunction. The generalized mixed equilibrium problem is to find $x \in C$ (see [15]) such that

$$F(x,y) + \varphi(y) - \varphi(x) + \langle Ax, y-x\rangle \geq 0, \tag{3.1}$$

for all $y \in C$.

We shall denote the set of solutions of this generalized mixed equilibrium problem by GMEP; that is

$$\mathrm{GMEP} := \left\{x \in C : F(x,y) + \varphi(y) - \varphi(x) + \langle Ax, y-x\rangle \geq 0, \forall y \in C\right\}.$$

We now discuss several special cases of GMEP as follows:

1. If $\varphi = 0$, then the problem (3.1) is reduced to generalized equilibrium problem, *i.e.*, finding $x \in C$ such that

   $$F(x,y) + \langle Ax, y-x\rangle \geq 0,$$

   for all $y \in C$.

2. If $A = 0$, then the problem (3.1) is reduced to the mixed equilibrium problem, that is, to find $x \in C$ such that
$$F(x,y) + \varphi(y) - \varphi(x) \geq 0,$$
for all $y \in C$. We shall write the set of solutions of the mixed equilibrium problem by MEP.
3. If $\varphi = 0$, $A = 0$, then the problem (3.1) is reduced to the equilibrium problem, which is to find $x \in C$ such that
$$F(x,y) \geq 0,$$
for all $y \in C$.
4. If $\varphi = 0$, $F = 0$, then the problem (3.1) is reduced to the variational inequality problem (1.2).

Now let $\varphi : C \to \mathbb{R}$ be a real-valued function. To solve the generalized mixed equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F, φ, and C satisfy the following conditions:

(A_1) $F(x,x) = 0$ for all $x \in C$;
(A_2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A_3) for each $x, y, z \in C$, $\lim_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x,y)$;
(A_4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous;
(B_1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for each $z \in C \backslash D_x$,
$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$
(B_2) C is a bounded set.

In what follows we state some lemmas which are useful to prove our convergence results.

Lemma 3.1 ([16]) *Assume that* $F : C \times C \to \mathbb{R}$ *satisfies* (A_1)-(A_4), *and let* $\varphi : C \to \mathbb{R}$ *be a lower semicontinuous and convex function. Assume that either* (B_1) *or* (B_2) *holds. For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r^{(F,\varphi)} : H \to C$ *as follows*:
$$T_r^{(F,\varphi)}(x) := \left\{ z \in C : F(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\},$$
for all $x \in H$. *Then the following assertions hold*:
(1) *For each* $x \in H$, $T_r^{(F,\varphi)} \neq \emptyset$;
(2) $T_r^{(F,\varphi)}$ *is single-valued*;
(3) $T_r^{(F,\varphi)}$ *is firmly nonexpansive, i.e., for any* $x, y \in H$,
$$\left\| T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y \right\|^2 \leq \left\langle T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y, x - y \right\rangle;$$
(4) $\mathrm{Fix}(T_r^{(F,\varphi)}) = \mathrm{MEP}$;
(5) MEP *is closed and convex*.

Lemma 3.2 ([17]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that S_1 is a nonexpansive mapping from C into H and S_2 a firmly nonexpansive mapping from H into C such that $\mathrm{Fix}(S_1) \cap \mathrm{Fix}(S_2) \neq \emptyset$. Then S_1S_2 is a nonexpansive mapping from H into itself and $\mathrm{Fix}(S_1S_2) = \mathrm{Fix}(S_1) \cap \mathrm{Fix}(S_2)$.*

Lemma 3.3 ([18, 19]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A_1)-(A_4) and $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Assume that either (B_1) or (B_2) holds. Let $\{r_n\}$ be a sequence in $(0,\infty)$, such that $\inf\{r_n : n \in N\} > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ and $T_{r_n}^{(F,\varphi)}$ be a mapping defined as in Lemma 3.1. Then*

(1) $\sum_{n=1}^{\infty} \sup\{\|T_{r_{n+1}}^{(F,\varphi)}x - T_{r_n}^{(F,\varphi)}x\| : x \in B\} < \infty$, *for any bounded subset B of C;*
(2) $\mathrm{Fix}(T_r^{(F,\varphi)}) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_{r_n}^{(F,\varphi)})$, *where $T_r^{(F,\varphi)}$ is a mapping defined by $T_r^{(F,\varphi)}x := \lim_{n\to\infty} T_{r_n}^{(F,\varphi)}x$, for all $x \in C$. Moreover, $\lim_{n\to\infty} \|T_{r_n}^{(F,\varphi)}x - T_r^{(F,\varphi)}x\| = 0$.*

