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Research article

Ulam stability of a functional equation deriving from quadratic and additive mappings in random normed spaces

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Abstract: The aim of this work is to introduce a new mixed type quadratic-additive functional equation, to obtain its general solution and to investigate Ulam stability by using Hyers method in random normed spaces.

Keywords: Ulam stability; random normed space; quadratic-additive functional equation
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1. Introduction

In 1940, Ulam [32] raised a question concerning the stability of homomorphisms: Given a group G_1, a metric group G_2 with the metric $d(\cdot,\cdot)$, and a nonnegative real number ϵ, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality

$$d\,(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G_1$ then there exists a homomorphism $F : G_1 \to G_2$ with

$$d\,(f(x), F(x)) < \epsilon$$

for all $x \in G_1$? As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable.

In 1941, Hyers [12] gave a partial solution of Ulam's problem for the case of approximately additive mappings $f : X \to Y$, where X and Y are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$. The limit

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in X$ and the mapping $A : X \to Y$ is a unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \varepsilon$$

for all $x \in X$.

The outcome declared that the Cauchy functional equation is stable for any pair of Banach space. The technique which was providing through Hyers, forming the additive function $A(x)$, is called direct method. This is called as Ulam stability for the Cauchy additive functional equation. We refer the interested readers for more information on such problems to the articles [6, 9, 11, 13, 15, 16, 18, 26, 33].

Hyers' result was generalized by Aoki [1] for additive mappings and Rassias [25] for linear mappings by considering the stability problem with unbounded Cauchy differences. Furthermore, in 1994, a generalization of Rassias' theorem was obtained by Gavruta [10] by replacing the bound $\epsilon\left(\|x\|^p + \|y\|^p\right)$ by a general control function $\varphi(x, y)$.

In 2008, Mihet and Radu [19] applied fixed point alternative method to prove the stability theorems of the *Cauchy functional equation:*

$$f(x + y) - f(x) - f(y) = 0$$

in random normed spaces. In 2008, Najati and Moghimi [22] obtained a stability of the functional equation deriving from quadratic and additive function:

$$f(2x + y) + f(2x - y) + 2f(x) - f(x + y) - f(x - y) - 2f(2x) = 0 \tag{1.1}$$

by using the direct method. After that, Jin and Lee [14] proved the stability of the above mentioned functional equation in random normed spaces.

In 2011, Saadati *et al.* [24] proved the nonlinear stability of the quartic functional equation of the form

$$\begin{aligned} 16f(x + 4y) + f(4x - y) = \quad & 306\left[9f\left(x + \frac{y}{3}\right) + f(x + 2y)\right] + 136f(x - y) \\ & -1394f(x + y) + 425f(y) - 1530f(x) \end{aligned}$$

in the setting of random normed spaces. Furthermore, the interdisciplinary relation among the theory of random spaces, the theory of non-Archimedean spaces, the fixed point theory, the theory of intuitionistic spaces and the theory of functional equations were also presented. Azadi Kenary [4] investigated the Ulam stability of the following nonlinear function equation

$$f\left(f(x) - f(y)\right) + f(x) + f(y) = f(x + y) + f(x - y),$$

in random normed spaces. Recently, the stability problems of several functional equations in various spaces such as random normed spaces, intuitionistic random normed spaces, quasi-Banach spaces, fuzzy normed spaces have been extensively investigated by a number of mathematicians such as Azadi Kenary [2, 3], Chang *et al.* [5], Eshaghi Gordji *et al.* [7, 8], Mihet *et al.* [20], Saadati *et al.* [21, 27, 28] and Tamilvanan *et al.* [23, 31].

In this paper, we introduce a new mixed type quadratic-additive functional equation of the form

$$\begin{aligned}\phi\left(\sum_{1\le a\le m} as_a\right)+\sum_{1\le a\le m}\phi\left(-as_a+\sum_{b=1;a\neq b}^{m} bs_b\right) &= (m-3)\sum_{1\le a<b\le m}\phi\left(as_a+bs_b\right)\\ &- \left(m^2-5m+2\right)\sum_{1\le a\le m} a^2\left[\frac{\phi(s_a)+\phi(-s_a)}{2}\right]\\ &- \left(m^2-5m+4\right)\sum_{1\le a\le m} a\left[\frac{\phi(s_a)-\phi(-s_a)}{2}\right],\end{aligned} \tag{1.2}$$

where $\phi(0)=0$ and m is an integer greater than 4. The main aim of this work is to obtain its general solution and to investigate the Ulam stability by using the Hyers method in random normed spaces. It is easy to see that the mapping $\phi(s)=as^2+bs$ is a solution of the functional equation (1.2). Every solution of the functional equation deriving from quadratic and additive function (1.2) is said to be a general quadratic mapping.

2. Preliminaries

In this section, we state the usual terminology, notions and conventions of the theory of random normed spaces as in [29].

Let Γ^+ denote the set of all probability distribution functions $F:\mathbb{R}\cup[-\infty,+\infty]\to[0,1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0)=0$, $F(+\infty)=1$. It is clear that the set $D^+=\{F\in\Gamma^+ : l^-F(-\infty)=1\}$, where $l^-f(x)=\lim_{t\to x^-}f(t)$, is a subset of Γ^+. The set Γ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F\le G$ if and only if $F(t)\le G(t)$ for all $t\in\mathbb{R}$. For any $a\ge 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t)=\begin{cases}0, & \text{if } t\le a,\\ 1, & \text{if } t>a.\end{cases}$$

We can easily show that the maximal element in Γ^+ is the distribution function $H_0(t)$.

Definition 2.1. [29] A function $T:[0,1]^2\to[0,1]$ is a continuous triangular norm (briefly, a t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(x,1)=x$ for all $x\in[0,1]$;
(d) $T(x,y)\le T(z,w)$ whenever $x\le z$ and $y\le w$ for all $x,y,z,w\in[0,1]$.

Three typical examples of continuous t-norms are $T_P(x,y)=xy$, $T_{\max}(x,y)=\max\{a+b-1,0\}$, $T_M(x,y)=\min\{a,b\}$.

Recall that, if T is a t-norm and $\{x_n\}$ is a sequence in $[0,1]$, then $T_{i=1}^{n}x_i$ is defined recursively by $T_{i=1}^{1}x_i=x_1$ and $T_{i=1}^{n}x_i=T\left(T_{i=1}^{n-1}x_i,x_n\right)$ for all $n\ge 2$. $T_{i=n}^{\infty}x_i$ is defined by $T_{i=1}^{\infty}x_{n+i}$.

