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Fuzzy stability of functional inequalities in matrix fuzzy normed spaces

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Abstract

Using the fixed point method, we prove the Hyers-Ulam stability of additive functional inequalities in matrix fuzzy normed spaces.

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1 Introduction and preliminaries

Katsaras [1] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [2–4]. In particular, Bag and Samanta [5], following Cheng and Mordeson [6], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [7]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [8].

We use the definition of fuzzy normed spaces given in [5, 9, 10] to investigate a fuzzy version of the Hyers-Ulam stability for the Cauchy additive functional inequality and for the Cauchy-Jensen additive functional inequality in the fuzzy normed vector space setting.

Definition 1.1 [5, 9–11] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [10, 11].

Definition 1.2 [5, 9–11] Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$

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for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

Definition 1.3 [5, 9, 10] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [8]).

We will use the following notations:

$M_n(X)$ is the set of all $n \times n$-matrices in X;

$e_j \in M_{1,n}(\mathbb{C})$ is that jth component is 1 and the other components are zero;

$E_{ij} \in M_n(\mathbb{C})$ is that (i,j)-component is 1 and the other components are zero;

$E_{ij} \otimes x \in M_n(X)$ is that (i,j)-component is x and the other components are zero.

For $x \in M_n(X)$, $y \in M_k(X)$,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \le \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}(\mathbb{C})$, $x = (x_{ij}) \in M_n(X)$ and $B \in M_{n,k}(\mathbb{C})$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

A matrix normed space $(X, \{\|\cdot\|_n\})$ is called an L^∞-*matrix normed space* if $\|x \oplus y\|_{n+k} = \max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

We introduce the concept of a matrix fuzzy normed space.

Definition 1.4 Let (X, N) be a fuzzy normed space.

(1) $(X, \{N_n\})$ is called a *matrix fuzzy normed space* if for each positive integer n, $(M_n(X), N_n)$ is a fuzzy normed space and $N_k(AxB, t) \ge N_n(x, \frac{t}{\|A\|\cdot\|B\|})$ for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\| \cdot \|B\| \ne 0$.

(2) $(X, \{N_n\})$ is called a *matrix fuzzy Banach space* if (X, N) is a fuzzy Banach space and $(X, \{N_n\})$ is a matrix fuzzy normed space.

Example 1.5 Let $(X, \{\|\cdot\|_n\})$ be a matrix normed space. Let $N_n(x,t) := \frac{t}{t+\|x\|_n}$ for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$. Then

$$N_k(AxB,t) = \frac{t}{t + \|AxB\|_k} \geq \frac{t}{t + \|A\| \cdot \|x\|_n \cdot \|B\|} = \frac{\frac{t}{\|A\|\cdot\|B\|}}{\frac{t}{\|A\|\cdot\|B\|} + \|x\|_n}$$

for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\| \cdot \|B\| \neq 0$. So, $(X, \{N_n\})$ is a matrix fuzzy normed space.

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [12] implies that quotients, mapping spaces, and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory (see [13]).

The proof given in [12] appealed to the theory of ordered operator spaces [14]. Effros and Ruan [15] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [16] and Haagerup [17] (as modified in [18]).

The stability problem of functional equations originated from a question of Ulam [19] concerning the stability of group homomorphisms. Hyers [20] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [21] for additive mappings and by Rassias [22] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [23] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In [24], Gilányi showed that if f satisfies the functional inequality

$$\left\|2f(x) + 2f(y) - f\left(xy^{-1}\right)\right\| \leq \left\|f(xy)\right\|, \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f\left(xy^{-1}\right).$$

See also [25]. Gilányi [26] and Fechner [27] proved the Hyers-Ulam stability of the functional inequality (1.1).

Park *et al.* [28] proved the Hyers-Ulam stability of the following functional inequalities:

$$\left\|f(x) + f(y) + f(z)\right\| \leq \left\|2f\left(\frac{x+y+z}{2}\right)\right\|,$$

$$\left\|f(x) + f(y) + f(z)\right\| \leq \left\|f(x+y+z)\right\|, \tag{1.2}$$

$$\left\|f(x) + f(y) + 2f(z)\right\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\|. \tag{1.3}$$

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.6 [29, 30] *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with a Lipschitz constant $\alpha < 1$. Then, for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [31] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [32–38]).

Throughout this paper, let $(X, \|\cdot\|_n)$ be a matrix normed space, $(Y, \|\cdot\|_n)$ be a matrix Banach space and let n be a fixed positive integer. Let (X, N_n) be a matrix fuzzy normed space and let (Y, N_n) be a matrix fuzzy Banach space.

In Section 2, we prove the Hyers-Ulam stability of the Cauchy additive functional inequality (1.2) in fuzzy normed spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the Cauchy additive functional equation in matrix fuzzy normed spaces by using the fixed point method.

In Section 4, we prove the Hyers-Ulam stability of the Cauchy-Jensen additive functional inequality (1.3) in fuzzy normed spaces by using the fixed point method.

In Section 5, we prove the Hyers-Ulam stability of the Cauchy additive functional inequality (1.2) in matrix normed spaces by using the direct method and by using the fixed point method.

2 Hyers-Ulam stability of the Cauchy functional inequality in fuzzy normed spaces

We need the following lemma to prove the main results.

Lemma 2.1 [16, 39] *Let (Y, N) be a fuzzy normed vector space. Let $f : X \to Y$ be a mapping such that*

$$N\left(f(x) + f(y) + 2f(z), t\right) \geq N\left(2f\left(\frac{x+y}{2} + z\right), \frac{2t}{3}\right)$$

for all $x, y, z \in X$ and all $t > 0$. Then f is Cauchy additive, i.e., $f(x + y) = f(x) + f(y)$ for all $x, y \in X$.

