Advances in Difference Equations
a SpringerOpen Journal

# Hermite–Hadamard-type inequalities for $\eta_h$-convex functions via $\psi$-Riemann–Liouville fractional integrals

Choonkil Park[1], Yu-Ming Chu[2*], Muhammad Shoaib Saleem[3], Sana Mukhtar[3] and Nasir Rehman[4]

*Correspondence:
chuyuming2005@126.com
[2]Department of Mathemetics,
Huzhou University, Huzhou, China
Full list of author information is
available at the end of the article

**Abstract**

In this paper, we establish some new Hermite–Hadamard type inequalities involving $\psi$-Riemann–Liouville fractional integrals via $\eta_h$-convex functions. Finally, we give some applications to the special means of real numbers.

**Keywords:** Hermite–Hadamard inequalities; $\psi$-Riemann–Liouville fractional integrals; $\eta_h$-convex functions

## 1 Introduction

The study of fractional calculus, differential equation and inequalities got rapid development in the last few decades. Comparatively, fractional derivatives and integrals express physical phenomena and hereditary properties of various materials in a more precise way than classical derivatives. Classical derivatives are not enough to solve modern problems of engineering, physics, and other applied sciences because of involvement of fractional equations and inequalities.

To overcome this difficulty, many researchers are working on this area of research, see e.g. [1–4]. For more about the topic, we refer to the book [5].

The Hermite–Hadamard type inequalities are considered as one of important inequalities in convex analysis.

The function $g : \varphi \to \mathbb{R}$ is convex if the following inequality holds:

$$g\big(t\beta + (1-t)\gamma\big) \leq tg(\beta) + (1-t)g(\gamma) \tag{1.1}$$

for all $\beta, \gamma \in \varphi$ and $t \in (0,1)$.

Hermite–Hadamard type inequalities have been studied extensively by many researchers, and a significant number of generalizations have appeared in a number of papers on convex analysis, inequality theory, and fractional integrals (see e.g. [6–9]).

The present paper is organized as follows:

In the second section we provide some preliminary material and basic lemmas. The third section is devoted to the main results, whereas in the last section we give some applications to means.

Springer

We will start with some preliminaries and basic lemmas.

## 2 Preliminaries

**Definition 2.1** ([10]) Let $g : \varphi \to \mathbb{R}$ be an extended real-valued function defined on a convex set $\varphi \subset \mathbb{R}^n$. Then the function $g$ is convex on $\varphi$ if

$$g\big(t\beta + (1-t)\gamma\big) \leq tg(\beta) + (1-t)g(\gamma) \tag{2.1}$$

for all $\beta, \gamma \in \varphi$ and $t \in (0,1)$.

**Definition 2.2** ([11]) Choose the functions $g, h : \varphi \subset \mathbb{R} \to \mathbb{R}$ to be nonnegative. Then $g$ is called $h$-convex function if

$$g\big(t\beta + (1-t)\gamma\big) \leq h(t)g(\beta) + h(1-t)g(\gamma) \tag{2.2}$$

for all $\beta, \gamma \in J$ and $t \in [0,1]$.

**Definition 2.3** ([12]) Let $\varphi$ be an interval in real line $\mathbb{R}$. A function $g : \varphi = [\beta, \gamma] \to \mathbb{R}$ is said to be generalized convex with respect to an arbitrary bifunction $\eta(\beta, \gamma) : \mathbb{E} \times \mathbb{E} \to \mathbb{F}$, where $E, F \in \mathbb{R}$, if

$$g\big(t\beta + (1-t)\gamma\big) \leq g(\gamma) + t\eta\big(g(\beta), g(\gamma)\big) \tag{2.3}$$

for all $\beta, \gamma \in \varphi, t \in [0,1]$.

**Definition 2.4** A function $\eta$ is said to be additive if

$$\eta(\beta_1, \gamma_1) + \eta(\beta_2, \gamma_2) = \eta(\beta_1 + \beta_2, \gamma_1 + \gamma_2) \tag{2.4}$$

for all $\beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{R}$.

**Definition 2.5** A function $\eta$ is said to be nonnegative homogeneous if

$$\eta(t\beta_1, t\beta_2) = t\eta(\beta_1, \beta_2) \tag{2.5}$$

for all $\beta_1, \beta_2 \in \mathbb{R}$.

**Definition 2.6** ([13]) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f$ and $g$ are real functions defined on $[\beta, \gamma]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[\beta, \gamma]$, $q > 1$, then the following inequality is called Holder inequality for integrals:

$$\int_\beta^\gamma \big|f(x)g(x)\big|\,dx \leq \left( \int_\beta^\gamma \big|f(x)\big|^p\,dx \right)^{\frac{1}{p}} \left( \int_\beta^\gamma \big|g(x)\big|^q\,dx \right)^{\frac{1}{q}}, \tag{2.6}$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

*Remark* 2.7 If we get $|f||g| = (|f|^{\frac{1}{p}})(|f|^{\frac{1}{q}}|g|)$ in the Holder inequality, then we obtain the following power-mean integral inequality as a simple result of the Holder inequality.

**Definition 2.8** Let $q > 1$, if $g$ and $h$ are real functions defined on $[\beta, \gamma]$ and if $|f|$, $|f||g|^q$ are integrable unctions on $[\beta, \gamma]$, then the following inequality is called power-mean integral inequality:

$$\int_\beta^\gamma |f(x)g(x)|\, dx \le \left(\int_\beta^\gamma |f(x)|\, dx\right)^{1-\frac{1}{q}} \left(\int_\beta^\gamma |f(x)||g(x)|^q\, dx\right)^{\frac{1}{q}}. \tag{2.7}$$

**Definition 2.9** ([14]) A function $g : \varphi = [\beta, \gamma] \to \mathbb{R}$ is called $\eta_h$ convex function if

$$g(t\beta + (1-t)\gamma) \le g(\gamma) + h(t)\eta(g(\beta), g(\gamma)) \tag{2.8}$$

for all $\beta, \gamma \in \varphi$, $t \in [0,1]$ and $h : J \to \mathbb{R}$ is a nonnegative function.

**Definition 2.10** ([15]) Let $(\beta, \gamma)$ $(-\infty \le \beta < \gamma \le \infty)$ be a finite or infinite interval on the real line $\mathbb{R}$, and let $\alpha > 0$. Also, let $\psi(x)$ be an increasing positive function on $(\beta, \gamma]$ with continuous derivative $\psi'(x)$ on $(\beta, \gamma)$. Then the left- and right-sided $\psi$-Riemann–Liouville fractional integrals of a function g with respect to the function $\psi$ on $[\beta, \gamma]$ are defined by

$$I_{\beta^+}^{\alpha:\psi} g(x) = \frac{1}{\Gamma(\alpha)} \int_\beta^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} g(t)\, dt, \tag{2.9}$$

$$I_{\gamma^-}^{\alpha:\psi} g(x) = \frac{1}{\Gamma(\alpha)} \int_x^\gamma \psi'(t)(\psi(t) - \psi(x))^{\alpha-1} g(t)\, dt. \tag{2.10}$$

The next remark provides the relations among convexities.