Definition 2.2. [30] A random normed space (briefly, RNS) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and $\mu:X\to D^+$ is a mapping such that the following conditions hold:

(RN1) $\mu_x(t) = H_0(t)$ for all $x \in X$ and $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \geq 0$;
(RN3) $\mu_{x+y}(t+s) \geq T\left(\mu_x(t), \mu_y(t)\right)$ for all $x, y \in X$ and $t, s \geq 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, μ, T_M), where $\mu_u(t) = \frac{t}{t+\|u\|}$ for all $t > 0$ and T_M is the minimum t-norm. This space X is called the induced random normed space. If the t-norm T is such that $\sup_{0<a<1} T(a,a) = 1$, then every random normed space (X, μ, T) is a metrizable linear topological space with the topology τ (called the μ-topology or the (ϵ, δ)-topology, where $\epsilon > 0$ and $\lambda \in (0,1)$) induced by the base $\{U(\epsilon, \lambda)\}$ of neighbourhoods of θ, where $U(\epsilon, \lambda) = \{x \in X : \Psi_x(\epsilon) > 1 - \lambda\}$.

Definition 2.3. Let (X, μ, T) be a random normed space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if $\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$ for all $t > 0$.
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence in X if $\lim_{n\to\infty} \mu_{x_n - x_m}(t) = 1$ for all $t > 0$.
(iii) The random normed space (X, μ, T) is said to be complete if every Cauchy sequence in X is convergent.

Theorem 2.4. *[29] If (X, μ, T) is a random normed space and $\{x_n\}$ is a sequence such that $x_n \to x$, then* $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.

3. Solution of the functional equation (1.2)

Throughout this section, assume that E and F are real vector spaces.

Theorem 3.1. *If $\phi : E \to F$ is an odd mapping which satisfies the functional equation (1.2) for all $s_1, s_2, \cdots, s_m \in E$, then ϕ is additive.*

Proof. In the sense of oddness of ϕ, $\phi(-s) = -\phi(s)$ for all $s \in E$. Then (1.2) turns into

$$\begin{aligned}\phi\left(\sum_{1\leq a\leq m} as_a\right) + \sum_{1\leq a\leq m} \phi\left(-as_a + \sum_{b=1;a\neq b}^{m} bs_b\right) &= (m-3)\sum_{1\leq a<b\leq m} \phi\left(as_a + bs_b\right) \\ &- \left(m^2 - 5m + 4\right)\sum_{1\leq a\leq m} a\phi(s_a) \end{aligned} \tag{3.1}$$

for all $s_1, s_2, \cdots, s_m \in E$. Now, setting $s_1 = s_2 = \cdots = s_m = 0$ in (3.1), we obtain that $\phi(0) = 0$. Replacing $(s_1, s_2, \cdots, s_m)$ by $(0, s, 0, \cdots)$ in (3.1), we obtain that

$$\phi(2s) = 2\phi(s) \tag{3.2}$$

for all $s \in E$. Again replacing s by $2s$ in (3.2), we get

$$\phi(2^2 s) = 2^2\phi(s) \tag{3.3}$$

for all $s \in E$. Also, changing s by $2s$ in (3.3), we have

$$\phi(2^3 s) = 2^3\phi(s) \tag{3.4}$$

for all $s \in E$. From (3.2)–(3.4), we conclude, for a positive integer m,

$$\phi(2^m s) = 2^m \phi(s)$$

for all $s \in E$. Now, replacing $(s_1, s_2, \cdots, s_m)$ by $(u, \frac{v}{2}, 0, \cdots, 0)$ in (3.1), we get

$$\phi(u + v) = \phi(u) + \phi(v)$$

for all $u, v \in E$. Therefore, the mapping ϕ is additive. □

Theorem 3.2. *If $\phi : E \to F$ is an even mapping which satisfies the functional equation (1.2) for all $s_1, s_2, \cdots, s_m \in E$, then ϕ is quadratic.*

Proof. In the sense of evenness of ϕ, $\phi(-s) = \phi(s)$ for all $s \in E$. Then (1.2) becomes

$$\begin{aligned} \phi\left(\sum_{1\le a\le m} as_a\right) + \sum_{1\le a\le m} \phi\left(-as_a + \sum_{b=1;a\neq b}^{m} bs_b\right) &= (m-3) \sum_{1\le a<b\le m} \phi\left(as_a + bs_b\right) \\ &- \left(m^2 - 5m + 2\right) \sum_{1\le a\le m} a^2 \phi(s_a) \end{aligned} \tag{3.5}$$

for all $s_1, s_2, \cdots, s_m \in E$. Now, setting $s_1 = s_2 = \cdots = s_m = 0$ in (3.5), we get $\phi(0) = 0$. Letting $(s_1, s_2, \cdots, s_m) = (0, s, 0, \cdots)$ in (3.5), we have

$$\phi(2s) = 2^2 \phi(s) \tag{3.6}$$

for all $s \in E$. Replacing s by $2s$ in (3.6), we obtain

$$\phi(2^2 s) = 2^4 \phi(s) \tag{3.7}$$

for all $s \in E$. Replacing s by $2s$ in (3.7), we get

$$\phi(2^3 s) = 2^6 \phi(s) \tag{3.8}$$

for all $s \in E$. From (3.6)–(3.8), we conclude, for a positive integer m,

$$\phi(2^m s) = 2^{2m} \phi(s)$$

for all $s \in E$. Now, replacing $(s_1, s_2, \cdots, s_m)$ by $(u, \frac{v}{2}, 0, \cdots, 0)$ in (3.5), we get

$$\phi(u + v) + \phi(u - v) = 2\phi(u) + 2\phi(v)$$

for all $u, v \in E$. Therefore, the mapping ϕ is quadratic. □

Theorem 3.3. *A mapping $\phi : E \to F$ satisfies $\phi(0) = 0$ and (1.2) for all $s_1, s_2, \cdots, s_m \in E$ if and only if there exist a symmetric bi-additive mapping $Q : E \times E \to F$ and an additive mapping $A : E \to F$ such that $\phi(s) = Q(s, s) + A(s)$ for all $s \in E$.*

Proof. Let ϕ satisfy (1.2) and $\phi(0) = 0$. We split ϕ into the odd part and even part as follows

$$\phi_o(s) = \frac{\phi(s) - \phi(-s)}{2}, \quad \phi_e(s) = \frac{\phi(s) + \phi(-s)}{2}$$

for all $s \in E$, respectively. It is clear that $\phi(s) = \phi_e(s) + \phi_o(s)$ for all $s \in E$. It is easy to show that the mappings ϕ_o and ϕ_e satisfy (1.2). Hence by Theorems 3.1 and 3.2, we have that ϕ_o and ϕ_e are additive and quadratic, respectively. So there exist a symmetric bi-additive mapping $Q : E \times E \to F$ such that $\phi_e(s) = Q(s, s)$ and an additive mapping $A : E \to F$ such that $\phi_o(s) = A(s)$ for all $s \in E$. Hence $\phi(s) = Q(s, s) + A(s)$ for all $s \in E$.