In this section, using the fixed point method, we prove the Hyers-Ulam stability of the Cauchy additive functional inequality (1.2) in fuzzy Banach spaces.

Theorem 2.2 *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \leq \frac{L}{2} \varphi(2x, 2y, 2z)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$N\big(f(x) + f(y) + f(z), t\big) \geq \min\left\{N\left(f(x + y + z), \frac{t}{2}\right), \frac{t}{t + \varphi(x, y, z)}\right\} \tag{2.1}$$

for all $x, y, z \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\big(f(x) - A(x), t\big) \geq \frac{(2 - 2L)t}{(2 - 2L)t + L\varphi(x, x, -2x)} \tag{2.2}$$

for all $x \in X$ and all $t > 0$.

Proof Since f is odd, $f(0) = 0$. So, $N(f(0), \frac{t}{2}) = 1$. Letting $y = x$ and replacing z by $-2x$ in (2.1), we get

$$N\big(f(2x) - 2f(x), t\big) \geq \frac{t}{t + \varphi(x, x, -2x)} \tag{2.3}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : N\big(g(x) - h(x), \mu t\big) \geq \frac{t}{t + \varphi(x, x, -2x)}, \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete. (See the proof of [40, Lemma 2.1].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N\big(g(x) - h(x), \varepsilon t\big) \geq \frac{t}{t + \varphi(x, x, -2x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N\big(Jg(x) - Jh(x), L\varepsilon t\big) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \end{aligned}$$

$$\begin{aligned}
&\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi(\frac{x}{2}, \frac{x}{2}, -x)} \\
&\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x, x, -2x)} \\
&= \frac{t}{t + \varphi(x, x, -2x)}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So, $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.3) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t + \varphi(x, x, -2x)}$$

for all $x \in X$ and all $t > 0$. So, $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.6, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, *i.e.*,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.4}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.4) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N\big(f(x) - A(x), \mu t\big) \geq \frac{t}{t + \varphi(x, x, -2x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (2.2) holds.

By (2.1),

$$\begin{aligned}
&N\left(2^n\left(f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)+f\left(\frac{z}{2^n}\right)\right),2^nt\right)\\
&\quad\geq \min\left\{N\left(2^nf\left(\frac{x+y+z}{2^n}\right),2^{n-1}t\right),\frac{t}{t+\varphi(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n})}\right\}
\end{aligned}$$

for all $x,y,z\in X$, all $t>0$, and all $n\in\mathbb{N}$. So,

$$\begin{aligned}
&N\left(2^n\left(f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)+f\left(\frac{z}{2^n}\right)\right),t\right)\\
&\quad\geq \min\left\{N\left(2^nf\left(\frac{x+y+z}{2^n}\right),\frac{t}{2}\right),\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y,z)}\right\}
\end{aligned}$$

for all $x,y,z\in X$, all $t>0$, and all $n\in\mathbb{N}$. Since $\lim_{n\to\infty}\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y,z)}=1$ for all $x,y,z\in X$ and all $t>0$,

$$N\big(A(x)+A(y)+A(z),t\big)\geq N\left(A(x+y+z),\frac{t}{2}\right)$$

for all $x,y,z\in X$ and all $t>0$. By [41, Lemma 2.1], the mapping $A:X\to Y$ is a Cauchy additive, as desired. □

Corollary 2.3 *Let $\theta\geq 0$ and let p be a real number with $p>1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f:X\to Y$ be an odd mapping satisfying*

$$N\big(f(x)+f(y)+f(z),t\big)\geq \min\left\{N\left(f(x+y+z),\frac{t}{2}\right),\frac{t}{t+\theta(\|x\|^p+\|y\|^p+\|z\|^p)}\right\}\tag{2.5}$$

for all $x,y,z\in X$ and all $t>0$. Then $A(x):=N\text{-}\lim_{n\to\infty}2^nf(\frac{x}{2^n})$ exists for each $x\in X$ and defines an additive mapping $A:X\to Y$ such that

$$N\big(f(x)-A(x),t\big)\geq\frac{(2^p-2)t}{(2^p-2)t+(2+2^p)\theta\|x\|^p}$$

for all $x\in X$ and all $t>0$.

Proof The proof follows from Theorem 2.2 by taking

$$\varphi(x,y):=\theta\big(\|x\|^p+\|y\|^p+\|z\|^p\big)$$

for all $x,y,z\in X$. Then we can choose $L=2^{1-p}$, and we get the desired result. □

Theorem 2.4 *Let $\varphi:X^3\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y,z)\leq 2L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.1). *Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\big(f(x) - A(x), t\big) \geq \frac{(2-2L)t}{(2-2L)t + \varphi(x, x, -2x)} \tag{2.6}$$

for all $x \in X$ and all $t > 0$.

Proof Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N\big(g(x) - h(x), \varepsilon t\big) \geq \frac{t}{t + \varphi(x, x, -2x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N\big(Jg(x) - Jh(x), L\varepsilon t\big) &= N\left(\frac{1}{2} g(2x) - \frac{1}{2} h(2x), L\varepsilon t\right) \\ &= N\big(g(2x) - h(2x), 2L\varepsilon t\big) \\ &\geq \frac{2Lt}{2Lt + \varphi(2x, 2x, -4x)} \geq \frac{2Lt}{2Lt + 2L\varphi(x, x, -2x)} \\ &= \frac{t}{t + \varphi(x, x, -2x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So, $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq L d(g, h)$$

for all $g, h \in S$.

It follows from (2.3) that

$$N\left(f(x) - \frac{1}{2} f(2x), \frac{1}{2} t\right) \geq \frac{t}{t + \varphi(x, x, -2x)}$$

for all $x \in X$ and all $t > 0$. So, $d(f, Jf) \leq \frac{1}{2}$.