*Remark* 2.11  1. If $\eta(\beta, \gamma) = \beta - \gamma$, then (2.3) reduces to (2.1).
  2. If $h(t) = t$, then (2.8) reduces to (2.3).
  3. If $h(t) = t$ and $\eta(\beta, \gamma) = \beta - \gamma$, then (2.8) reduces to (2.1).

The next lemmas are useful in proving the main results.

**Lemma 2.12** ([16]) *Let $g : [\beta, \gamma] \to \mathbb{R}$ be a differentiable mapping on $(\beta, \gamma)$ with $\beta < \gamma$. Also, let $g \in L_1[\beta, \gamma]$. Then we have the following equality for fractional integrals:*

$$\frac{g(\beta) + g(\gamma)}{2} - \frac{\Gamma(\alpha+1)}{2(\gamma-\beta)^\alpha} \left[ I_{\psi^{-1}(\beta)^+}^{\alpha:\psi} (g \circ \psi)(\psi^{-1}(\gamma)) + I_{\psi^{-1}(\gamma)^-}^{\alpha:\psi} (g \circ \psi)(\psi^{-1}(\beta)) \right]$$

$$= \frac{\gamma - \beta}{2} \int_0^1 ((1-t)^\alpha - t^\alpha) g'(t\beta + (1-t)\gamma)\, dt. \tag{2.11}$$

**Lemma 2.13** ([16]) *Let $g : [\beta, \gamma] \to \mathbb{R}$ be a differentiable mapping on $(\beta, \gamma)$ with $\beta < \gamma$. If $g \in L_1[\beta, \gamma]$, then we have the following equality for fractional integrals:*

$$\frac{\Gamma(\alpha+1)}{2(\gamma-\beta)^\alpha} \left[ I_{\psi^{-1}(\beta)^+}^{\alpha:\psi} (g \circ \psi)(\psi^{-1}(\gamma)) + I_{\psi^{-1}(\gamma)^-}^{\alpha:\psi} (g \circ \psi)(\psi^{-1}(\beta)) \right] - g\left(\frac{\beta+\gamma}{2}\right)$$

$$= \frac{\gamma - \beta}{2} \int_0^1 (k + t^\alpha - (1-t)^\alpha) g'(t\beta + (1-t)\gamma)\, dt,$$

*where*

$$
k = \begin{cases} 1, & 0 \le t < 1, \\ -1, & \frac{1}{2} \le t < 1. \end{cases}
\tag{2.12}
$$

## 3  Main results

We are now in a position to establish some inequalities of Hermite–Hadamard type involving $\psi$-Riemann–Liouville fractional integrals with $\alpha \in (0,1)$ via $\eta_h$ convex functions.

**Theorem 3.1** *Let $\alpha \in (0,1)$, let $g : [\beta, \gamma] \to \mathbb{R}$ be a positive function with $0 \le \beta < \gamma$ and $g \in L_1[\beta, \gamma]$, and let $\psi$ be an increasing positive function on $[\beta, \gamma]$ having a continuous derivative $\psi'$ on $(\beta, \gamma)$. If $g$ is an $\eta_h$-convex function on $[\beta, \gamma]$ such that $\eta$ is bounded above on $g([\beta, \gamma]) \times g([\beta, \gamma])$ i.e. $\eta(x, y) \le M$. Then we have the following inequality for fractional integrals*:

$$
g\left(\frac{\beta + \gamma}{2}\right) - h\left(\frac{1}{2}\right)M - \alpha M \int_0^1 t^{\alpha-1} h(t)\, dt
$$
$$
\le \frac{\Gamma(\alpha + 1)}{2(\gamma - \beta)^\alpha} \left[ I^{\alpha:\psi}_{\psi^{-1}(\beta)^+} (g \circ \psi)(\psi^{-1}(\gamma)) + I^{\alpha:\psi}_{\psi^{-1}(\gamma)^-} (g \circ \psi)(\psi^{-1}(\beta)) \right]
$$
$$
\le \frac{g(\beta) + g(\gamma)}{2} + \frac{\alpha}{2} M \int_0^1 t^{\alpha-1} h(t)\, dt.
\tag{3.1}
$$

*Proof* Let $x, y \in [\beta, \gamma]$. Since $g : [\beta, \gamma] \to \mathbb{R}$ is a $\eta_h$ convex function, from 4.1 of [14] we have

$$
g\left(\frac{x + y}{2}\right) - h\left(\frac{1}{2}\right)M \le \frac{g(x) + g(y)}{2} + M \int_0^1 h(t)\, dt,
\tag{3.2}
$$

where M is an upper bound of $\eta$,

$M \ge \eta(g(x), g(y))$ in the first inequality and $M \ge \eta(g(ta + (1 - t)b), g((1 - t)a + tb))$ in the third inequality.

Let $x = t\beta + (1 - t)\gamma$, $y = t\gamma + (1 - t)\beta$ and put $x$, $y$ into (3.2), so we have

$$
g\left(\frac{\beta + \gamma}{2}\right) - h\left(\frac{1}{2}\right)M \le \frac{g(t\beta + (1 - t)\gamma) + g(t\gamma + (1 - t)\beta)}{2} + M \int_0^1 h(t)\, dt
$$
$$
\times 2\left[ g\left(\frac{\beta + \gamma}{2}\right) - h\left(\frac{1}{2}\right)M - M \int_0^1 h(t)\, dt \right]
$$
$$
\le g(t\beta + (1 - t)\gamma) + g(t\gamma + (1 - t)\beta)
$$
$$
\times g(t\beta + (1 - t)\gamma) + g(t\gamma + (1 - t)\beta)
$$
$$
\ge 2\left[ g\left(\frac{\beta + \gamma}{2}\right) - h\left(\frac{1}{2}\right)M - M \int_0^1 h(t)\, dt \right].
\tag{3.3}
$$

Multiplying both sides of (3.3) by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to $t$ over $[0, 1]$, we obtain

$$\int_0^1 t^{\alpha-1} g\big(t\beta + (1-t)\gamma\big) + \int_0^1 t^{\alpha-1} g\big(t\gamma + (1-t)\beta\big)$$

$$\geq 2\left[\int_0^1 t^{\alpha-1} g\left(\frac{\beta+\gamma}{2}\right) - \int_0^1 t^{\alpha-1} h\left(\frac{1}{2}\right)M - M\int_0^1 t^{\alpha-1} h(t)\,dt\right],$$

$$\int_0^1 t^{\alpha-1} g\big(t\beta + (1-t)\gamma\big) + \int_0^1 t^{\alpha-1} g\big(t\gamma + (1-t)\beta\big)$$

$$\geq 2\left[\frac{1}{\alpha} g\left(\frac{\beta+\gamma}{2}\right) - \frac{1}{\alpha} h\left(\frac{1}{2}\right)M - M\int_0^1 t^{\alpha-1} h(t)\,dt\right].$$

(3.4)

