Radical Structures of Fuzzy Polynomial Ideals in a Ring

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1. Introduction


The present authors [13] introduced the notion of a fuzzy polynomial ideal \( \alpha_x \) of a polynomial ring \( R[x] \) induced by a fuzzy ideal \( \alpha \) of a ring \( R \) and obtained an isomorphism theorem of a ring of fuzzy cosets of \( \alpha_x \). It was shown that a fuzzy ideal \( \alpha \) of a ring is fuzzy prime if and only if \( \alpha_x \) is a fuzzy prime ideal of \( R[x] \). Moreover, we showed that if \( \alpha_x \) is a fuzzy maximal ideal of \( R[x] \), then \( \alpha \) is a fuzzy maximal ideal of \( R \).

In this paper we investigate the radical structure of a fuzzy polynomial ideal induced by a fuzzy ideal of a ring and study their properties.

2. Preliminaries

In this section, we review some definitions which will be used in the later section. Throughout this paper unless stated otherwise all rings are commutative rings with identity.

Definition 1 (see [3]). A fuzzy ideal of a ring \( R \) is a function \( \alpha : R \to [0, 1] \) satisfying the following axioms:

(1) \( \alpha(x + y) \geq \min\{\alpha(x), \alpha(y)\} \).
(2) \( \alpha(xy) \geq \max\{\alpha(x), \alpha(y)\} \).
(3) \( \alpha(-x) = \alpha(x) \) for any \( x, y \in R \).

Definition 2 (see [2]). Let \( f : R \to S \) be a homomorphism of rings and let \( \beta \) be a fuzzy subset of \( S \). We define a fuzzy subset \( f^{-1}\beta \) of \( R \) by \( f^{-1}\beta(x) = \beta(f(x)) \) for all \( x \in R \).

Definition 3 (see [2]). Let \( f : R \to S \) be a homomorphism of rings and let \( \alpha \) be a fuzzy subset of \( R \). We define a fuzzy subset \( f(\alpha) \) of \( S \) by

\[
 f(\alpha)(y) = \begin{cases} 
 \sup \{\alpha(t) \mid t \in R, f(t) = y\} & \text{if } f^{-1}(y) \neq \emptyset, \\
 0 & \text{if } f^{-1}(y) = \emptyset.
\end{cases}
\]
Definition 4 (see [2]). Let $R$ and $S$ be any sets and let $f : R \to S$ be a function. A fuzzy subset $\alpha$ of $R$ is called an $f$-invariant if $f(x) = f(y)$ implies $\alpha(x) = \alpha(y)$, where $x, y \in R$.

Zadeh [1] defined the following notions. The union of two fuzzy subsets $\alpha$ and $\beta$ of a set $S$, denoted by $\alpha \cup \beta$, is a fuzzy subset of $S$ defined by

$$
(\alpha \cup \beta)(x) = \max\{\alpha(x), \beta(x)\}
$$

for all $x \in S$.

The intersection of $\alpha$ and $\beta$, symbolized by $\alpha \cap \beta$, is a fuzzy subset of $S$, defined by

$$
(\alpha \cap \beta)(x) = \min\{\alpha(x), \beta(x)\}
$$

for all $x \in S$.

Theorem 5 (see [13]). Let $\alpha : R \to [0, 1]$ be a fuzzy ideal of a ring $R$ and let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial in $R[x]$. Define a fuzzy set $\alpha(f(x)) : R[x] \to [0, 1]$ by $\alpha(f(x)) = \min\{\alpha(a_i) \mid a_i$’s are coefficients of $f(x)\}$. Then $\alpha(f(x))$ is a fuzzy ideal of $R[x]$.

The fuzzy ideal $\alpha_x$ discussed in Theorem 5 is called the fuzzy polynomial ideal [13] of $R[x]$ induced by a fuzzy ideal $\alpha$.

Theorem 6 (see [13]). Let $\alpha : R \to [0, 1]$ be a fuzzy ideal of a ring $R$. Then $\alpha$ is a fuzzy prime ideal of $R$ if and only if $\alpha_x$ is a fuzzy prime ideal of $R[x]$.

Notation 1. Let $\alpha : R \to [0, 1]$ be a fuzzy subset of a set $R$. We denote a level set $\alpha_x$ by $\alpha_x = \{a \in R \mid \alpha(a) = \alpha(0)\}$, and we know that $\alpha(0) \geq \alpha(x)$ for all $x \in R$. The set of all polynomials $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ whose $\alpha$’s values $\alpha(a_i)$ are equal to $\alpha(0)$ for all $i = 0, 1, \ldots, n$, is denoted by $\alpha_x[x]$.