By Theorem 1.6, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, *i.e.*,

$$A(2x) = 2A(x) \tag{2.7}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \big\{g \in S : d(f, g) < \infty\big\}.$$

This implies that A is a unique mapping satisfying (2.7) such that there exists a $\mu \in (0,\infty)$ satisfying

$$N\big(f(x) - A(x), \mu t\big) \geq \frac{t}{t + \varphi(x, x, -2x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f\big(2^n x\big) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{1}{2 - 2L}.$$

This implies that the inequality (2.6) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5 *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (2.5). *Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\big(f(x) - A(x), t\big) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + (2 + 2^p)\theta \|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof The proof follows from Theorem 2.4 by taking

$$\varphi(x, y) := \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$$

for all $x, y, z \in X$. Then we can choose $L = 2^{p-1}$, and we get the desired result. □

3 Hyers-Ulam stability of the Cauchy additive functional equation in matrix fuzzy normed spaces

Using a fixed point method, we prove the Hyers-Ulam stability of the Cauchy additive functional equation in matrix fuzzy normed spaces.

We will use the following notations:

$M_n(X)$ is the set of all $n \times n$-matrices in X;

$e_j \in M_{1,n}(\mathbb{R})$ is that jth component is 1 and the other components are zero;

$E_{ij} \in M_n(\mathbb{R})$ is that (i, j)-component is 1 and the other components are zero;

$E_{ij} \otimes x \in M_n(X)$ is that (i, j)-component is x and the other components are zero.

Lemma 3.1 *Let $(X, \{N_n\})$ be a matrix fuzzy normed space.*

(1) *$N_n(E_{kl} \otimes x, t) = N(x, t)$ for all $t > 0$ and $x \in X$.*

(2) *For all* $[x_{ij}] \in M_n(X)$ *and* $t = \sum_{i,j=1}^{n} t_{ij}$,

$$N(x_{kl}, t) \geq N_n\big([x_{ij}], t\big) \geq \min\big\{N(x_{ij}, t_{ij}) : i,j = 1,2,\ldots,n\big\},$$
$$N(x_{kl}, t) \geq N_n\big([x_{ij}], t\big) \geq \min\left\{N\left(x_{ij}, \frac{t}{n^2}\right) : i,j = 1,2,\ldots,n\right\}.$$

(3) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} x_{ijn} = x_{ij}$ *for* $x_n = [x_{ijn}]$, $x = [x_{ij}] \in M_k(X)$.

Proof (1) Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1$, $N_n(E_{kl} \otimes x, t) \geq N(x,t)$. Since $e_k(E_{kl} \otimes x)e_l^* = x$, $N_n(E_{kl} \otimes x, t) \leq N(x,t)$. So, $N(E_{kl} \otimes x, t) = N(x,t)$.

(2) $N(x_{kl}, t) = N(e_k[x_{ij}]e_l^*, t) \geq N_n([x_{ij}], \frac{t}{\|e_k\|\cdot\|e_l\|}) = N_n([x_{ij}], t)$.

$$\begin{aligned} N_n\big([x_{ij}], t\big) &= N_n\left(\sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}, t\right) \geq \min\big\{N_n(E_{ij} \otimes x_{ij}, t_{ij}) : i,j = 1,2,\ldots,n\big\} \\ &= \min\big\{N(x_{ij}, t_{ij}) : i,j = 1,2,\ldots,n\big\}, \end{aligned}$$

where $t = \sum_{i,j=1}^{n} t_{ij}$. So, $N_n([x_{ij}], t) \geq \min\{N(x_{ij}, \frac{t}{n^2}) : i,j = 1,2,\ldots,n\}$.

(3) By $N(x_{kl}, t) \geq N_n([x_{ij}], t) \geq \min\{N(x_{ij}, \frac{t}{n^2}) : i,j = 1,2,\ldots,n\}$, we obtain the result. □

For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and $Df_n : M_n(X^2) \to M_n(Y)$ by

$$Df(a,b) = f(a+b) - f(a) - f(b),$$
$$Df_n\big([x_{ij}], [y_{ij}]\big) := f_n\big([x_{ij} + y_{ij}]\big) - f_n\big([x_{ij}]\big) - f_n\big([y_{ij}]\big)$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 3.2 *Let* $\varphi : X^2 \to [0,\infty)$ *be a function such that there exists an* $\alpha < 1$ *with*

$$\varphi(a,b) \leq \frac{\alpha}{2}\varphi(2a, 2b) \tag{3.1}$$

for all $a, b \in X$. *Let* $f : X \to Y$ *be a mapping satisfying*

$$N_n\big(Df_n\big([x_{ij}], [y_{ij}]\big), t\big) \geq \frac{t}{t + \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij})} \tag{3.2}$$

for all $t > 0$ *and* $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. *Then* $A(a) := N\text{-}\lim_{l\to\infty} 2^l f(\frac{a}{2^l})$ *exists for each* $a \in X$ *and defines an additive mapping* $A : X \to Y$ *such that*

$$N\big(f_n\big([x_{ij}]\big) - A_n\big([x_{ij}]\big), t\big) \geq \frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\alpha \sum_{i,j=1}^{n} \varphi(x_{ij}, x_{ij})} \tag{3.3}$$

for all $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$.

Proof Let $n = 1$. Then (3.2) is equivalent to

$$N\big(f(a+b) - f(a) - f(b), t\big) \geq \frac{t}{t + \varphi(a,b)} \tag{3.4}$$

for all $t > 0$ and $a, b \in X$.