Now let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $A : C \to H$ is called *monotone* if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in C$. It is called *α-inverse strongly monotone* if there exists a positive real number α such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$, for all $x, y \in C$. An α-inverse strongly monotone mapping is sometimes called *α-cocoercive*. A mapping A is said to be *relaxed α-cocoercive* if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq -\alpha\|Ax - Ay\|^2,$$

for all $x, y \in C$. The mapping A is said to be *relaxed (α,λ)-cocoercive* if there exist $\alpha, \lambda > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq -\alpha\|Ax - Ay\|^2 + \lambda\|x - y\|^2,$$

for all $x, y \in C$. A mapping $A : H \to H$ is said to be *μ-Lipschitzian* if there exists $\mu \geq 0$ such that

$$\|Ax - Ay\| \leq \mu\|x - y\|,$$

for all $x, y \in H$. It is clear that each α-inverse strongly monotone mapping is monotone and $\frac{1}{\alpha}$-Lipschitzian and that each μ-Lipschitzian, relaxed (α,λ)-cocoercive mapping with $\alpha\mu^2 \leq \lambda$ is monotone. Also, if A is an α-inverse strongly monotone, then $I - \lambda A$ is a nonexpansive mapping from C to H, provided that $\lambda \leq 2\alpha$ (see [20]).

Now we have the following theorem.

Theorem 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A_1)-(A_4) and $\varphi : C \to \mathbb{R}$ a lower semicontinuous and convex function. Assume that either (B_1) or (B_2) holds. Let A be a μ-Lipschitzian, relaxed (α,λ)-cocoercive mapping from C into H, B a β-inverse strongly monotone mapping from C into H and f a contraction from H into C with constant $k \leq \frac{1}{2}$. Suppose that S_n is a sequence of nonexpansive mappings from H into C such that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{VI}(C,A) \cap \mathrm{GMEP} \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ be real sequences in*

$(0,1)$ and $\{r_n\}, \{s_n\} \subset (0,\infty)$. Let $\{y_n\}$, $\{u_n\}$, and $\{x_n\}$ be generated by $x_1 \in C$,

$$\begin{cases} y_n := P_C(1-\lambda_n)x_n, \\ u_n := T_{r_n}^{(F,\varphi)}(y_n - r_n By_n), \\ x_{n+1} := \alpha_n x_n + \beta_n f(x_n) + \gamma_n S_n P_C(u_n - s_n Au_n), \quad n \in \mathbb{N}. \end{cases}$$

Suppose that the following conditions are satisfied:

(1) $\alpha_n + \beta_n + \gamma_n = 1$,

(2) $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$,

(3) $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$,

(4) $\lim_{n\to\infty}\lambda_n = 0$, $\sum_{n=1}^{\infty}\lambda_n = \infty$, $\sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$,

(5) $0 < a \le r_n \le 2\beta$, $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$,

(6) $0 < s_n \le \frac{2(\lambda - \alpha\mu^2)}{\mu^2}$, $\sum_{n=1}^{\infty}|s_{n+1} - s_n| < \infty$,

(7) $\sum_{n=1}^{\infty}\sup\{\|S_{n+1}x - S_n x\| : x \in E\} < \infty$, *for any bounded subset E of C*.

Then $\{x_n\}$ converges strongly to an element $\omega \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{VI}(C,A) \cap \mathrm{GMEP}$, where $\omega = P_{\bigcap_{n=1}^{\infty}\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)\cap\mathrm{GMEP}}f(\omega)$.