Next,

$$\frac{\Gamma(\alpha+1)}{2(\gamma-\beta)^\alpha}\Big[I^{\alpha:\psi}_{\psi^{-1}(\beta)^+}(g\circ\psi)\big(\psi^{-1}(\gamma)\big) + I^{\alpha:\psi}_{\psi^{-1}(\gamma)^-}(g\circ\psi)\big(\psi^{-1}(\beta)\big)\Big]$$

$$= \frac{\Gamma(\alpha+1)}{2(\gamma-\beta)^\alpha}\frac{1}{\Gamma(\alpha)}\left[\int_{\psi^{-1}(\beta)}^{\psi^{-1}(\gamma)} \psi'(v)\big(\gamma - \psi(v)\big)^{\alpha-1}(g\circ\psi)(v)\,dv\right.$$

$$+ \int_{\psi^{-1}(\beta)}^{\psi^{-1}(\gamma)} \psi'(v)\big(\psi(v) - \beta\big)^{\alpha-1}(g\circ\psi)(v)\,dv\bigg]$$

$$= \frac{\alpha}{2}\left[\int_{\psi^{-1}(\beta)}^{\psi^{-1}(\gamma)}\left(\frac{\gamma - \psi(v)}{\gamma-\beta}\right)^{\alpha-1} g\big(\psi(v)\big)\frac{\psi'(v)}{\gamma-\beta}\,dv\right.$$

$$+ \int_{\psi^{-1}(\beta)}^{\psi^{-1}(\gamma)}\left(\frac{\psi(v) - \beta}{\gamma-\beta}\right)^{\alpha-1} g\big(\psi(v)\big)\frac{\psi'(v)}{\gamma-\beta}\,dv\bigg]$$

$$= \frac{\alpha}{2}\left[\int_0^1 t^{\alpha-1} g\big(t\beta + (1-t)\gamma\big)\,dt + \int_0^1 s^{\alpha-1} g\big(s\gamma + (1-s)\beta\big)\,ds\right]$$

let $\big(t = \frac{\psi(v)\gamma}{\beta-\gamma}, s = \frac{\psi(v)-\beta}{\gamma-\beta}\big)$

$$= \frac{\alpha}{2}\left[\int_0^1 t^{\alpha-1} g\big(t\beta + (1-t)\gamma\big)\,dt + \int_0^1 t^{\alpha-1} g\big(t\gamma + (1-t)\beta\big)\,dt\right]$$

$$\geq \frac{\alpha}{2} 2\left[\left(\frac{1}{\alpha} g\left(\frac{\beta+\gamma}{2}\right) - \frac{1}{\alpha} h\left(\frac{1}{2}\right)M - M\int_0^1 t^{\alpha-1} h(t)\,dt\right)\right],$$

$$\frac{\Gamma(\alpha+1)}{2(\gamma-\beta)^\alpha}\Big[I^{\alpha:\psi}_{\psi^{-1}(\beta)^+}(g\circ\psi)\big(\psi^{-1}(\gamma)\big) + I^{\alpha:\psi}_{\psi^{-1}(\gamma)^-}(g\circ\psi)\big(\psi^{-1}(\beta)\big)\Big]$$

$$\geq g\left(\frac{\beta+\gamma}{2}\right) - h\left(\frac{1}{2}\right)M - \alpha M\int_0^1 t^{\alpha-1} h(t)\,dt,$$

where (3.4) is used, so the left-hand side inequality in (3.1) is proved.

To prove the right-hand side inequality in (3.1), since g is a $\eta_h$ convex function, then for $t[0, 1]$ we have

$$g\big(t\beta + (1-t)\gamma\big) \leq g(\gamma) + h(t)\eta\big(g(\beta), g(\gamma)\big)$$

and

$$g\big(t\gamma + (1 - t)\beta\big) \le g(\beta) + h(t)\eta\big(g(\gamma), g(\beta)\big).$$

Now,

$$g\big(t\beta + (1 - t)\gamma\big) + g\big(t\gamma + (1 - t)\beta\big) \le g(\beta) + g(\gamma) + h(t)\eta\big(g(\beta), g(\gamma)\big)$$
$$+ h(t)\eta\big(g(\gamma), g(\beta)\big). \tag{3.5}$$

Multiplying both sides of (3.5) by $t^{\alpha - 1}$ and then integrating, we obtain

$$\int_0^1 t^{\alpha - 1} g\big(t\beta + (1 - t)\gamma\big) + \int_0^1 t^{\alpha - 1} g\big(t\gamma + (1 - t)\beta\big)$$
$$\le \frac{g(\beta) + g(\gamma)}{\alpha} + \int_0^1 t^{\alpha - 1} h(t)\big[\eta\big(g(\beta), g(\gamma)\big) + \eta\big(g(\gamma), g(\beta)\big)\big]\, dt.$$

So then

$$\frac{\Gamma(\alpha + 1)}{2(\gamma - \beta)^\alpha}\big[I_{\psi^{-1}(\beta)^+}^{\alpha:\psi}(g \circ \psi)\big(\psi^{-1}(\gamma)\big) + I_{\psi^{-1}(\gamma)^-}^{\alpha:\psi}(g \circ \psi)\big(\psi^{-1}(\beta)\big)\big]$$
$$= \frac{\alpha}{2}\left[\int_0^1 t^{\alpha - 1} g\big(t\beta + (1 - t)\gamma\big)\, dt + \int_0^1 t^{\alpha - 1} g\big(t\gamma + (1 - t)\beta\big)\, dt\right]$$
$$\le \frac{\alpha}{2}\left[\frac{g(\beta) + g(\gamma)}{\alpha} + \int_0^1 t^{\alpha - 1} h(t)\big(\eta\big(g(\beta), g(\gamma)\big) + \eta\big(g(\gamma), g(\beta)\big)\big)\, dt\right]$$
$$\le \frac{g(\beta) + g(\gamma)}{2} + \frac{\alpha}{2}\int_0^1 t^{\alpha - 1} h(t) M\, dt.$$

The proof is completed.                                                                                                                $\square$

*Remark* 3.2  If we take $h(t) = t$ and $\eta(\beta, \gamma) = \beta - \gamma$, then inequality (3.1) reduces inequality (2) in [17].

**Theorem 3.3** *Let $g : [\beta, \gamma] \to \mathbb{R}$ be a positive function with $0 \le \beta < \gamma$ such that $g \in L_1[\beta, \gamma]$, and let $\psi$ be an increasing positive function on $[\beta, \gamma]$ having a continuous derivative $\psi'$ on $(\beta, \gamma)$. If $h'$ is an $\eta_h$-convex function on $[\beta, \gamma]$ for some fixed $h \in (0, 1]$, then we have the following inequality for fractional integrals:*

$$\left|\frac{\Gamma(\alpha + 1)}{2(\gamma - \beta)^\alpha}\big[I_{\psi^{-1}(\beta)^+}^{\alpha:\psi}(g \circ \psi)\big(\psi^{-1}(\gamma)\big) + I_{\psi^{-1}(\gamma)^-}^{\alpha:\psi}(g \circ \psi)\big(\psi^{-1}(\beta)\big)\big] - g\left(\frac{\beta + \gamma}{2}\right)\right|$$
$$\le \frac{\gamma - \beta}{2}\left\{|g'(\gamma)|\left(\frac{2}{2^\alpha(\alpha + 1)} - \frac{2}{\alpha + 1} + 1\right) + \eta\big(|g'(\beta)|, |g'(\gamma)|\big)\right.$$
$$\left.\times\left[\int_0^{\frac{1}{2}} h(t)\big(1 + t^\alpha - (1 - t)^\alpha\big)\, dt + \int_{\frac{1}{2}}^1 h(t)\big((1 - t)^\alpha + 1 - t^\alpha\big)\, dt\right]\right\}. \tag{3.6}$$