Letting $b = a$ in (3.4), we get

$$N\big(f(2a) - 2f(a), t\big) \geq \frac{t}{t + \varphi(a,a)} \tag{3.5}$$

and so

$$N\left(f(a) - 2f\left(\frac{a}{2}\right), t\right) \geq \frac{t}{t + \varphi(\frac{a}{2}, \frac{a}{2})} \geq \frac{t}{t + \frac{\alpha}{2}\varphi(a,a)} \tag{3.6}$$

for all $t > 0$ and $a \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N\big(g(a) - h(a), \mu t\big) \geq \frac{t}{t + \varphi(a,a)}, \forall a \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see the proof of [40, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 2g\left(\frac{a}{2}\right)$$

for all $a \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N\big(g(a) - h(a), \varepsilon t\big) \geq \frac{t}{t + \varphi(a,a)}$$

for all $a \in X$ and $t > 0$. Hence

$$\begin{aligned} N\big(Jg(a) - Jh(a), \alpha\varepsilon t\big) &= N\left(2g\left(\frac{a}{2}\right) - 2h\left(\frac{a}{2}\right), \alpha\varepsilon t\right) \\ &= N\left(g\left(\frac{a}{2}\right) - h\left(\frac{a}{2}\right), \frac{\alpha}{2}\varepsilon t\right) \\ &\geq \frac{\frac{\alpha t}{2}}{\frac{\alpha t}{2} + \varphi(\frac{a}{2}, \frac{a}{2})} \geq \frac{\frac{\alpha t}{2}}{\frac{\alpha t}{2} + \frac{\alpha}{2}\varphi(a,a)} = \frac{t}{t + \varphi(a,a)} \end{aligned}$$

for all $a \in X$ and $t > 0$. So, $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that

$$d(Jg, Jh) \leq \alpha d(g,h)$$

for all $g, h \in S$.

It follows from (3.6) that $d(f, Jf) \leq \frac{\alpha}{2}$.

By Theorem 1.6, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, *i.e.*,

$$A\left(\frac{a}{2}\right) = \frac{1}{2}A(a)$$

for all $a \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$N\text{-}\lim_{l\to\infty} 2^l f\left(\frac{a}{2^l}\right) = A(a)$$

for all $a \in X$.

(3) $d(f,A) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f,A) \leq \frac{\alpha}{2-2\alpha}. \tag{3.7}$$

By (3.4),

$$N\left(2^l f\left(\frac{a+b}{2^l}\right) - 2^l f\left(\frac{a}{2^l}\right) - 2^l f\left(\frac{b}{2^l}\right), 2^l t\right) \geq \frac{t}{t + \varphi(\frac{a}{2^l}, \frac{b}{2^l})}$$

for all $a, b \in X$ and $t > 0$. So,

$$N\left(2^l f\left(\frac{a+b}{2^l}\right) - 2^l f\left(\frac{a}{2^l}\right) - 2^l f\left(\frac{b}{2^l}\right), t\right) \geq \frac{\frac{t}{2^l}}{\frac{t}{2^l} + \frac{\alpha^l}{2^l}\varphi(a,b)}$$

for all $a, b \in X$ and $t > 0$. Since $\lim_{l\to\infty} \frac{\frac{t}{2^l}}{\frac{t}{2^l} + \frac{\alpha^l}{2^l}\varphi(a,b)} = 1$ for all $a, b \in X$ and $t > 0$,

$$N\big(A(a+b) - A(a) - A(b), t\big) = 1$$

for all $a, b \in X$ and $t > 0$. Thus $A(a+b) - A(a) - A(b) = 0$. So, the mapping $A : X \to Y$ is additive.

By Lemma 3.1 and (3.7),

$$\begin{aligned} N_n\big(f_n([x_{ij}]) - A_n([x_{ij}]), t\big) &\geq \min\left\{N\left(f(x_{ij}) - A(x_{ij}), \frac{t}{n^2}\right) : i,j = 1,2,\ldots,n\right\} \\ &\geq \min\left\{\frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\alpha\varphi(x_{ij}, x_{ij})} : i,j = 1,2,\ldots,n\right\} \\ &\geq \frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\alpha\sum_{i,j=1}^n \varphi(x_{ij}, x_{ij})} \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $A : X \to Y$ is a unique additive mapping satisfying (3.3), as desired. □

Corollary 3.3 *Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping satisfying*

$$N_n\big(Df_n([x_{ij}],[y_{ij}]),t\big) \geq \frac{t}{t + \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r)} \tag{3.8}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then $A(a) := N\text{-}\lim_{l\to\infty} 2^l f(\frac{a}{2^l})$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\big(f_n([x_{ij}]) - A_n([x_{ij}]),t\big) \geq \frac{(2-2^r)t}{(2-2^r)t + n^2 \cdot 2^r \sum_{i,j=1}^{n} \theta\|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 3.2 by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a,b \in X$. Then we can choose $\alpha = 2^{r-1}$, and we get the desired result. □

Theorem 3.4 *Let $f : X \to Y$ be a mapping satisfying* (3.2) *for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \leq 2\alpha\varphi\left(\frac{a}{2},\frac{b}{2}\right)$$

for all $a,b \in X$. Then $A(a) := N\text{-}\lim_{l\to\infty} \frac{1}{2^l} f(2^l a)$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\big(f_n([x_{ij}]) - A_n([x_{ij}]),t\big) \geq \frac{2(1-\alpha)t}{2(1-\alpha)t + n^2 \sum_{i,j=1}^{n} \varphi(x_{ij},x_{ij})}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof Let (S,d) be the generalized metric space defined in the proof of Theorem 3.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 2g\left(\frac{a}{2}\right)$$

for all $a \in X$.