Proof For all $x, y \in C$ and $s_n \in [0, \frac{2(\lambda-\alpha\mu^2)}{\mu^2}]$, we obtain

$$\begin{aligned} &\bigl\|(I - s_nA)x - (I - s_nA)y\bigr\|^2 \\ &\quad = \bigl\|x - y - s_n(Ax - Ay)\bigr\|^2 \\ &\quad = \|x-y\|^2 - 2s_n\langle x - y, Ax - Ay\rangle + s_n^2\|Ax - Ay\|^2 \\ &\quad \le \|x-y\|^2 - 2s_n\bigl[-\alpha\|Ax - Ay\|^2 + \lambda\|x-y\|^2\bigr] + s_n^2\|Ax - Ay\|^2 \\ &\quad \le \|x-y\|^2 + 2s_n\mu^2\alpha\|x-y\|^2 - 2s_n\lambda\|x-y\|^2 + \mu^2 s_n^2\|x-y\|^2 \\ &\quad = \bigl(1 + 2s_n\mu^2\alpha - 2s_n\lambda + \mu^2 s_n^2\bigr)\|x-y\|^2 \le \|x-y\|^2. \end{aligned}$$

This shows that $I - s_nA$ is nonexpansive for each $n \in \mathbb{N}$. By Lemma 3.3, it implies that

$$\sum_{n=1}^{\infty}\sup\bigl\{\bigl\|T_{r_{n+1}}^{(F,\varphi)}x - T_{r_n}^{(F,\varphi)}x\bigr\| : x \in E\bigr\} < \infty,$$

for any bounded subset E of C. In addition, the mapping $T_r^{(F,\varphi)}$, defined by $T_r^{(F,\varphi)}x := \lim_{n\to\infty}T_{r_n}^{(F,\varphi)}x$ for all $x \in C$, satisfies $\mathrm{Fix}(T_r^{(F,\varphi)}) = \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_{r_n}^{(F,\varphi)}) = \mathrm{MEP}$.

Put $T_n := S_nP_C(I - s_nA)T_{r_n}^{(F,\varphi)}(I - r_nB) = S_nP_C(I - s_nA)U_n$. Then, by Lemmas 3.2, 3.1, and (1.3), we find that T_n is a nonexpansive mapping from C into itself and $\mathrm{Fix}(T_n) = \mathrm{Fix}(S_n) \cap \mathrm{VI}(C,A) \cap \mathrm{Fix}(T_{r_n}^{F,\varphi}(I - r_nB)) = \mathrm{Fix}(S_n) \cap \mathrm{VI}(C,A) \cap \mathrm{GMEP}$, for all $n \in \mathbb{N}$, and so

$$\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n) = \bigcap_{n=1}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{VI}(C,A) \cap \mathrm{GMEP}.$$

Also we note that

$$\begin{aligned} \|T_{n+1}x - T_nx\| &= \bigl\|S_{n+1}P_C(I - s_{n+1}A)U_{n+1}x - S_nP_C(I - s_nA)U_nx\bigr\| \\ &\le \bigl\|S_{n+1}P_C(I - s_{n+1}A)U_{n+1}x - S_nP_C(I - s_{n+1}A)U_{n+1}x\bigr\| \end{aligned}$$

$$\begin{aligned}
&\quad + \left\| S_n P_C(I - s_{n+1}A)U_{n+1}x - S_n P_C(I - s_nA)U_nx \right\| \\
&\leq \|S_{n+1}v_n - S_nv_n\| + \left\| (I - s_{n+1}A)U_{n+1}x - (I - s_nA)U_nx \right\| \\
&\leq \|S_{n+1}v_n - S_nv_n\| + \left\| (I - s_{n+1}A)U_{n+1}x - (I - s_{n+1}A)U_nx \right\| \\
&\quad + \left\| (I - s_{n+1}A)U_nx - (I - s_nA)U_nx \right\| \\
&\leq \|S_{n+1}v_n - S_nv_n\| + \left\| T_{r_{n+1}}^{(F,\varphi)}(I - r_{n+1}B)x - T_{r_n}^{(F,\varphi)}(I - r_nB)x \right\| \\
&\quad + |s_{n+1} - s_n| \|AU_nx\| \\
&\leq \|S_{n+1}v_n - S_nv_n\| + \left\| T_{r_{n+1}}^{(F,\varphi)}(I - r_{n+1}B)x - T_{r_n}^{(F,\varphi)}(I - r_{n+1}B)x \right\| \\
&\quad + \left\| T_{r_n}^{(F,\varphi)}(I - r_{n+1}B)x - T_{r_n}^{(F,\varphi)}(I - r_nB)x \right\| \\
&\quad + |s_{n+1} - s_n| \|AU_nx\| \\
&\leq \|S_{n+1}v_n - S_nv_n\| + \left\| T_{r_{n+1}}^{(F,\varphi)}w_n - T_{r_n}^{(F,\varphi)}w_n \right\| \\
&\quad + |r_{n+1} - r_n| \|Bx\| + |s_{n+1} - s_n| \|AU_nx\|,
\end{aligned}$$