Conversely, assume that there exist a symmetric bi-additive mapping $Q : E \times E \to F$ and an additive mapping $A : E \to F$ such that $\phi(s) = Q(s, s) + A(s)$ for all $s \in E$. One can easily show that the mappings $s \mapsto Q(s, s)$ and the mapping $A : E \to F$ satisfy the functional equation (1.2). Therefore, the mapping $\phi : E \to F$ satisfies the functional equation (1.2). □

For our notational handiness, for a mapping $\phi : E \to F$, we define

$$\begin{aligned} D\phi(s_1, s_2, \cdots, s_m) = \quad & \phi\left(\sum_{1\le a\le m} as_a\right) + \sum_{1\le a\le m} \phi\left(-as_a + \sum_{b=1;a\neq b}^{m} bs_b\right) \\ & - (m-3) \sum_{1\le a<b\le m} \phi\left(as_a + bs_b\right) \\ & + \left(m^2 - 5m + 2\right) \sum_{1\le a\le m} a^2 \left[\frac{\phi(s_a) + \phi(-s_a)}{2}\right] \\ & + \left(m^2 - 5m + 4\right) \sum_{1\le a\le m} a \left[\frac{\phi(s_a) - \phi(-s_a)}{2}\right] \end{aligned}$$

for all $s_1, s_2, \cdots, s_m \in E$.

4. Main results for odd case

In this section, we investigate the Ulam stability of the finite variable functional equation (1.2) for odd case in random normed spaces by using the Hyers method.

Theorem 4.1. *Let E be a real linear space, $\left(Z, \mu', \min\right)$ be a random normed space and $\varphi : E^m \to Z$ be a function such that there exists $0 < \rho < \frac{1}{2}$ such that*

$$\mu'_{\varphi\left(\frac{s_1}{2}, \frac{s_2}{2}, \cdots, \frac{s_m}{2}\right)}(t) \ge \mu'_{\rho\varphi(s_1, s_2, \cdots, s_m)}(t) \tag{4.1}$$

for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$ and $\lim_{m\to\infty} \mu'_{\varphi\left(\frac{s_1}{2^m}, \frac{s_2}{2^m}, \cdots, \frac{s_m}{2^m}\right)}\left(\frac{t}{2^m}\right) = 1$ for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$. Let $(F, \mu, \min)$ be a complete random normed space. If $\phi : E \to F$ is a mapping such that

$$\mu_{D\phi(s_1, s_2, \cdots, s_m)}(t) \ge \mu'_{\varphi(s_1, s_2, \cdots, s_m)}(t) \tag{4.2}$$

for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$, then the limit $A_1(s) = \lim_{m\to\infty} 2^m \phi\left(\frac{s}{2^m}\right)$ exists for all $s \in E$ and defines a unique additive mapping $A_1 : E \to F$ such that

$$\mu_{\phi(s)-A_1(s)}(t) \ge \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2 - 5m + 4)\left(\frac{1-2\rho}{\rho}\right)t\right) \tag{4.3}$$

for all $s \in E$ and $t > 0$.

Proof. Replacing $(s_1, s_2, \cdots, s_m)$ by $(0, s, 0, \cdots, 0)$ in (4.2), we get

$$\mu_{(m^2-5m+4)\phi(2s)-2(m^2-5m+4)\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}(t) \tag{4.4}$$

for all $s \in E$. From (4.4), we get

$$\mu_{\phi(2s)-2\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}((m^2-5m+4)t) \tag{4.5}$$

for all $s \in E$. Again, replacing s by $\frac{s}{2}$ in (4.5), we have

$$\mu_{2\phi\left(\frac{s}{2}\right)-\phi(s)}(t) \geq \mu'_{\varphi(0,\frac{s}{2},0,\cdots,0)}((m^2-5m+4)t) \tag{4.6}$$

for all $s \in E$. Replacing s by $\frac{s}{2^n}$ in (4.6) and using (4.1), we obtain

$$\begin{aligned} \mu_{2^{n+1}\phi\left(\frac{s}{2^{n+1}}\right)-2^n\phi\left(\frac{s}{2^n}\right)}(t) &\geq \mu'_{\varphi(0,\frac{s}{2^{n+1}},0,\cdots,0)}\left((m^2-5m+4)\frac{t}{2^n}\right) \\ &\geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2-5m+4)\frac{t}{2^n\rho^{n+1}}\right) \end{aligned} \tag{4.7}$$

for all $s \in E$. We know that

$$2^n\phi\left(\frac{s}{2^n}\right) - \phi(s) = \sum_{l=0}^{n-1} 2^{l+1}\phi\left(\frac{s}{2^{l+1}}\right) - 2^l\phi\left(\frac{s}{2^l}\right)$$

and so

$$\begin{aligned} \mu_{2^n\phi\left(\frac{s}{2^n}\right)-\phi(s)}\left(\frac{\sum_{l=0}^{n-1} 2^l\rho^{l+1}}{(m^2-5m+4)}t\right) &= \mu_{\sum_{l=0}^{n-1} 2^{l+1}\phi\left(\frac{s}{2^{l+1}}\right)-2^l\phi\left(\frac{s}{2^l}\right)}\left(\frac{\sum_{l=0}^{n-1} 2^l\rho^{l+1}}{(m^2-5m+4)}t\right) \\ &\geq T_{l=0}^{n-1}\left(\mu_{2^{l+1}\phi\left(\frac{s}{2^{l+1}}\right)-2^l\phi\left(\frac{s}{2^l}\right)}\left(\frac{2^l\rho^{l+1}}{(m^2-5m+4)}t\right)\right) \\ &\geq T_{l=0}^{n-1}\left(\mu'_{\varphi(0,s,0,\cdots,0)}(t)\right) \\ &\geq \mu'_{\varphi(0,s,0,\cdots,0)}(t) \end{aligned}$$

$$\Rightarrow \mu_{2^n\phi\left(\frac{s}{2^n}\right)-\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left(\frac{(m^2-5m+4)}{\sum_{l=0}^{n-1} 2^l\rho^{l+1}}t\right) \tag{4.8}$$