*Proof* Using Lemma (2.13) and the $\eta_h$-convexity of $h$, we have

$$
\frac{\Gamma(\alpha+1)}{2(\gamma-\beta)^\alpha}\Big[I^{\alpha:\psi}_{\psi^{-1}(\beta)^+}(g\circ\psi)\big(\psi^{-1}(\gamma)\big)+I^{\alpha:\psi}_{\psi^{-1}(\gamma)^-}(g\circ\psi)\big(\psi^{-1}(\beta)\big)\Big]-g\left(\frac{\beta+\gamma}{2}\right)
$$

$$
=\frac{\gamma-\beta}{2}\int_0^1\big(k+t^\alpha-(1-t)^\alpha\big)g'\big(t\beta+(1-t)\gamma\big)\,dt
$$

$$
\leq\frac{\gamma-\beta}{2}\left\{\int_0^{\frac{1}{2}}\big(1+t^\alpha-(1-t)^\alpha\big)\Big[\big|g'(\gamma)\big|+h(t)\eta\big(\big|g'(\beta)\big|,\big|g'(\gamma)\big|\big)\Big]\,dt\right.
$$

$$
\left.+\int_{\frac{1}{2}}^1\big(1-t^\alpha+(1-t)^\alpha\big)\Big[\big|g'(\gamma)\big|+h(t)\eta\big(\big|g'(\beta)\big|,\big|g'(\gamma)\big|\big)\Big]\,dt\right\}
$$

$$
=\frac{\gamma-\beta}{2}\left\{\big|g'(\gamma)\big|\int_0^{\frac{1}{2}}\big[1+t^\alpha-(1-t)^\alpha\big]\,dt+\eta\big(\big|g'(\beta)\big|,\big|g'(\gamma)\big|\big)\right.
$$

$$
\times\int_0^{\frac{1}{2}}h(t)\big[1+t^\alpha-(1-t)^\alpha\big]\,dt
$$

$$
+\big|g'(\gamma)\big|\int_{\frac{1}{2}}^1\big[(1-t)^\alpha+1+t^\alpha\big]\,dt+\eta\big(\big|g'(\beta)\big|,\big|g'(\gamma)\big|\big)
$$

$$
\left.\times\int_{\frac{1}{2}}^1h(t)\big[(1-t)^\alpha+1+t^\alpha\big]\,dt\right\}
$$

$$
\leq\frac{\gamma-\beta}{2}\left\{\big|g'(\gamma)\big|\left(\frac{1}{2}-\frac{1}{\alpha+1}+\frac{1}{2^\alpha(\alpha+1)}\right)+\eta\big(\big|g'(\beta)\big|,\big|g'(\gamma)\big|\big)\right.
$$

$$
\times\int_0^{\frac{1}{2}}h(t)\big[1+t^\alpha-(1-t)^\alpha\big]\,dt
$$

$$
+\big|g'(\gamma)\big|\left(\frac{1}{2^\alpha(\alpha+1)}-\frac{1}{\alpha+1}+\frac{1}{2}\right)
$$

$$
\left.+\eta\big(\big|g'(\beta)\big|,\big|g'(\gamma)\big|\big)\int_{\frac{1}{2}}^1h(t)\big[(1-t)^\alpha+1+t^\alpha\big]\,dt\right\}
$$

$$
\leq\frac{\gamma-\beta}{2}\left\{\big|g'(\gamma)\big|\left(\frac{1}{2}-\frac{1}{\alpha+1}+\frac{1}{2^\alpha(\alpha+1)}+\frac{1}{2^\alpha(\alpha+1)}-\frac{1}{\alpha+1}+\frac{1}{2}\right)\right.
$$

$$
+\eta\big(\big|g'(\beta)\big|,\big|g'(\gamma)\big|\big)\left[\int_0^{\frac{1}{2}}h(t)\big(1+t^\alpha-(1-t)^\alpha\big)\,dt\right.
$$

$$
\left.\left.+\int_{\frac{1}{2}}^1h(t)\big((1-t)^\alpha+1-t^\alpha\big)\,dt\right)\right]\right\},
$$

$$
\left|\frac{\Gamma(\alpha+1)}{2(\gamma-\beta)^\alpha}\Big[I^{\alpha:\psi}_{\psi^{-1}(\beta)^+}(g\circ\psi)\big(\psi^{-1}(\gamma)\big)+I^{\alpha:\psi}_{\psi^{-1}(\gamma)^-}(g\circ\psi)\big(\psi^{-1}(\beta)\big)\Big]-g\left(\frac{\beta+\gamma}{2}\right)\right|
$$

$$
\leq\frac{\gamma-\beta}{2}\left\{\big|g'(\gamma)\big|\left(\frac{2}{2^\alpha(\alpha+1)}-\frac{2}{\alpha+1}+1\right)+\eta\big(\big|g'(\beta)\big|,\big|g'(\gamma)\big|\big)\right.
$$

$$
\left.\times\left[\int_0^{\frac{1}{2}}h(t)\big(1+t^\alpha-(1-t)^\alpha\big)\,dt+\int_{\frac{1}{2}}^1h(t)\big((1-t)^\alpha+1-t^\alpha\big)\,dt\right]\right\}.
$$

The proof is completed.                                                                                               □

*Remark* 3.4 If we take $h(t) = t$ and $\eta(\beta, \gamma) = \beta - \gamma$, then Theorem 3.3 will be reduced as a result of classical convexity.

**Theorem 3.5** *Let* $g : [\beta, \gamma] \to \mathbb{R}$ *be a positive function with* $0 \leq \beta < \gamma$ *such that* $g \in L_1[\beta, \gamma]$, *and let* $\psi$ *be an increasing positive function on* $[\beta, \gamma]$ *having a continuous derivative* $\psi'$ *on* $(\beta, \gamma)$. *If* $|g'|^q(q > 1)$ *is an* $\eta_h$-*convex function on* $[\beta, \gamma]$ *for some fixed* $h \in (0, 1]$, *then we have the following inequality for fractional integrals:*

$$
\left| \frac{\Gamma(\alpha + 1)}{2(\gamma - \beta)^\alpha} \left[ I^{\alpha:\psi}_{\psi^{-1}(\beta)^+}(g \circ \psi)\left(\psi^{-1}(\gamma)\right) + I^{\alpha:\psi}_{\psi^{-1}(\gamma)^-}(g \circ \psi)\left(\psi^{-1}(\beta)\right) \right] - g\left( \frac{\beta + \gamma}{2} \right) \right|
$$

$$
\leq (\gamma - \beta) \left( \frac{1}{(\alpha p + 1)2^{\alpha p + 1}} \right)^{\frac{1}{p}} \left\{ \left( \frac{1}{2}|g'(\gamma)|^q + \eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \int_0^{\frac{1}{2}} h(t)\, dt \right)^{\frac{1}{q}} \right.
$$

$$
\left. \times \left( \frac{1}{2}|g'(\gamma)|^q + \eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \int_{\frac{1}{2}}^1 h(t)\, dt \right)^{\frac{1}{q}} \right\}, \tag{3.7}
$$

*where* $\frac{1}{p} = 1 - \frac{1}{q}$.