where $v_n = P_C(I - s_{n+1}A)U_{n+1}x$ and $w_n = (I - r_{n+1}B)x$. Moreover, for any bounded subset E of C, $F = \{P_C(I - s_{n+1}A)U_{n+1}x : x \in E, n \in \mathbb{N}\}$ and $G = \{(I - r_{n+1}B)x : x \in E, n \in \mathbb{N}\}$ are bounded and

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sup\left\{ \|T_{n+1}x - T_nx\| : x \in E \right\} \\
&\quad \leq \sum_{n=1}^{\infty} \sup\left\{ \|S_{n+1}y - S_ny\| : y \in F \right\} + \sum_{n=1}^{\infty} \sup\left\{ \left\| T_{r_{n+1}}^{(F,\varphi)}z - T_{r_n}^{(F,\varphi)}z \right\| : z \in G \right\} \\
&\qquad + \sum_{n=1}^{\infty} |r_{n+1} - r_n| \sup\left\{ \|Bx\| : x \in E \right\} + \sum_{n=1}^{\infty} |s_{n+1} - s_n| \sup\left\{ \|AU_nx\| : x \in E \right\} < \infty.
\end{aligned}$$

Therefore, by Theorem 2.1, $\{x_n\}$ converges strongly to an element $\omega \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{VI}(C, A) \cap \mathrm{GMEP}$, where $\omega = P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{VI}(C,A) \cap \mathrm{GMEP}} f(\omega)$. This completes the proof. $\square$

3.2 *W*-Mappings

The concept of W-mappings was introduced in [21, 22]. It is now one of the main tools in studying convergence of iterative methods to approach a common fixed point of nonlinear mapping; more recent progress can be found in [23] and the references cited therein.

Let $\{S_n\}$ be a countable family of nonexpansive mappings $S_n : H \to H$ and $\delta_1, \delta_2, \ldots$ be real numbers such that $0 \leq \delta_n \leq 1$ for every $n \in \mathbb{N}$. We consider the mapping W_n defined by

$$\begin{aligned}
&U_{n,n+1} := I, \\
&U_{n,n} := \delta_n S_n U_{n,n+1} + (1 - \delta_n)I, \\
&U_{n,n-1} := \delta_{n-1} S_{n-1} U_{n,n} + (1 - \delta_{n-1})I, \\
&\vdots \\
&U_{n,k} := \delta_k S_k U_{n,k+1} + (1 - \delta_k)I, \qquad (3.2)
\end{aligned}$$

$$
\begin{aligned}
&U_{n,k-1} := \delta_{k-1} S_{k-1} U_{n,k} + (1 - \delta_{k-1}) I, \\
&\vdots \\
&U_{n,2} := \delta_2 S_2 U_{n,3} + (1 - \delta_2) I, \\
&W_n := U_{n,1} = \delta_1 S_1 U_{n,2} + (1 - \delta_1) I.
\end{aligned}
$$

One can find the proof of the following lemma in [24].

Lemma 3.5 *Let H be a real Hilbert space. Let $\{S_n\}$ be a sequence of nonexpansive mappings from H into itself such that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \neq \emptyset$. Let $\delta_1, \delta_2, \ldots$ be real numbers such that $0 < \delta_n \leq b < 1$ for all $n \in \mathbb{N}$. Then*

1. *W_n is nonexpansive and $\mathrm{Fix}(W_n) = \bigcap_{i=1}^{n} \mathrm{Fix}(S_i)$ for all $n \in \mathbb{N}$;*
2. *$\lim_{n\to\infty} U_{n,k} x$ exists, for all $x \in H$ and $k \in \mathbb{N}$;*
3. *the mapping $W : C \to C$ defined by $Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$, for all $x \in C$ is a nonexpansive mapping satisfying $\mathrm{Fix}(W) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$; and it is called W-mapping generated by $S_1, S_2, \ldots, S_n$, and $\delta_1, \delta_2, \ldots, \delta_n$.*