Theorem 7 (see [13]). If $\alpha$ and $\beta$ are fuzzy ideals of a ring $R$, then

(i) $(\alpha \cap \beta)_x = \alpha_x \cap \beta_x$,

(ii) $\alpha_x \cup \beta_x \subseteq (\alpha \cup \beta)_x$.

Let $f : R \to R'$ be a homomorphism of rings. A map $f_x : R[x] \to R'[x]$ defined by $f_x(a_0 + a_1x + \cdots + a_nx^n) = f(a_0) + f(a_1)x + \cdots + f(a_n)x^n$ is obviously a ring homomorphism, and we call it an induced homomorphism [13] by $f$.

Theorem 8 (see [13]). Let $f : R \to R'$ be an epimorphism of rings and let $f_x$ be an induced homomorphism of $f$. If $\alpha$ is an $f$-invariant fuzzy ideal of $R$, then $(f(\alpha))_x = f_x(\alpha_x)$.

3. Fuzzy Polynomial Ideals

In this section, we study some relations between the radical of the fuzzy polynomial induced by a fuzzy ideal and the radical of a fuzzy ideal of a ring.

A fuzzy ideal $\alpha : R \to [0, 1]$ of a ring $R$ is called a fuzzy prime ideal [14] of $R$ if $\alpha_x$ is a prime ideal of $R$. A fuzzy set $\sqrt{\alpha} : R \to [0, 1]$, defined as $\sqrt{\alpha}(a) := \{a(a^n) \mid n > 0\}$, is called a fuzzy nil radical [15] of $\alpha$.

Theorem 9 (see [15]). If $\alpha : R \to [0, 1]$ is a fuzzy ideal of $R$, then the fuzzy set $\sqrt{\alpha}$ is a fuzzy ideal of $R$.

Lemma 10 (see [15]). If $\alpha$ and $\beta$ are fuzzy ideals of $R$, then $\sqrt{\alpha \cap \beta} = \sqrt{\alpha} \cap \sqrt{\beta}$.

Lemma 11. If $\alpha$ and $\beta$ are fuzzy ideals of $R$, then $\sqrt{\alpha \cup \beta} = \sqrt{\alpha} \cup \sqrt{\beta}$.

Proof. If $a$ is an element of $R$, then

$$
\sqrt{\alpha \cup \beta}(a) = \big\{ (\alpha \cup \beta)(a^n) \mid n > 0 \big\}
$$

$$
= \big\{ \max\{\alpha(a^n), \beta(a^n)\} \mid n > 0 \big\}
$$

$$
= \max\big\{ \sqrt{\alpha}(a^n), \sqrt{\beta}(a^n) \big\} = \big( \sqrt{\alpha} \cup \sqrt{\beta} \big)(a).
$$

This proves that $\sqrt{\alpha \cup \beta} = \sqrt{\alpha} \cup \sqrt{\beta}$.

Since $\alpha_x$ is a fuzzy ideal of a polynomial ring $R[x]$ by Theorem 5, the fuzzy set $\sqrt{\alpha_x} : R[x] \to [0, 1]$ is the fuzzy nil radical of $\alpha_x$. The following theorem gives that the two fuzzy nil radicals have the same value.

Theorem 12. If $\alpha : R \to [0, 1]$ is a fuzzy ideal of $R$, then

$$
(\sqrt{\alpha_x})_x = (\sqrt{\alpha})_x.
$$

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ be any element of $R[x]$. Then, by Theorem 5, we have $\alpha_x(a_i^n) = \alpha_x(a_0^n + a_1 x + a_2 x^2 + \cdots + a_n x^n) = \min\{\alpha(a_i^n), \alpha(0), \ldots, \alpha(0)\} = (\alpha(a_i))$. Since $\sqrt{\alpha_x}$ is a fuzzy ideal of $R[x]$, we obtain

$$
(\sqrt{\alpha_x})(f(x)) = \min_{i=0}^{m} \{ (\sqrt{\alpha_x})(a_i^n) \}
$$

$$
= \min_{i=0}^{m} \{ \sqrt{\alpha_x}(a_i^n) \mid n > 0 \}
$$

$$
= \min_{i=0}^{m} \{ \sqrt{\alpha}(a_i^n) \mid n > 0 \}
$$

$$
= (\sqrt{\alpha})_x(f(x)).
$$

This proves that $\sqrt{\alpha_x}(x) = (\sqrt{\alpha_x})_x$.