It follows from (3.5) that $d(f,Jf) \leq \frac{1}{2}$. So,

$$d(f,A) \leq \frac{1}{2-2\alpha}.$$

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5 *Let r, θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be a mapping satisfying* (3.8)*. Then $A(a) := N\text{-}\lim_{l\to\infty} 2^l f(\frac{a}{2^l})$ exists for each $a \in X$ and defines an additive* *mapping $A : X \to Y$ such that*

$$N\big(f_n([x_{ij}]) - A_n([x_{ij}]), t\big) \geq \frac{(2^r - 2)t}{(2^r - 2)t + n^2 \cdot 2^r \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 3.4 by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{1-r}$, and we get the desired result. □

4 Fuzzy stability of the Cauchy-Jensen additive functional inequality (1.3) in fuzzy normed spaces

In this section, using the fixed point method, we prove the generalized Hyers-Ulam stability of the Cauchy-Jensen additive functional inequality (1.3) in fuzzy Banach spaces.

Theorem 4.1 *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z) \leq \frac{L}{2}\varphi(2x,2y,2z)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$N\big(f(x) + f(y) + f(2z), t\big) \geq \min\left\{N\left(2f\left(\frac{x+y}{2} + z\right), \frac{2t}{3}\right), \frac{t}{t + \varphi(x,y,z)}\right\} \tag{4.1}$$

for all $x, y, z \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\big(f(x) - A(x), t\big) \geq \frac{(2 - 2L)t}{(2 - 2L)t + L\varphi(x,x,-x)} \tag{4.2}$$

for all $x \in X$ and all $t > 0$.

Proof Letting $y = x = -z$ in (4.1), we get

$$N\big(f(2x) - 2f(x), t\big) \geq \frac{t}{t + \varphi(x,x,-x)} \tag{4.3}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N\big(g(x) - h(x), \mu t\big) \geq \frac{t}{t + \varphi(x,x,-x)}, \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete. (See the proof of [40, Lemma 2.1].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N\big(g(x) - h(x), \varepsilon t\big) \geq \frac{t}{t + \varphi(x, x, -x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N\big(Jg(x) - Jh(x), L\varepsilon t\big) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2})} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x, x, -x)} \\ &= \frac{t}{t + \varphi(x, x, -x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So, $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (4.3) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t + \varphi(x, x, -x)}$$

for all $x \in X$ and all $t > 0$. So, $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.6, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, *i.e.*,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{4.4}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (4.4) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N\big(f(x) - A(x), \mu t\big) \geq \frac{t}{t + \varphi(x, x, -x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (4.2) holds.

The rest of proof is similar to the proof of Theorem 2.2. □

Corollary 4.2 *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned} &N\big(f(x) + f(y) + f(2z), t\big) \\ &\quad \geq \min\left\{N\left(f\left(\frac{x+y}{2} + z\right), \frac{2t}{3}\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}\right\} \end{aligned} \tag{4.5}$$

for all $x, y, z \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\big(f(x) - A(x), t\big) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 3\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof The proof follows from Theorem 4.1 by taking

$$\varphi(x, y) := \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$$

for all $x, y, z \in X$. Then we can choose $L = 2^{1-p}$, and we get the desired result. □

Theorem 4.3 *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping satisfying (4.1). *Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\big(f(x) - A(x), t\big) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \varphi(x, x, -x)} \tag{4.6}$$

for all $x \in X$ and all $t > 0$.

Proof Let (S,d) be the generalized metric space defined in the proof of Theorem 4.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

Let $g,h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N\bigl(g(x) - h(x), \varepsilon t\bigr) \geq \frac{t}{t + \varphi(x,x,-x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N\bigl(Jg(x) - Jh(x), L\varepsilon t\bigr) &= N\left(\frac{1}{2}g(2x) - \frac{1}{2}h(2x), L\varepsilon t\right) \\ &= N\bigl(g(2x) - h(2x), 2L\varepsilon t\bigr) \\ &\geq \frac{2Lt}{2Lt + \varphi(2x,2x,-2x)} \geq \frac{2Lt}{2Lt + 2L\varphi(x,x,-x)} \\ &= \frac{t}{t + \varphi(x,x,-x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So, $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leq L\varepsilon$. This means that

$$d(Jg,Jh) \leq Ld(g,h)$$

for all $g,h \in S$.

It follows from (4.3) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x,x,-x)}$$

for all $x \in X$ and all $t > 0$. So, $d(f,Jf) \leq \frac{1}{2}$.

By Theorem 1.6, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, *i.e.*,

$$A(2x) = 2A(x) \tag{4.7}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \bigl\{g \in S : d(f,g) < \infty\bigr\}.$$

This implies that A is a unique mapping satisfying (4.7) such that there exists a $\mu \in (0,\infty)$ satisfying

$$N\bigl(f(x) - A(x), \mu t\bigr) \geq \frac{t}{t + \varphi(x,x,-x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{1}{2 - 2L}.$$

This implies that the inequality (4.6) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 4.4 *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (4.5). *Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 3\theta \|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof The proof follows from Theorem 4.3 by taking

$$\varphi(x, y) := \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right)$$

for all $x, y, z \in X$. Then we can choose $L = 2^{p-1}$, and we get the desired result. □

5 Hyers-Ulam stability of the additive functional inequality (1.2) in matrix normed spaces

In this section, we prove the Hyers-Ulam stability of the additive functional inequality (1.2) in matrix normed spaces by using the direct method and by using the fixed point method.

Lemma 5.1 *Let $(X, \{\|\cdot\|_n\})$ be a matrix normed space.*

(1) $\|E_{kl} \otimes x\|_n = \|x\|$ *for* $x \in X$;

(2) $\|x_{kl}\| \leq \|[x_{ij}]\|_n \leq \sum_{i,j=1}^{n} \|x_{ij}\|$ *for* $[x_{ij}] \in M_n(X)$;

(3) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} x_{nij} = x_{ij}$ *for* $x_n = [x_{nij}], x = [x_{ij}] \in M_k(X)$.