for all $s \in E$. Replacing s by $\frac{s}{2^q}$ in (4.8), we have

$$\mu_{2^{n+q}\phi\left(\frac{s}{2^{n+q}}\right)-2^q\phi\left(\frac{s}{2^q}\right)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left(\frac{(m^2-5m+4)}{\sum_{l=q}^{n+q-1} 2^l\rho^{l+1}}t\right) \tag{4.9}$$

for all $s \in E$. Since $\lim_{q,n\to\infty} \mu'_{\varphi(0,s,0,\cdots,0)}\left(\frac{(m^2-5m+4)}{\sum_{l=q}^{n+q-1} 2^l\rho^{l+1}}t\right) = 1$, it follows that $\{2^n\phi\left(\frac{s}{2^n}\right)\}_{n=1}^{\infty}$ is a Cauchy sequence in a complete random normed space $(F, \mu, \min)$ and so there exists a point $A_1(s) \in F$ such that $\lim_{n\to\infty} 2^n\phi\left(\frac{s}{2^n}\right) = A_1(s)$. Fix $s \in E$ and put $q = 0$ in (4.9). Then we obtain

$$\mu_{2^n\phi\left(\frac{s}{2^n}\right)-\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left(\frac{(m^2-5m+4)}{\sum_{l=0}^{n-1} 2^l\rho^{l+1}}t\right)$$

and so, for any $\delta > 0$,

$$\begin{aligned}\mu_{A_1(s)\pm 2^n\phi(\frac{s}{2^n})-\phi(s)}(t+\delta) \quad &\geq T\left(\mu_{A_1(s)-2^n\phi(\frac{s}{2^n})}(\delta), \mu_{2^n\phi(\frac{s}{2^n})-\phi(s)}(t)\right) \\ &\geq T\left(\mu_{A_1(s)-2^n\phi(\frac{s}{2^n})}(\delta), \mu'_{\varphi(0,s,0,\cdots,0)}\left(\frac{(m^2-5m+4)}{\sum_{l=0}^{n-1} 2^l\rho^{l+1}} t\right)\right)\end{aligned} \tag{4.10}$$

for all $s \in E$ and $t > 0$. Taking $n \to \infty$ in (4.10), we have

$$\mu_{A_1(s)-\phi(s)}(t+\delta) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2-5m+4)\left(\frac{1-2\rho}{\rho}\right)t\right) \tag{4.11}$$

for all $s \in E$. Since δ is arbitrary, by taking $\delta \to 0$ in (4.11), we obtain

$$\mu_{A_1(s)-\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2-5m+4)\left(\frac{1-2\rho}{\rho}\right)t\right)$$

for all $s \in E$. Now, replacing $(s_1, s_2, \cdots, s_m)$ by $\left(\frac{s_1}{2^n}, \frac{s_2}{2^n}, \cdots, \frac{s_m}{2^n}\right)$ in (4.2), we have

$$\mu_{2^n D\phi(\frac{s_1}{2^n}, \frac{s_2}{2^n}, \cdots, \frac{s_m}{2^n})}(t) \geq \mu'_{\varphi(\frac{s_1}{2^n}, \frac{s_2}{2^n}, \cdots, \frac{s_m}{2^n})}\left(\frac{t}{2^n}\right)$$

for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$. Since $\lim_{n\to\infty} \mu'_{\varphi(\frac{s_1}{2^n}, \frac{s_2}{2^n}, \cdots, \frac{s_m}{2^n})}\left(\frac{t}{2^n}\right) = 1$, we conclude that A_1 satisfies the functional equation (1.2). On the other hand,

$$2A_1\left(\frac{s}{2}\right) - A_1(s) = \lim_{n\to\infty} 2^{n+1}\phi\left(\frac{s}{2^{n+1}}\right) - \lim_{n\to\infty} 2^n\phi\left(\frac{s}{2^n}\right) = 0$$

for all $s \in E$. This implies that $A_1 : E \to F$ is an additive mapping. To prove the uniqueness of the additive mapping A_1, assume that there exists another additive mapping $A_2 : E \to F$ which satisfies the inequality (4.3). Then we get

$$\begin{aligned}\mu_{A_1(s)-A_2(s)}(t) \quad &= \lim_{n\to\infty} \mu_{2^n A_1(\frac{s}{2^n})-2^n A_2(\frac{s}{2^n})}(t) \\ &\geq \lim_{n\to\infty} \min\left\{\mu_{2^n A_1(\frac{s}{2^n})-2^n\phi(\frac{s}{2^n})}\left(\frac{t}{2}\right), \mu_{2^n\phi(\frac{s}{2^n})-2^n A_2(\frac{s}{2^n})}\left(\frac{t}{2}\right)\right\} \\ &\geq \lim_{n\to\infty} \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2-5m+4)\left(\frac{1-2\rho}{2^{n+1}\rho^n}\right)t\right)\end{aligned}$$

for all $s \in E$ and $t > 0$. Since $\lim_{n\to\infty}(m^2-5m+4)\left(\frac{1-2\rho}{2^{n+1}\rho^n}\right)t = \infty$, we have

$$\lim_{n\to\infty} \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2-5m+4)\left(\frac{1-2\rho}{2^{n+1}\rho^n}\right)t\right) = 1.$$

It follows that $\mu_{A_1(s)-A_2(s)}(t) = 1$ for all $t > 0$ and so $A_1(s) = A_2(s)$. This completes the proof. □

Corollary 4.2. *Let E be a real normed linear space, $(Z, \mu', \min)$ be a random normed space and $(F, \mu, \min)$ be a complete random normed space. Let p be a positive real number with $p > 1$, $z_0 \in Z$ and $\phi : E \to F$ be a mapping satisfying*

$$\mu_{D\phi(s_1,s_2,\cdots,s_m)}(t) \geq \mu'_{(\sum_{j=1}^m \|s_j\|^p)z_0}(t) \tag{4.12}$$

for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$. Then the limit $A_1(s) = \lim_{n\to\infty} 2^n\phi\left(\frac{s}{2^n}\right)$ exists for all $s \in E$ and defines a unique additive mapping $A_1 : E \to F$ such that

$$\mu_{\phi(s)-A_1(s)}(t) \geq \mu'_{\|s\|^p z_0}\left((m^2-5m+4)(2^p-2)t\right),$$

for all $s \in E$ and $t > 0$.