*Proof* Using Lemma (2.13) and the Holder inequality via the $\eta_h$-convexity of $|g'|^q(q > 1)$, we have

$$
\frac{\Gamma(\alpha + 1)}{2(\gamma - \beta)^\alpha} \left[ I^{\alpha:\psi}_{\psi^{-1}(\beta)^+}(g \circ \psi)\left(\psi^{-1}(\gamma)\right) + I^{\alpha:\psi}_{\psi^{-1}(\gamma)^-}(g \circ \psi)\left(\psi^{-1}(\beta)\right) \right] - g\left( \frac{\beta + \gamma}{2} \right)
$$

$$
= \frac{\gamma - \beta}{2} \int_0^1 \left( k + t^\alpha - (1 - t)^\alpha \right) g'\left( t\beta + (1 - t)\gamma \right) dt
$$

$$
\leq \frac{\gamma - \beta}{2} \left\{ \int_0^{\frac{1}{2}} \left( 1 + t^\alpha - (1 - t)^\alpha \right) \left| g'\left( t\beta + (1 - t)\gamma \right) dt \right| dt \right.
$$

$$
\left. + \int_{\frac{1}{2}}^1 \left( 1 - t^\alpha + (1 - t)^\alpha \right) \left| g'\left( t\beta + (1 - t)\gamma \right) dt \right| dt \right\}
$$

$$
\leq \frac{\gamma - \beta}{2} \left\{ \left( \int_0^{\frac{1}{2}} \left( 1 + t^\alpha - (1 - t)^\alpha \right)^p dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{2}} \left| g'\left( t\beta + (1 - t)\gamma \right) dt \right|^q dt \right)^{\frac{1}{q}} \right.
$$

$$
\left. + \left( \int_{\frac{1}{2}}^1 \left( (1 - t)^\alpha + 1 - t^\alpha \right)^p dt \right)^{\frac{1}{p}} \left( \int_{\frac{1}{2}}^1 \left| g'\left( t\beta + (1 - t)\gamma \right) dt \right|^q dt \right)^{\frac{1}{q}} \right\}
$$

$$
\leq \frac{\gamma - \beta}{2} \left( \int_0^{\frac{1}{2}} \left( 1 + t^\alpha - (1 - t)^\alpha \right)^p dt \right)^{\frac{1}{p}}
$$

$$
\times \left\{ \left( \int_0^{\frac{1}{2}} \left[ |g'(\gamma)|^q + h(t)\eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \right] dt \right)^{\frac{1}{q}} \right.
$$

$$
\left. + \left( \int_{\frac{1}{2}}^1 \left[ |g'(\gamma)|^q + h(t)\eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \right] dt \right)^{\frac{1}{q}} \right\}
$$

$$
\leq \frac{\gamma - \beta}{2} \left( \int_0^{\frac{1}{2}} \left( 1 + t^\alpha - (1 - t)^\alpha \right)^p dt \right)^{\frac{1}{p}}
$$

$$\times \left\{ \left( \int_0^{\frac{1}{2}} |g'(\gamma)|^q \, dt + \eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \int_0^{\frac{1}{2}} h(t)\,dt \right)^{\frac{1}{q}} \right.$$
$$\left. + \left( \int_{\frac{1}{2}}^1 |g'(\gamma)|^q \, dt + \eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \int_{\frac{1}{2}}^1 h(t)\,dt \right)^{\frac{1}{q}} \right\}.$$

Next,

$$\leq \frac{\gamma - \beta}{2} \left( \int_0^{\frac{1}{2}} \left(1 + t^\alpha - (1-t)^\alpha\right)^p \, dt \right)^{\frac{1}{p}}$$
$$\times \left\{ \left( \frac{1}{2}|g'(\gamma)|^q \, dt + \eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \int_0^{\frac{1}{2}} h(t)\,dt \right)^{\frac{1}{q}} \right.$$
$$\left. + \left( \frac{1}{2}|g'(\gamma)|^q \, dt + \eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \int_{\frac{1}{2}}^1 h(t)\,dt \right)^{\frac{1}{q}} \right\}$$
$$\leq \frac{\gamma - \beta}{2} \left( 2^p \int_0^{\frac{1}{2}} t^{\alpha p} \, dt \right)^{\frac{1}{p}} \left\{ \left( \frac{1}{2}|g'(\gamma)|^q \, dt + \eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \int_0^{\frac{1}{2}} h(t)\,dt \right)^{\frac{1}{q}} \right.$$
$$\left. + \left( \frac{1}{2}|g'(\gamma)|^q \, dt + \eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \int_{\frac{1}{2}}^1 h(t)\,dt \right)^{\frac{1}{q}} \right\}$$
$$\leq (\gamma - \beta) \left( \frac{1}{(\alpha p + 1)2^{\alpha p + 1}} \right)^{\frac{1}{p}} \left\{ \left( \frac{1}{2}|g'(\gamma)|^q + \eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \int_0^{\frac{1}{2}} h(t)\,dt \right)^{\frac{1}{q}} \right.$$
$$\left. \left( \frac{1}{2}|g'(\gamma)|^q + \eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \int_{\frac{1}{2}}^1 h(t)\,dt \right)^{\frac{1}{q}} \right\}.$$

The proof is completed. □

*Remark* 3.6 When we take $h(t) = t$ and $\eta(\beta, \gamma) = \beta - \gamma$, Theorem 3.5 will be reduced as a result of classical convexity.

**Corollary 3.7** *Let $g : [\beta, \gamma] \to \mathbb{R}$ be a positive function with $0 \leq \beta < \gamma$ such that $g \in L_1[\beta, \gamma]$, and let $\psi$ be an increasing positive function on $[\beta, \gamma]$ having a continuous derivative $\psi'$ on $(\beta, \gamma)$. If $|g'|^q (q > 1)$ is an $\eta_h$-convex function on $[\beta, \gamma]$ for some fixed $h \in (0, 1]$, then we have the following inequality for fractional integrals:*

$$\left| \frac{\Gamma(\alpha + 1)}{2(\gamma - \beta)^\alpha} \left[ I^{\alpha:\psi}_{\psi^{-1}(\beta)^+} (g \circ \psi)\left(\psi^{-1}(\gamma)\right) + I^{\alpha:\psi}_{\psi^{-1}(\gamma)^-} (g \circ \psi)\left(\psi^{-1}(\beta)\right) \right] - g\left( \frac{\beta + \gamma}{2} \right) \right|$$
$$\leq (\gamma - \beta) \left( \frac{1}{(\alpha p + 1)2^{\alpha p + 1}} \right)^{\frac{1}{p}} \left\{ \left( \eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \int_0^{\frac{1}{2}} h(t)\,dt \right)^{\frac{1}{q}} \right.$$
$$\left. + \left( \eta\left(|g'(\beta)|^q, |g'(\gamma)|^q\right) \int_{\frac{1}{2}}^1 h(t)\,dt \right)^{\frac{1}{q}} + 2^{1 - \frac{1}{q}} |g'(\gamma)| \right\}, \tag{3.8}$$