Proof (1) Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1$, $\|E_{kl} \otimes x\|_n \leq \|x\|$. Since $e_k(E_{kl} \otimes x)e_l^* = x$, $\|x\| \leq \|E_{kl} \otimes x\|_n$. So, $\|E_{kl} \otimes x\|_n = \|x\|$.

(2) Since $e_k x e_l^* = x_{kl}$ and $\|e_k\| = \|e_l^*\| = 1$, $\|x_{kl}\| \leq \|[x_{ij}]\|_n$.

Since $[x_{ij}] = \sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}$,

$$\left\|[x_{ij}]\right\|_n = \left\|\sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}\right\|_n \leq \sum_{i,j=1}^{n} \|E_{ij} \otimes x_{ij}\|_n = \sum_{i,j=1}^{n} \|x_{ij}\|.$$

(3) By (2), we have

$$\|x_{nkl} - x_{kl}\| \le \left\| [x_{nij} - x_{ij}] \right\|_n = \left\| [x_{nij}] - [x_{ij}] \right\|_n \le \sum_{i,j=1}^{n} \|x_{nij} - x_{ij}\|.$$

So, we get the result. □

We need the following result.

Lemma 5.2 [28, Proposition 2.2] *Let $f : X \to Y$ be a mapping such that*

$$\|f(a) + f(b) + f(c)\| \le \|f(a + b + c)\|$$

for all $a, b, c \in X$. Then $f : X \to Y$ is additive.

Theorem 5.3 *Let $f : X \to Y$ be a mapping and let $\phi : X^3 \to [0, \infty)$ be a function such that*

$$\Phi(a, b, c) := \frac{1}{2} \sum_{l=0}^{\infty} \frac{1}{2^l} \phi\left(2^l a, 2^l b, 2^l c\right) < +\infty, \tag{5.1}$$

$$\begin{aligned} &\left\|f_n\left([x_{ij}]\right) + f_n\left([y_{ij}]\right) + f_n\left([z_{ij}]\right)\right\|_n \\ &\quad \le \left\|f_n\left([x_{ij}] + [y_{ij}] + [z_{ij}]\right)\right\|_n + \sum_{i,j=1}^{n} \phi(x_{ij}, y_{ij}, z_{ij}) \end{aligned} \tag{5.2}$$

for all $a, b, c \in X$ and all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\left\|f_n\left([x_{ij}]\right) - A_n\left([x_{ij}]\right)\right\|_n \le \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij}, -2x_{ij}) \tag{5.3}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof When $n = 1$, (5.2) is equivalent to

$$\|f(a) + f(b) + f(c)\| \le \|f(a + b + c)\| + \phi(a, b, c)$$

for all $a, b, c \in X$. By the same reasoning as in the proof of [28, Theorem 3.2], one can show that there is a unique additive mapping $A : X \to Y$ such that

$$\|f(a) - A(a)\| \le \Phi(a, a, -2a)$$

for all $a \in X$. The mapping $A : X \to Y$ is given by

$$A(a) = \lim_{l \to \infty} \frac{1}{2^l} f\left(2^l a\right)$$

for all $a \in X$. By Lemma 5.1,

$$\left\|f_n([x_{ij}]) - A_n([x_{ij}])\right\|_n \le \sum_{i,j=1}^{n} \left\|f(x_{ij}) - A(x_{ij})\right\| \le \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij}, -2x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $A : X \to Y$ is a unique additive mapping satisfying (5.3), as desired. □

Corollary 5.4 *Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}\left\|f_n([x_{ij}]) + f_n([y_{ij}]) + f_n([z_{ij}])\right\|_n &\le \left\|f_n([x_{ij}] + [y_{ij}] + [z_{ij}])\right\|_n \\ &\quad + \sum_{i,j=1}^{n} \theta\left(\|x_{ij}\|^r + \|y_{ij}\|^r + \|z_{ij}\|^r\right)\end{aligned} \tag{5.4}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\left\|f_n([x_{ij}]) - A_n([x_{ij}])\right\|_n \le \sum_{i,j=1}^{n} \frac{2 + 2^r}{2 - 2^r} \theta \|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof Letting $\phi(a, b, c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ in Theorem 5.3, we obtain the result. □

Theorem 5.5 *Let $f : X \to Y$ be a mapping and let $\phi : X^3 \to [0, \infty)$ be a function satisfying* (5.2) *and*

$$\Phi(a, b, c) := \frac{1}{2} \sum_{l=1}^{\infty} 2^l \phi\left(\frac{a}{2^l}, \frac{b}{2^l}, \frac{c}{2^l}\right) < +\infty, \tag{5.5}$$

for all $a, b, c \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\left\|f_n([x_{ij}]) - A_n([x_{ij}])\right\|_n \le \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij}, -2x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof The proof is similar to the proof of Theorem 5.3. □

Corollary 5.6 *Let r, θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be a mapping satisfying* (5.4). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\left\|f_n([x_{ij}]) - A_n([x_{ij}])\right\|_n \le \sum_{i,j=1}^{n} \frac{2^r + 2}{2^r - 2} \theta \|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof Letting $\phi(a,b,c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ in Theorem 5.5, we obtain the result. □

We need the following result.

Lemma 5.7 [42] *If E is an L^∞-matrix normed space, then $\|[x_{ij}]\|_n \leq \|[\|x_{ij}\|]\|_n$ for all $[x_{ij}] \in M_n(E)$.*

Theorem 5.8 *Let Y be an L^∞-normed Banach space. Let $f : X \to Y$ be a mapping and let $\phi : X^3 \to [0,\infty)$ be a function satisfying* (5.1) *and*

$$\|f_n([x_{ij}]) + f_n([y_{ij}]) + f_n([z_{ij}])\|_n \leq \|f_n([x_{ij}] + [y_{ij}] + [z_{ij}])\|_n + \|[\phi(x_{ij}, y_{ij}, z_{ij})]\|_n \tag{5.6}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|[f(x_{ij}) - A(x_{ij})]\|_n \leq \|[\Phi(x_{ij}, x_{ij}, -2x_{ij})]\|_n \tag{5.7}$$

for all $x = [x_{ij}] \in M_n(X)$. Here Φ is given in Theorem 5.3.