Proof. Let $\rho = 2^{-p}$ and $\varphi : E^m \to Z$ be a mapping defined by $\varphi(s_1, s_2, \cdots, s_m) = \left(\sum_{j=1}^m \|s_j\|^p\right) z_0$. Then, from Theorem 4.1, the conclusion follows. □

Theorem 4.3. *Let E be a real linear space, $\left(Z, \mu', \min\right)$ be a random normed space and $\varphi : E^m \to Z$ be a function for which t there exists $0 < \rho < 2$ such that*

$$\mu'_{\varphi(2s_1,2s_2,\cdots,2s_m)}(t) \geq \mu'_{\rho\varphi(s_1,s_2,\cdots,s_m)}(t) \tag{4.13}$$

for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$ and $\lim_{n\to\infty} \mu'_{\varphi(2^n s_1, 2^n s_2,\cdots,2^n s_m)}(2^n t) = 1$ for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$. Let $(F, \mu, \min)$ be a complete random normed space. If $\phi : E \to F$ is a mapping satisfying (4.2), then the limit $A_1(s) = \lim_{n\to\infty} \frac{\phi(2^n s)}{2^n}$ exists for all $s \in E$ and defines a unique additive mapping $A_1 : E \to F$ such that

$$\mu_{\phi(s)-A_1(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2-5m+4)(2-\rho)t\right)$$

for all $s \in E$ and $t > 0$.

Proof. Replacing $(s_1, s_2, \cdots, s_m)$ by $(0, s, 0, \cdots, 0)$ in (4.2), we get

$$\mu_{(m^2-5m+4)\phi(2s)-2(m^2-5m+4)\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}(t) \tag{4.14}$$

for all $s \in E$. From (4.14), we obtain

$$\mu_{\frac{\phi(2s)}{2}-\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}(2(m^2-5m+4)t)$$

for all $s \in E$. Replacing s by $2^n s$ in (4) and using (4.13), we obtain

$$\begin{aligned}
\mu_{\frac{\phi\left(2^{n+1}s\right)}{2^{n+1}}-\frac{\phi(2^n s)}{2^n}}(t) &\geq \mu'_{\varphi(0,2^n s,0,\cdots,0)}\left(2^{n+1}(m^2-5m+4)t\right)\\
&\geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2-5m+4)\frac{2^{n+1}}{\rho^n}t\right)
\end{aligned}$$

for all $s \in E$. The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let E be a real normed linear space, $(Z, \mu', \min)$ be a random normed space and $(F, \mu, \min)$ be a complete random normed space. Let p be a positive real number with $0 < p < 1$, $z_0 \in Z$ and $\phi : E \to F$ be a mapping satisfying (4.12). Then the limit $A_1(s) = \lim_{n\to\infty} \frac{\phi(2^n s)}{2^n}$ exists for all $s \in E$ and defines a unique additive mapping $A_1 : E \to F$ such that*

$$\mu_{\phi(s)-A_1(s)}(t) \geq \mu'_{\|s\|^p z_0}\left((m^2-5m+4)(2-2^p)t\right)$$

for all $s \in E$ and $t > 0$.

Proof. Let $\rho = 2^p$ and $\varphi : E^m \to Z$ be a mapping defined by $\varphi(s_1, s_2, \cdots, s_m) = \left(\sum_{j=1}^m \|s_j\|^p\right) z_0$. Then, from Theorem 4.3, the conclusion follows. □

5. Main results for even case

In this section, we investigate the Ulam stability of the finite variable functional equation (1.2) for even case in random normed spaces by using the Hyers method.

Theorem 5.1. *Let E be a real linear space, $\left(Z, \mu', \min\right)$ be a random normed space and $\varphi : E^m \to Z$ be a function for which there exists $0 < \rho < \frac{1}{2^2}$ such that*

$$\mu'_{\varphi\left(\frac{s_1}{2}, \frac{s_2}{2}, \cdots, \frac{s_m}{2}\right)}(t) \geq \mu'_{\rho\varphi(s_1, s_2, \cdots, s_m)}(t) \tag{5.1}$$

for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$ and $\lim_{n\to\infty} \mu'_{\varphi\left(\frac{s_1}{2^n}, \frac{s_2}{2^n}, \cdots, \frac{s_m}{2^n}\right)}\left(\frac{t}{2^{2n}}\right) = 1$ for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$. Let $(F, \mu, \min)$ be a complete random normed space. If $\phi : E \to F$ is a mapping with $\phi(0) = 0$ such that

$$\mu_{D\phi(s_1, s_2, \cdots, s_m)}(t) \geq \mu'_{\varphi(s_1, s_2, \cdots, s_m)}(t) \tag{5.2}$$

for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$, then the limit $Q_2(s) = \lim_{n\to\infty} 2^{2n}\phi\left(\frac{s}{2^n}\right)$ exists for all $s \in E$ and defines a unique quadratic mapping $Q_2 : E \to F$ such that

$$\mu_{\phi(s)-Q_2(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2 - 5m + 2)\left(\frac{1 - 2^2\rho}{\rho}\right)t\right) \tag{5.3}$$

for all $s \in E$ and $t > 0$.

Proof. Replacing $(s_1, s_2, \cdots, s_m)$ by $(0, s, 0, \cdots, 0)$ in (5.2), we obtain

$$\mu_{(m^2-5m+2)\phi(2s)-2^2(m^2-5m+2)\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}(t) \tag{5.4}$$

for all $s \in E$. From (5.4), we have

$$\mu_{\phi(2s)-2^2\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}((m^2 - 5m + 2)t) \tag{5.5}$$

for all $s \in E$. Replacing s by $\frac{s}{2}$ in (5.5), we get

$$\mu_{2^2\phi\left(\frac{s}{2}\right)-\phi(s)}(t) \geq \mu'_{\varphi(0,\frac{s}{2},0,\cdots,0)}((m^2 - 5m + 2)t) \tag{5.6}$$

for all $s \in E$. Again, replacing s by $\frac{s}{2^n}$ in (5.6) and using (5.1), we have

$$\begin{aligned} \mu_{2^{2(n+1)}\phi\left(\frac{s}{2^{n+1}}\right)-2^{2n}\phi\left(\frac{s}{2^n}\right)}(t) \quad &\geq \mu'_{\varphi(0,\frac{s}{2^{n+1}},0,\cdots,0)}\left((m^2 - 5m + 2)\frac{t}{2^{2n}}\right) \\ &\geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2 - 5m + 2)\frac{t}{2^{2n}\rho^{n+1}}\right) \end{aligned}$$

for all $s \in E$. We know that

$$2^{2n}\phi\left(\frac{s}{2^n}\right) - \phi(s) = \sum_{l=0}^{n-1} 2^{2(l+1)}\phi\left(\frac{s}{2^{l+1}}\right) - 2^{2l}\phi\left(\frac{s}{2^l}\right)$$