Theorem 13. If $\alpha$ and $\beta$ are fuzzy ideals of $R$, then

$$
(\sqrt{\alpha \cap \beta})_x = (\sqrt{\alpha})_x \cap (\sqrt{\beta})_x.
$$

(7)
Proof. If \( \alpha \) and \( \beta \) are fuzzy ideals of \( R \), then \( \alpha_x \) and \( \beta_x \) are fuzzy ideals of \( R[x] \) by Theorem 5. It follows from Theorems 12 and 7(i) and Lemma 10 that

\[
\left( \sqrt{\alpha \cap \beta} \right)_x = \left( \sqrt{\alpha \cap \beta}_x \right) \quad \text{[Theorem 12]}
\]

\[
= \left( \sqrt{\alpha_x \cap \sqrt{\beta}_x} \right) \quad \text{[Theorem 7 (i)]}
\]

\[
= \left( \sqrt{\alpha_x} \cap \sqrt{\beta_x} \right) \quad \text{[Theorem 7 (ii)]}
\]

\[
= \left( \sqrt{\alpha} \right)_x \cap \left( \sqrt{\beta} \right)_x \quad \text{[Theorem 10]}
\]

proving the theorem. □

**Theorem 14.** If \( \alpha \) and \( \beta \) are fuzzy ideals of \( R \), then

\[
\left( \sqrt{\alpha} \right)_x \cup \left( \sqrt{\beta} \right)_x \subseteq \left( \sqrt{\alpha \cup \beta} \right)_x.
\]  

**Proof.** Since \( \alpha_x \) and \( \beta_x \) are fuzzy ideals of \( R[x] \) by Theorem 5, we obtain

\[
\left( \sqrt{\alpha} \right)_x \cup \left( \sqrt{\beta} \right)_x = \left( \sqrt{\alpha_x} \cup \sqrt{\beta_x} \right)_x \quad \text{[Theorem 12]}
\]

\[
\subseteq \left( \sqrt{\alpha_x \cup \sqrt{\beta_x}} \right) \quad \text{[Theorem 7 (ii)]}
\]

\[
= \left( \sqrt{\alpha_x \cup \beta_x} \right)_x \quad \text{[Lemma 11]}
\]

\[
\subseteq \left( \sqrt{\alpha \cup \beta} \right)_x \quad \text{[Theorem 7 (ii)]}
\]

\[
= \left( \sqrt{\alpha \cup \beta} \right)_x \quad \text{[Theorem 12]}
\]

This proves that \( \left( \sqrt{\alpha} \right)_x \cup \left( \sqrt{\beta} \right)_x \subseteq \left( \sqrt{\alpha \cup \beta} \right)_x \). □

**Theorem 15.** Let \( \beta \) be a fuzzy ideal of \( R \) and let \( f : R \to R' \) be a homomorphism of rings. If \( f_x \) is the induced homomorphism of \( f \), that is, \( f_x \left( \sum_{i=0}^n a_i x^i \right) = \sum_{i=0}^n f(a_i) x^i \), then

\[
f_x^{-1} \left( \sqrt{\beta} \right)_x = \left( \sqrt{f^{-1}(\beta)} \right)_x.
\]  

**Proof.** Given a polynomial \( g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x] \), we have

\[
\left( \sqrt{f^{-1}(\beta)} \right)_x (g(x)) = \min \left\{ \sqrt{f^{-1}(\beta)}(b_i) \right\}
\]

\[
= \min \{ \sqrt{f^{-1}(\beta)}(b_0) \mid n > 0 \},
\]

\[
\sqrt{f^{-1}(\beta)}(b_0) + \cdots + \sqrt{f^{-1}(\beta)}(b_m)
\]

\[
= \min \{ \sqrt{f^{-1}(\beta)}(b_0) \mid n > 0 \},
\]

\[
\sqrt{f^{-1}(\beta)}(b_0) + \cdots + \sqrt{f^{-1}(\beta)}(b_m)
\]

\[
\left( \sqrt{f^{-1}(\beta)} \right)_x (g(x)) = \left( \sqrt{f^{-1}(\beta)}(b_0) \right).
\]  

This proves that \( f_x^{-1} \left( \sqrt{\beta} \right)_x = \left( \sqrt{f^{-1}(\beta)} \right)_x \). □
that \( \gamma \) is a prime ideal of \( R \). Define a fuzzy subset \( \alpha_0 : R \to [0, 1] \) by
\[
\alpha_0(a) = \begin{cases} 
\beta (0) & \text{if } a \in \gamma, \\
\beta (a) & \text{if } a \notin \gamma.
\end{cases}
\] (14)

Then, by routine calculations, we show that \( \alpha_0 \) is a fuzzy ideal of \( R \) satisfying \( (\alpha_0)_\gamma = \gamma \). We claim that \( \alpha \subseteq \alpha_0 \). Given \( a \in R \), if \( a \in \gamma \), then \( \alpha(a) = \alpha_0(a) \leq \beta(a) \leq \beta(0) = \alpha_0(\gamma) \). If \( a \notin \gamma \), then \( \alpha(a) = \alpha_0(a) \leq \beta(a) = \alpha_0(\gamma) \). Since \( \gamma \) is a prime ideal of \( R \), \( (\alpha_0)_\gamma \) is a prime ideal of \( R \). This shows that \( \alpha_0 \) is a fuzzy prime ideal of \( R \), proving the theorem.