*where $\frac{1}{p} = 1 - \frac{1}{q}$*

*Proof* We consider inequality (3.7), and we let $a_1 = \frac{1}{2}|g'(\gamma)|^q$, $b_1 = \eta(|g'(\beta)|^q, |g'(\gamma)|^q) \times \int_0^{\frac{1}{2}} h(t)\,dt$, $a_2 = \frac{1}{2}|g'(\gamma)|^q$, $b_2 = \eta(|g'(\beta)|^q, |g'(\gamma)|^q) \int_{\frac{1}{2}}^1 h(t)\,dt$. Here, $0 < \frac{1}{q} < 1$ for $q > 1$. Using

the inequality $\Sigma_{i=1}^n (a_i + b_i)^r \leq \Sigma_{i=1}^n a_i^r + \Sigma_{i=1}^n b_i^r$ for $0 < r < 1$, $a_i > 0$, $b_i > 0$, $i = 1, 2, 3, \ldots, n$, we obtain the required result. This completes the proof. $\qquad\qquad\qquad\qquad\square$

**Theorem 3.8** *Let $g : [\beta, \gamma] \rightarrow \mathbb{R}$ be a positive function with $0 \leq \beta < \gamma$ such that $g \in L_1[\beta, \gamma]$, and let $\psi$ be an increasing positive function on $[\beta, \gamma]$ having a continuous derivative $\psi'$ on $(\beta, \gamma)$. If $|g'|^q (q > 1)$ is an $\eta_h$-convex function on $[\beta, \gamma]$ for some fixed $h \in (0, 1]$, then we have the following inequality for fractional integrals:*

$$\left| \frac{\Gamma(\alpha + 1)}{2(\gamma - \beta)^\alpha} \left[ I_{\psi^{-1}(\beta)^+}^{\alpha:\psi} (g \circ \psi)(\psi^{-1}(\gamma)) + I_{\psi^{-1}(\gamma)^-}^{\alpha:\psi} (g \circ \psi)(\psi^{-1}(\beta)) \right] - g\left( \frac{\beta + \gamma}{2} \right) \right|$$

$$\leq \frac{\gamma - \beta}{2} \left( \frac{1}{\alpha + 1} \right)^{1 - \frac{1}{q}} \left( \frac{\alpha - 1}{2} + \frac{1}{2^\alpha} \right)^{1 - \frac{1}{q}} \left\{ \left( |g'(\gamma)|^q \left( \frac{1}{2} - \frac{1}{\alpha + 1} + \frac{1}{2^\alpha(\alpha + 1)} \right) \right. \right.$$

$$+ \eta\left( |g'(\beta)|^q, |g'(\gamma)|^q \right) \int_0^{\frac{1}{2}} h(t) \left[ 1 + t^\alpha - (1 - t)^\alpha \right] dt \right)^{\frac{1}{q}}$$

$$+ \left( |g'(\gamma)|^q \left( \frac{1}{2^\alpha(\alpha + 1)} - \frac{1}{\alpha + 1} + \frac{1}{2} \right) \right.$$

$$\left. \left. + \eta\left( |g'(\beta)|^q, |g'(\gamma)|^q \right) \int_{\frac{1}{2}}^1 h(t) \left[ (1 - t)^\alpha + 1 - t^\alpha \right] dt \right)^{\frac{1}{q}} \right\}. \qquad (3.9)$$

*Proof* Using Lemma (2.13) and the power-mean inequality via the $\eta_h$-convexity of $|g'|^q$ $(q > 1)$, we have

$$\frac{\Gamma(\alpha + 1)}{2(\gamma - \beta)^\alpha} \left[ I_{\psi^{-1}(\beta)^+}^{\alpha:\psi} (g \circ \psi)(\psi^{-1}(\gamma)) + I_{\psi^{-1}(\gamma)^-}^{\alpha:\psi} (g \circ \psi)(\psi^{-1}(\beta)) \right] - g\left( \frac{\beta + \gamma}{2} \right)$$

$$= \frac{\gamma - \beta}{2} \int_0^1 \left( k + t^\alpha - (1 - t)^\alpha \right) g'\left( t\beta + (1 - t)\gamma \right) dt$$

$$\leq \frac{\gamma - \beta}{2} \left\{ \int_0^{\frac{1}{2}} \left( 1 + t^\alpha - (1 - t)^\alpha \right) \left| g'\left( t\beta + (1 - t)\gamma \right) dt \right| dt \right.$$

$$\left. + \int_{\frac{1}{2}}^1 \left( 1 - t^\alpha + (1 - t)^\alpha \right) \left| g'\left( t\beta + (1 - t)\gamma \right) dt \right| dt \right\}$$

$$\leq \frac{\gamma - \beta}{2} \left\{ \left( \int_0^{\frac{1}{2}} \left( 1 + t^\alpha - (1 - t)^\alpha \right) dt \right)^{1 - \frac{1}{q}} \right.$$

$$\times \left( \int_0^{\frac{1}{2}} \left( 1 + t^\alpha - (1 - t)^\alpha \right) \left| g'\left( t\beta + (1 - t)\gamma \right) dt \right|^q dt \right)^{\frac{1}{q}}$$

$$+ \left( \int_{\frac{1}{2}}^1 \left( 1 - t^\alpha + (1 - t)^\alpha \right) dt \right)^{1 - \frac{1}{q}}$$

$$\left. \times \left( \int_{\frac{1}{2}}^1 \left( 1 - t^\alpha + (1 - t)^\alpha \right) \left| g'\left( t\beta + (1 - t)\gamma \right) dt \right|^q dt \right)^{\frac{1}{q}} \right\}$$

$$\leq \frac{\gamma - \beta}{2} \left( \int_0^{\frac{1}{2}} \left( 1 + t^\alpha - (1 - t)^\alpha \right) dt \right)^{1 - \frac{1}{q}}$$

$$\times \left\{ \left( \int_0^{\frac{1}{2}} \left( 1 + t^\alpha - (1 - t)^\alpha \right) \left[ |g'(\gamma)|^q + h(t)\eta\left( |g'(\beta)|^q, |g'(\gamma)|^q \right) \right] dt \right)^{\frac{1}{q}} \right.$$

$$+ \left( \int_{\frac{1}{2}}^{1} \left( (1-t)^{\alpha} + 1 - t^{\alpha} \right) \left[ \left| g'(\gamma) \right|^{q} + h(t)\eta \left( \left| g'(\beta) \right|^{q}, \left| g'(\gamma) \right|^{q} \right) \right] dt \right)^{\frac{1}{q}} \right\}$$

$$\leq \frac{\gamma - \beta}{2} \left( \frac{1}{\alpha + 1} \right)^{1 - \frac{1}{q}} \left( \frac{\alpha - 1}{2} + \frac{1}{2^{\alpha}} \right)^{1 - \frac{1}{q}} \left\{ \left( \left| g'(\gamma) \right|^{q} \int_{0}^{\frac{1}{2}} \left[ 1 + t^{\alpha} - (1-t)^{\alpha} \right] dt \right. \right.$$