and so

$$\begin{aligned}\mu_{2^{2n}\phi\left(\frac{s}{2^n}\right)-\phi(s)}\left(\frac{\sum_{l=0}^{n-1}2^{2l}\rho^{l+1}}{(m^2-5m+2)}t\right) &= \mu_{\sum_{l=0}^{n-1}2^{2(l+1)}\phi\left(\frac{s}{2^{l+1}}\right)-2^{2l}\phi\left(\frac{s}{2^l}\right)}\left(\frac{\sum_{l=0}^{n-1}2^{2l}\rho^{l+1}}{(m^2-5m+2)}t\right)\\ &\geq T_{l=0}^{n-1}\left(\mu_{2^{2(l+1)}\phi\left(\frac{s}{2^{l+1}}\right)-2^{2l}\phi\left(\frac{s}{2^l}\right)}\left(\frac{2^{2l}\rho^{l+1}}{(m^2-5m+2)}t\right)\right)\\ &\geq T_{l=0}^{n-1}\left(\mu'_{\varphi(0,s,0,\cdots,0)}(t)\right)\\ &\geq \mu'_{\varphi(0,s,0,\cdots,0)}(t)\end{aligned}$$

$$\Rightarrow \mu_{2^{2n}\phi\left(\frac{s}{2^n}\right)-\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left(\frac{(m^2-5m+2)}{\sum_{l=0}^{n-1}2^{2l}\rho^{l+1}}t\right) \tag{5.7}$$

for all $s \in E$. Replacing s by $\frac{s}{2^q}$ in (5.7), we get

$$\mu_{2^{2(n+q)}\phi\left(\frac{s}{2^{n+q}}\right)-2^{2q}\phi\left(\frac{s}{2^q}\right)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left(\frac{(m^2-5m+2)}{\sum_{l=q}^{n+q-1}2^{2l}\rho^{l+1}}t\right) \tag{5.8}$$

for all $s \in E$. Since $\lim_{q,n\to\infty}\mu'_{\varphi(0,s,0,\cdots,0)}\left(\frac{(m^2-5m+2)}{\sum_{l=q}^{n+q-1}2^{2l}\rho^{l+1}}t\right) = 1$, it follows that $\{2^{2n}\phi\left(\frac{s}{2^n}\right)\}_{n=1}^{\infty}$ is a Cauchy sequence in a complete random normed space $(F, \mu, \min)$ and so there exists a point $Q_2(s) \in F$ such that $\lim_{n\to\infty} 2^{2n}\phi\left(\frac{s}{2^n}\right) = Q_2(s)$. Fix $s \in E$ and put $q = 0$ in (5.8). Then we have

$$\mu_{2^{2n}\phi\left(\frac{s}{2^n}\right)-\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left(\frac{(m^2-5m+2)}{\sum_{l=0}^{n-1}2^{2l}\rho^{l+1}}t\right)$$

and so, for any $\delta > 0$,

$$\begin{aligned}\mu_{Q_2(s)\pm 2^{2n}\phi\left(\frac{s}{2^n}\right)-\phi(s)}(t+\delta) &\geq T\left(\mu_{Q_2(s)-2^{2n}\phi\left(\frac{s}{2^n}\right)}(\delta), \mu_{2^{2n}\phi\left(\frac{s}{2^n}\right)-\phi(s)}(t)\right)\\ &\geq T\left(\mu_{Q_2(s)-2^{2n}\phi\left(\frac{s}{2^n}\right)}(\delta), \mu'_{\varphi(0,s,0,\cdots,0)}\left(\frac{(m^2-5m+2)}{\sum_{l=0}^{n-1}2^{2l}\rho^{l+1}}t\right)\right)\end{aligned} \tag{5.9}$$

for all $s \in E$ and $t > 0$. Passing the limit $n \to \infty$ in (5.9), we get

$$\mu_{Q_2(s)-\phi(s)}(t+\delta) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2-5m+2)\frac{(1-2^2\rho)}{\rho}t\right) \tag{5.10}$$

for all $s \in E$. Since δ is arbitrary, by taking $\delta \to 0$ in (5.10), we obtain

$$\mu_{Q_2(s)-\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2-5m+2)\frac{(1-2^2\rho)}{\rho}t\right)$$

for all $s \in E$. Now, replacing $(s_1, s_2, \cdots, s_m)$ by $\left(\frac{s_1}{2^n}, \frac{s_2}{2^n}, \cdots, \frac{s_m}{2^n}\right)$ in (5.2), we have

$$\mu_{2^{2n}D\phi\left(\frac{s_1}{2^n},\frac{s_2}{2^n},\cdots,\frac{s_m}{2^n}\right)}(t) \geq \mu'_{\varphi\left(\frac{s_1}{2^n},\frac{s_2}{2^n},\cdots,\frac{s_m}{2^n}\right)}\left(\frac{t}{2^{2n}}\right)$$

for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$. Since $\lim_{n\to\infty} \mu'_{\varphi(\frac{s_1}{2^n}, \frac{s_2}{2^n}, \cdots, \frac{s_m}{2^n})}\left(\frac{t}{2^{2n}}\right) = 1$, we conclude that Q_2 satisfies the functional equation (1.2). On the other hand,

$$2^2 Q_2\left(\frac{s}{2}\right) - Q_2(s) = \lim_{n\to\infty} 2^{2(n+1)}\phi\left(\frac{s}{2^{n+1}}\right) - \lim_{n\to\infty} 2^{2n}\phi\left(\frac{s}{2^n}\right) = 0$$

for all $s \in E$. This implies that Q_2 is a quadratic mapping. To prove the uniqueness of the quadratic mapping Q_2, assume that there exists another quadratic mapping $Q_2' : E \to F$ which satisfies (5.3). Then we have

$$\begin{aligned}
\mu_{Q_2(s)-Q_2'(s)}(t) &= \lim_{n\to\infty} \mu_{2^{2n}Q_2(\frac{s}{2^n})-2^{2n}Q_2'(\frac{s}{2^n})}(t) \\
&\geq \lim_{n\to\infty} \min\left\{\mu_{2^{2n}Q_2(\frac{s}{2^n})-2^{2n}\phi(\frac{s}{2^n})}\left(\frac{t}{2}\right), \mu_{2^{2n}\phi(\frac{s}{2^n})-2^{2n}Q_2'(\frac{s}{2^n})}\left(\frac{t}{2}\right)\right\} \\
&\geq \lim_{n\to\infty} \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2 - 5m + 2)\frac{(1-2^2\rho)}{2^{2(n+1)}\rho^n}t\right)
\end{aligned}$$

for all $s \in E$ and $t > 0$. Since $\lim_{n\to\infty}(m^2 - 5m + 2)\frac{(1-2^2\rho)}{2^{2(n+1)}\rho^n}t = \infty$, we have

$$\lim_{n\to\infty} \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2 - 5m + 2)\frac{(1-2^2\rho)}{2^{2(n+1)}\rho^n}t\right) = 1.$$