**Definition 20** (see [15]). Let \( \alpha : R \to [0, 1] \) be a fuzzy ideal of \( R \). The fuzzy ideal \( r(\alpha) \) defined by
\[
r(\alpha) = \bigcap \{ \beta | \alpha \subseteq \beta, \beta : \text{a fuzzy prime ideal of } R \} \tag{15}
\]
is called the prime fuzzy radical of \( \alpha \).

**Theorem 21.** Let \( \alpha \) be a fuzzy ideal of \( R \) and let \( \alpha_x \) be its fuzzy polynomial ideal of \( R[x] \). Then
\[
r(\alpha_x) \subseteq (r(\alpha))_x . \tag{16}
\]

**Proof.** By Theorem 6, \( \beta_0 \) is a fuzzy prime ideal of \( R \) with \( \alpha \subseteq \beta \), if and only if \( (\beta_0)_\gamma \) is a fuzzy prime ideal of \( R[x] \) with \( \alpha_x \subseteq (\beta_0)_x \). It follows from Theorem 7(i) that
\[
(r(\alpha))_x = \bigcap \{ \beta | \alpha \subseteq \beta, \beta : \text{a fuzzy prime ideal of } R \} \tag{17}
\]
is a fuzzy prime ideal of \( R[x] \). It follows from Theorem 7(i) that
\[
(r(\alpha))_x = \bigcap \{ \beta | \alpha \subseteq \beta, \beta : \text{a fuzzy prime ideal of } R \} \tag{18}
\]
proving the theorem.

**Notation 2.** Let \( \alpha \) be a fuzzy ideal of \( R \) and let \( \alpha_x \) be its fuzzy polynomial ideal of \( R[x] \). We denote \( \text{FPI}(\alpha) \) by
\[
\text{FPI}(\alpha) = \{ \beta | \alpha \subseteq \beta, \beta : \text{a fuzzy prime ideal of } R \} \tag{19}
\]
and \( \text{FPI}(\alpha_x) \) by
\[
\text{FPI}(\alpha_x) = \{ y | \alpha_x \subseteq y, y : \text{a fuzzy prime ideal of } R[x] \} . \tag{20}
\]

**Theorem 22.** Let \( \alpha \) be a fuzzy ideal of \( R \) and let \( \alpha_x \) be its fuzzy polynomial ideal of \( R[x] \). Then a map \( \phi : \text{FPI}(\alpha) \to \text{FPI}(\alpha_x) \) defined by \( \phi(\beta) = \beta_x \) is one-one.

**Proof.** If \( \beta, \gamma \in \text{FPI}(\alpha) \) such that \( \phi(\beta) = \phi(\gamma) \), then \( \beta_x = \gamma_x \). It follows that \( \beta_x(a) = \gamma_x(a) \) for all \( a \in R \), and hence \( \beta(a) = \gamma(a) \) for all \( a \in R \), proving that \( \beta = \gamma \). Hence \( \phi \) is one-one.

**Corollary 23.** Let \( \alpha \) be a fuzzy ideal of \( R \) and let \( \alpha_x \) be its fuzzy polynomial ideal of \( R[x] \). If the map \( \phi \) defined in Theorem 22 is an onto map, then
\[
(r(\alpha))_x = r(\alpha_x) . \tag{21}
\]

**Proof.** If \( \beta \) is any element of \( \text{FPI}(\alpha) \), then there exists \( y \in \text{FPI}(\alpha) \) such that \( \gamma_x = \phi(y) = \beta \) with \( \alpha \subseteq y \). Thus \( (r(\alpha))_x = r(\alpha_x) \). This shows that the reverse inclusion in Theorem 21 holds.

**Example 24.** Let \( Z \) be set of all integers. Let
\[
\alpha(x) = \begin{cases} 1 & \text{if } x \in 2Z, \\
0 & \text{if } x \notin 2Z.
\end{cases} \tag{22}
\]

Then \( \alpha \) is a fuzzy prime ideal of \( Z \), since \( \alpha_x = 2Z \) is a prime ideal of \( Z \), and its induced polynomial ideal \( \alpha_x \) is a fuzzy prime ideal of \( Z[x] \). Hence \( r(\alpha)_x = \alpha_x = r(\alpha_x) \).

**Competing Interests**

The authors declare that they have no competing interests.

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