Theorem 3.6 *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $\{S_n\}$ is a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \neq \emptyset$, and f a contraction from H into C with constant $k \leq \frac{1}{2}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ be real sequences in $(0,1)$. Also, suppose W_n are the W-mappings from C into itself generated by $S_1, S_2, \ldots, S_n$, and $\delta_1, \delta_2, \ldots, \delta_n$ such that $0 < \delta_n \leq b < 1$ for every $n \in \mathbb{N}$. Set $x_1 \in C$ and let $\{x_n\}$ be the iterative sequence defined by*

$$
\begin{cases}
y_n := P_C(1 - \lambda_n) x_n, \\
x_{n+1} := \alpha_n x_n + \beta_n f(x_n) + \gamma_n W_n y_n, \quad n \in \mathbb{N},
\end{cases}
$$

satisfying the following conditions:

1. $\alpha_n + \beta_n + \gamma_n = 1$,
2. $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$,
3. $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,
4. $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Let W be a mapping from C into itself defined by $Wx := \lim_{n\to\infty} W_n x$ for all $x \in C$. Then $\{x_n\}$ converges strongly to an element $\omega \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$, where $\omega = P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)} f(\omega)$.

Proof Since S_i and $U_{n,i}$ are nonexpansive, by (3.2), we deduce that, for each $n \in \mathbb{N}$,

$$
\begin{aligned}
\|W_{n+1}x - W_n x\| &= \|\delta_1 S_1 U_{n+1,2} x - \delta_1 S_1 U_{n,2} x\| \\
&\leq \delta_1 \|U_{n+1,2} x - U_{n,2} x\| \\
&= \delta_1 \|\delta_2 S_2 U_{n+1,3} x - \delta_2 S_2 U_{n,3} x\| \\
&\leq \delta_1 \delta_2 \|U_{n+1,3} x - U_{n,3} x\| \\
&\leq \cdots \\
&\leq \delta_1 \delta_2 \cdots \delta_n \|U_{n+1,n+1} x - U_{n,n+1} x\| \leq M \prod_{i=1}^{n} \delta_i,
\end{aligned}
$$

where $M > 0$ is a constant such that $\sup\{\|U_{n+1,n+1}x - U_{n,n+1}x\| : x \in B\} \le M$, for any bounded subset B of C. Then

$$\sum_{n=1}^{\infty} \sup\big\{\|W_{n+1}x - W_n x\| : x \in B\big\} < \infty.$$

Now, by setting $S_n := W_n$ in Theorem 2.1 and using Lemma 3.5, we obtain the result. □

Applying Lemma 3.5 and Theorem 3.4, we obtain the following result.

Corollary 3.7 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A$_1$)-(A$_4$) *and $\varphi : C \to \mathbb{R}$ a lower semicontinuous and convex function. Assume that either* (B$_1$) *or* (B$_2$) *holds. Let A be a μ-Lipschitzian, relaxed (α, λ)-cocoercive mapping from C into H, B a β-inverse strongly monotone mapping from C into H and f a contraction from H into C with constant $k \le \frac{1}{2}$. Suppose that S_n is a sequence of nonexpansive mappings from H into C such that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{VI}(C,A) \cap \mathrm{GMEP} \neq \emptyset$. Let $\{y_n\}$, $\{u_n\}$, and $\{x_n\}$ be generated by $x_1 \in C$,*

$$\begin{cases} y_n := P_C(1-\lambda_n)x_n, \\ u_n := T_{r_n}^{(F,\varphi)}(y_n - r_n B y_n), \\ x_{n+1} := \alpha_n x_n + \beta_n f(x_n) + \gamma_n W_n P_C(u_n - s_n A u_n), \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ are real sequences in $(0,1)$ and $\{r_n\}, \{s_n\} \subset (0,\infty)$ satisfying the following conditions:

(1) $\alpha_n + \beta_n + \gamma_n = 1$,
(2) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$,
(3) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$,
(4) $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,
(5) $0 < a \le r_n \le 2\beta$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,
(6) $0 < s_n \le \frac{2(\lambda - \alpha\mu^2)}{\mu^2}$, $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$.

Then $\{x_n\}$ converges strongly to an element $\omega \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{VI}(C,A) \cap \mathrm{GMEP}$, where $\omega = P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{VI}(C,A) \cap \mathrm{GMEP}} f(\omega)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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