It follows that $\mu_{Q_2(s)-Q_2'(s)}(t) = 1$ for all $t > 0$ and so $Q_2(s) = Q_2'(s)$. This completes the proof. □

Corollary 5.2. *Let E be a real normed linear space, $(Z, \mu', \min)$ be a random normed space and $(F, \mu, \min)$ be a complete random normed space. Let p be a positive real number with $p > 2$, $z_0 \in Z$ and $\phi : E \to F$ be a mapping satisfying (4.12) for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$. Then the limit $Q_2(s) = \lim_{n\to\infty} 2^{2n}\phi\left(\frac{s}{2^n}\right)$ exists for all $s \in E$ and defines a unique quadratic mapping $Q_2 : E \to F$ such that*

$$\mu_{\phi(s)-Q_2(s)}(t) \geq \mu'_{\|s\|^p z_0}\left((m^2 - 5m + 2)(2^p - 2^2)t\right)$$

for all $s \in E$ and $t > 0$.

Proof. Let $\rho = 2^{-p}$ and $\varphi : E^m \to Z$ be a mapping defined by $\varphi(s_1, s_2, \cdots, s_m) = \left(\sum_{j=1}^m \|s_j\|^p\right) z_0$. Then, from Theorem 5.1, the conclusion follows. □

Theorem 5.3. *Let E be a real linear space, $\left(Z, \mu', \min\right)$ be a random normed space and $\varphi : E^m \to Z$ be a function such that there exists $0 < \rho < 2^2$ such that*

$$\mu'_{\varphi(2s_1,2s_2,\cdots,2s_m)}(t) \geq \mu'_{\rho\varphi(s_1,s_2,\cdots,s_m)}(t) \tag{5.11}$$

for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$ and $\lim_{n\to\infty} \mu'_{\varphi(2^n s_1, 2^n s_2, \cdots, 2^n s_m)}\left(2^{2n}t\right) = 1$ for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$. Let $(F, \mu, \min)$ be a complete random normed space. If $\phi : E \to F$ is a mapping with $\phi(0) = 0$ sstisfyinf (4.2), then the limit $Q_2(s) = \lim_{n\to\infty} \frac{\phi(2^n s)}{2^{2n}}$ exists for all $s \in E$ and defines a unique quadratic mapping $Q_2 : E \to F$ such that

$$\mu_{\phi(s)-Q_2(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2 - 5m + 2)(2^2 - \rho)t\right) \tag{5.12}$$

for all $s \in E$ and $t > 0$.

Proof. Replacing $(s_1, s_2, \cdots, s_m)$ by $(0, s, 0, \cdots, 0)$ in (5.2), we obtain

$$\mu_{(m^2-5m+2)\phi(2s)-2^2(m^2-5m+2)\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}(t) \tag{5.13}$$

for all $s \in E$. From (5.13), we have

$$\mu_{\frac{\phi(2s)}{2^2}-\phi(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}(2^2(m^2-5m+2)t) \tag{5.14}$$

for all $s \in E$. Replacing s by $2^n s$ in (5.14) and using (5.11), we get

$$\begin{aligned}\mu_{\frac{\phi\left(2^{n+1}s\right)}{2^{2(n+1)}}-\frac{\phi(2^n s)}{2^{2n}}}(t) &\geq \mu'_{\varphi(0,2^n s,0,\cdots,0)}\left(2^{2(n+1)}(m^2-5m+2)t\right)\\ &\geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2-5m+2)\frac{2^{2(n+1)}}{\rho^n}t\right)\end{aligned}$$

for all $s \in E$. The rest of the proof is similar to the proof of Theorem 5.1. □

Corollary 5.4. *Let E be a real normed linear space, $(Z, \mu', \min)$ be a random normed space and $(F, \mu, \min)$ be a complete random normed space. Let p be a positive real number with $0 < p < 2$, $z_0 \in Z$ and $\phi : E \to F$ be a mapping satisfying (4.12). Then the limit $Q_2(s) = \lim_{n\to\infty} \frac{\phi(2^n s)}{2^{2n}}$ exists for all $s \in E$ and defines a unique quadratic mapping $Q_2 : E \to F$ such that*

$$\mu_{\phi(s)-Q_2(s)}(t) \geq \mu'_{\|s\|^p z_0}\left((m^2-5m+2)(2^2-2^p)t\right)$$

for all $s \in E$ and $t > 0$.

Proof. Let $\rho = 2^p$ and $\varphi : E^m \to Z$ be a mapping defined by $\varphi(s_1, s_2, \cdots, s_m) = \left(\sum_{j=1}^m \|s_j\|^p\right) z_0$. Then, from Theorem 5.3, the conclusion follows. □

6. Main results for mixed case

In this section, we investigate the Ulam stability of the finite variable functional equation (1.2) for mixed case in random normed spaces by using the Hyers method.

Theorem 6.1. *Let E be a real linear space, $\left(Z, \mu', \min\right)$ be a random normed space and $\varphi : E^m \to Z$ be a function for which there exists $0 < \rho < \frac{1}{2^2}$ such that (4.1) and $\lim_{n\to\infty} \mu'_{\varphi\left(\frac{s_1}{2^n}, \frac{s_2}{2^n}, \cdots, \frac{s_m}{2^n}\right)}\left(\frac{t}{2^n}\right) = 1$ and $\lim_{n\to\infty} \mu'_{\varphi\left(\frac{s_1}{2^n}, \frac{s_2}{2^n}, \cdots, \frac{s_m}{2^n}\right)}\left(\frac{t}{2^{2n}}\right) = 1$ for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$. Let $(F, \mu, \min)$ be a complete random normed space. If $\phi : E \to F$ is a mapping with $\phi(0) = 0$ satisfying (4.2), then the limits $Q_2(s) = \lim_{n\to\infty} 2^{2n}\phi\left(\frac{s}{2^n}\right)$ and $A_1(s) = \lim_{n\to\infty} 2^n\phi\left(\frac{s}{2^n}\right)$ exist for all $s \in E$ and define a unique quadratic mapping $Q_2 : E \to F$ and a unique additive mapping $A_1 : E \to F$ such that*

$$\mu_{\phi(s)-Q_2(s)-A_1(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2-5m+2)\frac{(1-2^2\rho)}{\rho}t + (m^2-5m+4)\frac{(1-2\rho)}{\rho}t\right)$$

for all $s \in E$ and $t > 0$.