Proof By the same reasoning as in the proof of Theorem 5.3, there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(a) - A(a)\| \leq \Phi(a, a, -2a)$$

for all $a \in X$. The mapping $A : X \to Y$ is given by

$$A(a) = \lim_{l\to\infty} \frac{1}{2^l} f(2^l a)$$

for all $a \in X$.

It is easy to show that if $0 \leq a_{ij} \leq b_{ij}$ for all i, j, then

$$\|[a_{ij}]\|_n \leq \|[b_{ij}]\|_n. \tag{5.8}$$

By Lemma 5.7 and (5.8),

$$\|[f(x_{ij}) - A(x_{ij})]\|_n \leq \|[\|f(x_{ij}) - A(x_{ij})\|]\|_n \leq \|[\Phi(x_{ij}, x_{ij}, -2x_{ij})]\|_n$$

for all $x = [x_{ij}] \in M_n(X)$. So, we obtain the inequality (5.7). □

Corollary 5.9 *Let Y be an L^∞-normed Banach space. Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}\|f_n([x_{ij}]) + f_n([y_{ij}]) + f_n([z_{ij}])\|_n &\leq \|f_n([x_{ij}] + [y_{ij}] + [z_{ij}])\|_n \\ &\quad + \|[\theta(\|x_{ij}\|^r + \|y_{ij}\|^r + \|z_{ij}\|^r)]\|_n \end{aligned} \tag{5.9}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$*. Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\left\| f_n([x_{ij}]) - A_n([x_{ij}]) \right\|_n \leq \left\| \left[\frac{2-2^r}{2-2^r} \theta \|x_{ij}\|^r \right] \right\|_n$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof Letting $\phi(a,b,c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ in Theorem 5.8, we obtain the result. □

Theorem 5.10 *Let* Y *be an* L^∞*-normed Banach space. Let* $f : X \to Y$ *be a mapping and let* $\phi : X^3 \to [0,\infty)$ *be a function satisfying* (5.5) *and* (5.6)*. Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\left\| \left[f(x_{ij}) - A(x_{ij}) \right] \right\|_n \leq \left\| \left[\Phi(x_{ij}, x_{ij}, -2x_{ij}) \right] \right\|_n$$

for all $x = [x_{ij}] \in M_n(X)$*. Here* Φ *is given in Theorem* 5.5.

Proof The proof is similar to the proof of Theorem 5.8. □

Corollary 5.11 *Let* Y *be an* L^∞*-normed Banach space. Let* r, θ *be positive real numbers with* $r > 1$*. Let* $f : X \to Y$ *be a mapping satisfying* (5.9)*. Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\left\| f_n([x_{ij}]) - A_n([x_{ij}]) \right\|_n \leq \left\| \left[\frac{2^r+2}{2^r-2} \theta \|x_{ij}\|^r \right] \right\|_n$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof Letting $\phi(a,b,c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ in Theorem 5.10, we obtain the result. □

Theorem 5.12 *Let* $\phi : X^3 \to [0,\infty)$ *be a function such that there exists an* $\alpha < 1$ *with*

$$\phi(a,b,c) \leq 2\alpha\phi\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right) \tag{5.10}$$

for all $a, b, c \in X$*. Let* $f : X \to Y$ *be a mapping satisfying* (5.2)*. Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\left\| f_n([x_{ij}]) - A_n([x_{ij}]) \right\|_n \leq \sum_{i,j=1}^{n} \frac{1}{2-2\alpha} \phi(x_{ij}, x_{ij}, -2x_{ij}) \tag{5.11}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof When $n = 1$, (5.2) is equivalent to

$$\left\| f(a) + f(b) + f(c) \right\| \leq \left\| f(a+b+c) \right\| + \phi(a,b,c) \tag{5.12}$$

for all $a, b, c \in X$. It follows from (5.12) that

$$\|2f(a) - f(2a)\| \leq \phi(a, a, -2a) \tag{5.13}$$

for all $a \in X$. So,

$$\left\| f(a) - \frac{1}{2} f(2a) \right\| \leq \frac{1}{2} \phi(a, a, -2a) \tag{5.14}$$

for all $a \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(a) - h(a)\| \leq \mu\phi(a, a, -2a), \forall a \in X\},$$

where, as usual, $\inf\{\} = +\infty$. It is easy to show that (S, d) is complete (see [40]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := \frac{1}{2} g(2a)$$

for all $a \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(a) - h(a)\| \leq \phi(a, a, -2a)$$

for all $a \in X$. Hence

$$\|Jg(a) - Jh(a)\| = \left\| \frac{1}{2} g(2a) - \frac{1}{2} h(2a) \right\| \leq \alpha\phi(a, a, -2a)$$

for all $a \in X$. So, $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that

$$d(Jg, Jh) \leq \alpha d(g, h)$$

for all $g, h \in S$.

It follows from (5.14) that $d(f, Jf) \leq \frac{1}{2}$.

By Theorem 1.6, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, *i.e.*,

$$A(2a) = 2A(x) \tag{5.15}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(h, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (5.15) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(a) - A(a)\| \le \mu\phi(a,a,-2a)$$

for all $a \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} \frac{1}{2^l} f(2^l a) = A(a)$$

for all $a \in X$;

(3) $d(f,A) \le \frac{1}{1-\alpha} d(f,Jf)$, which implies the inequality

$$d(f,A) \le \frac{1}{2-2\alpha}.$$

So,

$$\|f(a) - A(a)\| \le \frac{1}{2-2\alpha}\phi(a,a,-2a) \tag{5.16}$$

for all $a \in X$.