$$+ \eta \left( \left| g'(\beta) \right|^{q}, \left| g'(\gamma) \right|^{q} \right) \int_{0}^{\frac{1}{2}} h(t) \left[ 1 + t^{\alpha} - (1-t)^{\alpha} \right] dt \right)^{\frac{1}{q}}$$

$$+ \left( \left| g'(\gamma) \right|^{q} \int_{\frac{1}{2}}^{1} \left[ (1-t)^{\alpha} + 1 - t^{\alpha} \right] dt \right.$$

$$+ \eta \left( \left| g'(\beta) \right|^{q}, \left| g'(\gamma) \right|^{q} \right) \int_{\frac{1}{2}}^{1} h(t) \left[ (1-t)^{\alpha} + 1 - t^{\alpha} \right] dt \right)^{\frac{1}{q}} \right\}$$

$$\leq \frac{\gamma - \beta}{2} \left( \frac{1}{\alpha + 1} \right)^{1 - \frac{1}{q}} \left( \frac{\alpha - 1}{2} + \frac{1}{2^{\alpha}} \right)^{1 - \frac{1}{q}} \left\{ \left( \left| g'(\gamma) \right|^{q} \left( \frac{1}{2} - \frac{1}{\alpha + 1} + \frac{1}{2^{\alpha}(\alpha + 1)} \right) \right. \right.$$

$$+ \eta \left( \left| g'(\beta) \right|^{q}, \left| g'(\gamma) \right|^{q} \right) \int_{0}^{\frac{1}{2}} h(t) \left[ 1 + t^{\alpha} - (1-t)^{\alpha} \right] dt \right)^{\frac{1}{q}}$$

$$+ \left( \left| g'(\gamma) \right|^{q} \left( \frac{1}{2^{\alpha}(\alpha + 1)} - \frac{1}{\alpha + 1} + \frac{1}{2} \right) \right.$$

$$+ \eta \left( \left| g'(\beta) \right|^{q}, \left| g'(\gamma) \right|^{q} \right) \int_{\frac{1}{2}}^{1} h(t) \left[ (1-t)^{\alpha} + 1 - t^{\alpha} \right] dt \right)^{\frac{1}{q}} \right\}. \tag{3.10}$$

This completes the proof. □

*Remark* 3.9　If we take $h(t) = t$ and $\eta(\beta, \gamma) = \beta - \gamma$, then Theorem 3.8 will be reduced as a result of classical convexity.

**Corollary 3.10**　*Let $g : [\beta, \gamma] \to \mathbb{R}$ be a positive function with $0 \leq \beta < \gamma$ such that $g \in L_1[\beta, \gamma]$, and let $\psi$ be an increasing positive function on $[\beta, \gamma]$ having a continuous derivative $\psi'$ on $(\beta, \gamma)$. If $|g'|^q (q > 1)$ is an $\eta_h$-convex function on $[\beta, \gamma]$ for some fixed $h \in (0, 1]$, then we have the following inequality for fractional integrals:*

$$\left| \frac{\Gamma(\alpha + 1)}{2(\gamma - \beta)^{\alpha}} \left[ I_{\psi^{-1}(\beta)^{+}}^{\alpha:\psi} (g \circ \psi) \left( \psi^{-1}(\gamma) \right) + I_{\psi^{-1}(\gamma)^{-}}^{\alpha:\psi} (g \circ \psi) \left( \psi^{-1}(\beta) \right) \right] - g \left( \frac{\beta + \gamma}{2} \right) \right|$$

$$\leq \frac{\gamma - \beta}{2} \left( \frac{1}{\alpha + 1} \right)^{1 - \frac{1}{q}} \left( \frac{\alpha - 1}{2} + \frac{1}{2^{\alpha}} \right)^{1 - \frac{1}{q}} \left\{ \left| g'(\gamma) \right| \left[ \left( \frac{1}{2} - \frac{1}{\alpha + 1} + \frac{1}{2^{\alpha}(\alpha + 1)} \right)^{\frac{1}{q}} \right. \right.$$

$$+ \left( \frac{1}{2^{\alpha}(\alpha + 1)} - \frac{1}{\alpha + 1} + \frac{1}{2} \right)^{\frac{1}{q}} \right] + b_1^{\frac{1}{q}} + b_2^{\frac{1}{q}} \right\}, \tag{3.11}$$

*where*

$$b_1 = \eta \left( \left| g'(\beta) \right|^{q}, \left| g'(\gamma) \right|^{q} \right) \int_{0}^{\frac{1}{2}} h(t) \left[ 1 + t^{\alpha} - (1-t)^{\alpha} \right] dt,$$

$$b_2 = \eta \left( \left| g'(\beta) \right|^{q}, \left| g'(\gamma) \right|^{q} \right) \int_{\frac{1}{2}}^{1} h(t) \left[ (1-t)^{\alpha} + 1 - t^{\alpha} \right] dt.$$

*Proof* We can obtain the result using the technique in the proof of Corollary 3.7 by considering inequality (3.9).    □

## 4  Application to some special means

Bivariate means are with respect to two elements. Consider the following bivariate means (see [18]) for arbitrary $m, n \in \mathbb{R}$, $m \neq n$:

the harmonic mean

$$H(m, n) = \frac{2}{\frac{1}{m} + \frac{1}{n}}, \quad m, n \in \mathbb{R}\backslash\{0\},$$

the arithmetic mean

$$A(m, n) = \frac{m + n}{2}, \quad m, n \in \mathbb{R},$$

the logarithmic mean

$$L(m, n) = \frac{n - m}{\ln|n| - \ln|m|}, \quad |m| \neq |n|, mn \neq 0,$$

and the r-logarithmic mean

$$L_r(m, n) = \left[ \frac{n^{r+1} - m^{r+1}}{(r+1)(n-m)} \right]^{\frac{1}{r}}, \quad r \in Z\backslash\{-1, 0\}, m, n \in \mathbb{R}, m \neq n.$$

Now we give some applications to the special means of a real number.

**Proposition 4.1** *Let $m, n \in \mathbb{R}$, $m < n$, $r \in Z$, $|r| \geq 2$, $h \in (0, 1]$, and $q > 1$. Then*

$$
\left| L_r^r(m, n) - A^r(m, n) \right| \leq
\begin{cases}
(\frac{n-m}{2})\{\frac{1}{2}|rn^{r-1}| + \eta(|rm^{r-1}|, |rn^{r-1}|) \\
\quad \times [\int_0^{\frac{1}{2}} h(t)(1 + t^\alpha - (1-t)^\alpha)\, dt \\
\quad + \int_{\frac{1}{2}}^1 h(t)((1-t)^\alpha + 1 - t^\alpha)\, dt]\}, \\
(n-m)(\frac{1}{(p+1)2^{p+1}})^{\frac{1}{p}}\{2^{\frac{1}{p}}(|rn^{r-1}|) + (\eta(|rm^{r-1}|, |rn^{r-1}|) \\
\quad \times \int_0^{\frac{1}{2}} h(t)\, dt)^{\frac{1}{q}} + (\eta(|rm^{r-1}|, |rn^{r-1}|) \int_{\frac{1}{2}}^1 h(t)\, dt)^{\frac{1}{q}}\}, \\
(\frac{n-m}{2})(\frac{1}{4})^{\frac{1}{p}}\{2|rn^{r-1}|A((\frac{1}{4})^{\frac{1}{q}}, (\frac{1}{4})^{\frac{1}{q}}) \\
\quad + (\eta(|rm^{r-1}|, |rn^{r-1}|) \int_0^{\frac{1}{2}} h(t)[1 + t^\alpha - (1-t)^\alpha]\, dt)^{\frac{1}{q}} \\
\quad + (\eta(|rm^{r-1}|, |rn^{r-1}|) \int_{\frac{1}{2}}^1 h(t)[(1-t)^\alpha + 1 - t^\alpha]\, dt)^{\frac{1}{q}}\},
\end{cases}
$$