Proof. By Theorem 3.3, $\phi(s) = \phi_e(s) + \phi_o(s)$, where

$$\phi_e(s) = \frac{\phi(s) + \phi(-s)}{2}, \quad \phi_o(s) = \frac{\phi(s) - \phi(-s)}{2}$$

for all $s \in E$, respectively. So

$$\mu_{D\phi_e(s_1,s_2,\cdots,s_m)}(t) \geq \frac{1}{2}\left[\mu_{D\phi(s_1,s_2,\cdots,s_m)}(t) + \mu_{D\phi(-s_1,-s_2,\cdots,-s_m)}(t)\right]$$

and

$$\mu_{D\phi_o(s_1,s_2,\cdots,s_m)}(t) \geq \frac{1}{2}\left[\mu_{D\phi(s_1,s_2,\cdots,s_m)}(t) - \mu_{D\phi(-s_1,-s_2,\cdots,-s_m)}(t)\right]$$

for all $s_1, s_2, \cdots, s_m \in E$. Now,

$$\mu_{\phi(s)-Q_2(s)-A_1(s)}(t) = \mu_{\phi_e(s)+\phi_o(s)-Q_2(s)-A_1(s)}(t) \geq \mu_{\phi_e(s)-Q_2(s)}(t) + \mu_{\phi_o(s)-A_1(s)}(t)$$

for all $s \in E$ and $t > 0$. Using Theorems 4.1 and 5.1, we can complete the remaining proof of the theorem. □

Corollary 6.2. *Let E be a real normed linear space, $(Z, \mu', \min)$ be a random normed space and $(F, \mu, \min)$ be a complete random normed space. Let p be a positive real number with $p > 2$, $z_0 \in Z$ and $\phi : E \to F$ be a mapping with $\phi(0) = 0$ satisfying (4.12) for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$. Then the limits $Q_2(s) = \lim_{n\to\infty} 2^{2n}\phi\left(\frac{s}{2^n}\right)$ and $A_1(s) = \lim_{n\to\infty} 2^n\phi\left(\frac{s}{2^n}\right)$ exist for all $s \in E$ and define a unique quadratic mapping $Q_2 : E \to F$ and a unique additive mapping $A_1 : E \to F$ such that*

$$\mu_{\phi(s)-Q_2(s)-A_1(s)}(t) \geq \mu'_{\|s\|^p z_0}\left((m^2 - 5m + 2)(2^p - 2^2)t + (m^2 - 5m + 4)(2^p - 2)t\right)$$

for all $s \in E$ and $t > 0$.

Proof. Let $\rho = 2^{-p}$ and $\varphi : E^m \to Z$ be a mapping defined by $\varphi(s_1, s_2, \cdots, s_m) = \left(\sum_{j=1}^m \|s_j\|^p\right) z_0$. Then, from Theorem 6.1, the conclusion follows. □

Theorem 6.3. *Let E be a real linear space, $\left(Z, \mu', \min\right)$ be a random normed space and $\varphi : E^m \to Z$ be a function for which there exists $0 < \rho < 2$ such that $\mu'_{\varphi(2s_1,2s_2,\cdots,2s_m)}(t) \geq \mu'_{\rho\varphi(s_1,s_2,\cdots,s_m)}(t)$ for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$ and $\lim_{n\to\infty} \mu'_{\varphi(2^n s_1,2^n s_2,\cdots,2^n s_m)}\left(2^{2n}t\right) = 1$ and $\lim_{n\to\infty} \mu'_{\varphi(2^n s_1,2^n s_2,\cdots,2^n s_m)}(2^n t) = 1$ for all $s_1, s_2, \cdots, s_m \in E$ and $t > 0$. Let $(F, \mu, \min)$ be a complete random normed space. If $\phi : E \to F$ is a mapping with $\phi(0) = 0$ satisfying (4.2), then the limits $Q_2(s) = \lim_{n\to\infty} \frac{\phi(2^n s)}{2^{2n}}$ and $A_1(s) = \lim_{n\to\infty} \frac{\phi(2^n s)}{2^n}$ exist for all $s \in E$ and define a unique quadratic mapping $Q_2 : E \to F$ and a unique additive mapping $A_1 : E \to F$ such that*

$$\mu_{\phi(s)-Q_2(s)-A_1(s)}(t) \geq \mu'_{\varphi(0,s,0,\cdots,0)}\left((m^2 - 5m + 2)(2^2 - \rho)t + (m^2 - 5m + 4)(2 - \rho)t\right)$$

for all $s \in E$ and $t > 0$.

Proof. Using Theorems 4.3 and 5.3, in a similar manner of Theorem 6.1, we can complete the proof of the theorem. □

Corollary 6.4. *Let E be a real normed linear space, $(Z, \mu^{'}, \min)$ be a random normed space and $(F, \mu, \min)$ be a complete random normed space. Let p be a positive real number with $0 < p < 1$, $z_0 \in Z$ and $\phi : E \to F$ be a mapping with $\phi(0) = 0$ satisfying (4.12). Then the limits $Q_2(s) = \lim_{n\to\infty} \frac{\phi(2^n s)}{2^{2n}}$ and $A_1(s) = \lim_{n\to\infty} \frac{\phi(2^n s)}{2^n}$ exist for all $s \in E$ and define a unique quadratic mapping $Q_2 : E \to F$ and a unique additive mapping $A_1 : E \to F$ such that*

$$\mu_{\phi(s)-Q_2(s)-A_1(s)}(t) \geq \mu^{'}_{||s||^p z_0}\left((m^2 - 5m + 2)(2^2 - 2^p)t + (m^2 - 5m + 4)(2 - 2^p)t\right)$$

for all $s \in E$ and $t > 0$.

Proof. Let $\rho = 2^p$ and $\varphi : E^m \to Z$ be a mapping defined by $\varphi(s_1, s_2, \cdots, s_m) = \left(\sum_{j=1}^m ||s_j||^p\right) z_0$. Then, from Theorem 6.3, the conclusion follows. □

7. Conclusions

We have dealt with a new finite variable mixed type quadratic-additive functional equation (1.2) to obtain its solution. We employed the algorithm of the powerful tool (direct method) devised by Hyers to achieve our main results of Ulam stability of a finite variable mixed type functional equation (1.2).

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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Conflict of interest

The authors declare that they have no competing interests.

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