It follows from (5.10) and (5.12) that

$$\begin{aligned} &\lim_{l\to\infty} \frac{1}{2^l} \|f(2^l a) + f(2^l b) + f(2^l c)\| \\ &\quad \le \lim_{l\to\infty} \left(\frac{1}{2^l} \|f(2^l(a+b+c))\| + \frac{1}{2^l}\phi(2^l a, 2^l b, 2^l c) \right) \end{aligned} \tag{5.17}$$

for all $a,b,c \in X$.

It follows from (5.17) that

$$\|A(a) + A(b) + A(c)\| \le \|A(a+b+c)\|$$

for all $a,b,c \in X$. By Lemma 5.2, $A : X \to Y$ is additive.

By Lemma 5.7 and (5.16),

$$\|f_n([x_{ij}]) - A_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \|f(x_{ij}) - A(x_{ij})\| \le \sum_{i,j=1}^{n} \frac{1}{2-2\alpha}\phi(x_{ij}, x_{ij}, -2x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $A : X \to Y$ is a unique additive mapping satisfying (5.11), as desired. □

Corollary 5.13 *Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping satisfying* (5.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f_n([x_{ij}]) - A_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2+2^r}{2-2^r}\theta\|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 5.12 by taking $\phi(a,b,c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ for all $a,b,c \in X$. Then we can choose $\alpha = 2^{r-1}$, and we get the desired result. □

Theorem 5.14 *Let $f : X \to Y$ be a mapping satisfying* (5.2) *for which there exists a function $\phi : X^3 \to [0,\infty)$ such that there exists an $\alpha < 1$ with*

$$\phi(a,b,c) \le \frac{\alpha}{2}\phi(2a,2b,2c) \tag{5.18}$$

for all $a,b,c \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\left\|f_n([x_{ij}]) - A_n([x_{ij}])\right\|_n \le \sum_{i,j=1}^{n} \frac{\alpha}{2-2\alpha}\phi(x_{ij}, x_{ij}, -2x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof Let (S,d) be the generalized metric space defined in the proof of Theorem 5.12.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 2g\left(\frac{a}{2}\right)$$

for all $a \in X$.

It follows from (5.13) that

$$\left\|f(a) - 2f\left(\frac{a}{2}\right)\right\| \le \phi\left(\frac{a}{2}, \frac{a}{2}, -a\right) \le \frac{\alpha}{2}\phi(a,a,-2a)$$

for all $a \in X$. Thus $d(f, Jf) \le \frac{\alpha}{2}$. So,

$$d(f,A) \le \frac{\alpha}{2-2\alpha}.$$

The rest of the proof is similar to the proof of Theorem 5.12. □

Corollary 5.15 *Let r, θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be a mapping satisfying* (5.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\left\|f_n([x_{ij}]) - A_n([x_{ij}])\right\|_n \le \sum_{i,j=1}^{n} \frac{2^r+2}{2^r-2}\theta\|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 5.14 by taking $\phi(a,b,c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ for all $a,b,c \in X$. Then we can choose $\alpha = 2^{1-r}$, and we get the desired result. □

From now on, assume that Y is an L^∞-normed Banach space.

Theorem 5.16 *Let $f : X \to Y$ be a mapping and let $\phi : X^3 \to [0,\infty)$ be a function satisfying* (5.10) *and* (5.6). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\left\| \left[f(x_{ij}) - A(x_{ij}) \right] \right\|_n \le \left\| \left[\frac{1}{2-2\alpha} \phi(x_{ij}, x_{ij}, -2x_{ij}) \right] \right\|_n \tag{5.19}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof By the same reasoning as in the proof of Theorem 5.12, there exists a unique additive mapping $A : X \to Y$ such that

$$\left\| f(a) - A(a) \right\| \le \frac{1}{2-2\alpha} \phi(a, a, -2a)$$

for all $a \in X$.

By Lemma 5.7 and (5.8),

$$\left\| \left[f(x_{ij}) - A(x_{ij}) \right] \right\|_n \le \left\| \left[\left\| f(x_{ij}) - A(x_{ij}) \right\| \right] \right\|_n \le \left\| \left[\frac{1}{2-2\alpha} \phi(x_{ij}, x_{ij}) \right] \right\|_n$$

for all $x = [x_{ij}] \in M_n(X)$. So, we obtain the inequality (5.19). □

Corollary 5.17 *Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping satisfying* (5.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\left\| f_n([x_{ij}]) - A_n([x_{ij}]) \right\|_n \le \left\| \left[\frac{2-2^r}{2-2^r} \theta \|x_{ij}\|^r \right] \right\|_n$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 5.16 by taking $\phi(a,b,c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ for all $a, b, c \in X$. Then we can choose $\alpha = 2^{r-1}$, and we get the desired result. □

Theorem 5.18 *Let $f : X \to Y$ be a mapping and let $\phi : X^3 \to [0,\infty)$ be a function satisfying* (5.6) *and* (5.18). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\left\| \left[f(x_{ij}) - A(x_{ij}) \right] \right\|_n \le \left\| \left[\frac{\alpha}{2-2\alpha} \phi(x_{ij}, x_{ij}, -2x_{ij}) \right] \right\|_n$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof The proof is similar to the proof of Theorem 5.16. □

Corollary 5.19 *Let r, θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be a mapping satisfying* (5.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\left\| f_n([x_{ij}]) - A_n([x_{ij}]) \right\|_n \le \left\| \left[\frac{2^2+2}{2^r-2} \theta \|x_{ij}\|^r \right] \right\|_n$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 5.18 by taking $\phi(a,b,c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ for all $a, b, c \in X$. Then we can choose $\alpha = 2^{1-r}$, and we get the desired result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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