*where $\frac{1}{p} = 1 - \frac{1}{q}$*

*Proof* Applying Theorem 3.3, Corollary 3.7, and Corollary 3.10 respectively for $g(x) = x^r$, $\psi)(x) = x$, and $\alpha = 1$, we immediately obtain the result.    □

**Proposition 4.2** *Let $m, n \in \mathbb{R}$, $m < n$, $r \in Z$, $|r| \geq 2$, $h \in (0,1]$, and $q > 1$. Then*

$$\left| L^{-1}(m,n) - H^r\left(m^{-1}, n^{-1}\right) \right|$$

$$\leq \begin{cases} (\frac{n-m}{2})\{\frac{1}{2}|\frac{1}{n^2}| + \eta(|\frac{1}{m^2}|, |\frac{1}{n^2}|) \\ \quad \times [\int_0^{\frac{1}{2}} h(t)(1 + t^\alpha - (1-t)^\alpha)\, dt + \int_{\frac{1}{2}}^1 h(t)((1-t)^\alpha + 1 - t^\alpha)\, dt]\}, \\ (n-m)(\frac{1}{(p+1)2^{p+1}})^{\frac{1}{p}}\{2^{\frac{1}{p}}(|\frac{1}{n^2}|) + (\eta(|\frac{1}{m^2}|, |\frac{1}{n^2}|) \\ \quad \times \int_0^{\frac{1}{2}} h(t)\, dt)^{\frac{1}{q}} + (\eta(|\frac{1}{m^2}|, |\frac{1}{n^2}|) \int_{\frac{1}{2}}^1 h(t)\, dt)^{\frac{1}{q}}\}, \\ (\frac{n-m}{2})(\frac{1}{4})^{\frac{1}{p}}\{2|\frac{1}{n^2}|A((\frac{1}{4})^{\frac{1}{q}}, (\frac{1}{4})^{\frac{1}{q}}) \\ \quad + (\eta(|\frac{1}{m^2}|, |\frac{1}{n^2}|) \int_0^{\frac{1}{2}} h(t)[1 + t^\alpha - (1-t)^\alpha]\, dt)^{\frac{1}{q}} \\ \quad + (\eta(|\frac{1}{m^2}|, |\frac{1}{n^2}|) \int_{\frac{1}{2}}^1 h(t)[(1-t)^\alpha + 1 - t^\alpha]\, dt)^{\frac{1}{q}}\}, \end{cases}$$

*where $\frac{1}{p} = 1 - \frac{1}{q}$.*

*Proof* Applying Theorem 3.3, Corollary 3.7, and Corollary 3.10 respectively for $g(x) = \frac{1}{x}$, $\psi)(x) = x$, and $\alpha = 1$, we immediately obtain the result. □

## 5 Conclusion

In this article we established the Hermite–Hadamard type inequalities for $\eta_h$ convex functions. The main motivation of the article is [16]. The Hermite–Hadamard inequality derived here involved $\psi$-Riemann–Liouville fractional integrals. We also give some application of our results. Hopefully, the idea used in this paper will be interesting for the research of integral inequality and fractional calculus.

**Author details**
[1]Department of Mathematics, Hanyang University, Seoul 04763, Korea. [2]Department of Mathemetics, Huzhou University, Huzhou, China. [3]Department of Mathematics, University of Okara, Okara, Pakistan. [4]Department of Mathematics, Allama Iqbal Open University H-8, Islamabad, Pakistan.

**Publisher's Note**
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

**References**
1. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)
2. Chen, S.B., Rashid, S., Noor, M.A., Hammouch, Z., Chu, Y.M.: New fractional approaches for n-polynomial P-convexity with applications in special function theory. Adv. Differ. Equ. **2020**, 543 (2020). https://doi.org/10.1186/s13662-020-03000-5

3.  Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives. Theory and Application. Gordon & Breach, Yverdon (1993)
4.  Ozdemir, M.E., Avci, M., Kavrmaci, H.: Hermite-Hadamard-type inequalities via ($\alpha$,m)-convexity. Comput. Math. Appl. **61**(9), 2614–2620 (2011)
5.  Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
6.  Wang, J., Li, X., Zhu, C.: Refinements of Hermite-Hadamard type inequalities involving fractional integrals. Bull. Belg. Math. Soc. Simon Stevin **20**, 655–666 (2013)
7.  Dragomir, S.S., Fitzpatrick, S.: The Hadamard's inequality for *s*-convex functions in the second sense. Demonstr. Math. **32**, 687–696 (1999)
8.  Mitrinovic, D.S., Lackovic, I.B.: Hermite and convexity. Aequ. Math. **28**, 229–232 (1985)
9.  Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N.: Hermite-Hadamard's inequalities for fractional integral and related fractional inequalities. Math. Comput. Model. **57**, 2403–2407 (2013)
10. Mordukhovich, B.S., Nam, N.M.: An Easy Path to Convex Analysis and Applications. Morgan and Claypool (2014)
11. Varošanec, S.: On *h*-convexity. J. Math. Anal. Appl. **326**(1), 303–311 (2007)
12. Gordji, M.E., Delavar, M.R., Sen, M.D.: On $\psi$ convex functions. J. Math. Inequal. **10**(1), 173–183 (2016)
13. Mitrinovic, D.S., Pecaric, J.E., Fink, A.M.: Classical and New Inequalities in Analysis. Kluwer Academic, Dordrecht (1993)
14. Khurshid, Y., Adil Khan, M., Chu, Y.-M.: Conformable integral version of Hermite–Hadamard–Fejér inequalities via $\eta$-convex functions. AIMS Math. **5**(5), 5106–5120 (2020)
15. da Sousa, J.V., de Oliveira, E.C.: On the $\psi$-Hilfer fractional derivative. Commun. Nonlinear Sci. Numer. Simul. **60**, 72–91 (2018)
16. Zhao, Y., Sang, H., Xiong, W.: Hermite-Hadamard-type inequalities involving $\psi$-Riemann-Liouville fractional integrals via *s*-convex functions. J. Inequal. Appl. **2020**, 128 (2020)
17. Liu, K., Wang, J.: On the Hermite-Hadamard-type inequalities involving $\psi$-Riemann-Liouville fractional integrals via convex functions. J. Inequal. Appl. **2019**, 27 (2019)
18. Pearce, C.E., Pecaric, J.: Inequalities for differentiable mappings with application to special means and quadrature formulae. Appl. Math. Lett. **13**(2), 51–55 (2000)