Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 7161–7174.
DOI: 10.3934/math.2020458
Received: 10 June 2020
Accepted: 02 September 2020
Published: 10 September 2020

Research article

Fuzzy normed spaces and stability of a generalized quadratic functional equation

Choonkil Park[1], **K. Tamilvanan**[2], **Batool Noori**[3], **M. B. Moghimi**[3] **and Abbas Najati**[3,*]

[1] Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
[2] Department of Mathematics, Government Arts College for Men, Krishnagiri, Tamilnadu 635001, India
[3] Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran

* **Correspondence:** Email: a.nejati@yahoo.com.

Abstract: In this paper, we acquire the general solution of the generalized quadratic functional equation

$$\sum_{1\leq a<b<c\leq m} \varphi(r_a+r_b+r_c) = (m-2)\sum_{1\leq a<b\leq m} \varphi(r_a+r_b) - \left(\frac{m^2-3m+2}{2}\right)\sum_{a=1}^{m} \frac{\varphi(r_a)+\varphi(-r_a)}{2}$$

where $m \geqslant 3$ is an integer. We also investigate Hyers-Ulam stability results by means of using alternative fixed point theorem for this generalized quadratic functional equation.

Keywords: Hyers-Ulam stability; fuzzy Banach spaces; quadratic mapping
Mathematics Subject Classification: 46S40, 39B52, 39B82, 26E50, 46S50

1. Introduction

A function $f : \mathcal{X} \to \mathcal{Y}$ between real vector spaces is called a quadratic function if

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x, y \in \mathcal{X}. \tag{1.1}$$

The functional equation (1.1) is called a quadratic functional equation. Quadratic functional equations play an important role in the characterization of inner product spaces. The quadratic functional equation arises from the parallelogram equality in inner product spaces. For more general information on this subject, we refer the reader to [1, 9, 11, 15, 17, 22].

The examination of stability issues for functional equations is identified with an inquiry of Ulam [26] regarding the stability of group homomorphisms, which was positively replied for Banach spaces by Hyers [8]. Later, the consequence of Hyers was generalized by Aoki [2] and Rassias [21] for additive and linear mappings, respectively, by permitting the Cauchy difference to be unbounded. Găvruta [7] stated a generalization of the Rassias theorem by replacing the unbounded Cauchy difference by a general control function. The Hyers-Ulam stability and the generalized Hyers-Ulam stability problem for the quadratic functional equation (1.1) were studied by several mathematicians (cf. [4, 10, 25]).

The stability problems of several functional equations in the setting of fuzzy normed spaces have been extensively investigated by a number of authors. We refer the interested reader to [12, 13, 18].

In this paper, we acquire the general solution of the generalized quadratic functional equation

$$\sum_{1\leq a<b<c\leq m} \varphi(r_a + r_b + r_c) = (m-2) \sum_{1\leq a<b\leq m} \varphi(r_a + r_b) - \left(\frac{m^2-3m+2}{2}\right) \sum_{a=1}^{m} \frac{\varphi(r_a) + \varphi(-r_a)}{2} \tag{1.2}$$

where $m \geqslant 3$ is an integer. We also investigate a fuzzy version of the Hyers-Ulam stability for the functional equation (1.2) in fuzzy normed spaces by using the direct method and the fixed point method.

2. Preliminaries

We recall some basic facts concerning fuzzy normed spaces and some preliminary results. We use the definition of fuzzy normed spaces given in [3].

Definition 2.1. [3] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x, t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N_5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N_6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

Example 2.2. [14] Let $(X, ||.||)$ be a normed linear space and $\alpha, \beta > 0$. Define $N : X \times \mathbb{R} \to [0, 1]$ by

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta ||x||}, & t > 0,\ x \in X; \\ 0, & t \leq 0,\ x \in X. \end{cases}$$

It is easy to check that N is fuzzy norm on X.

Example 2.3. [14] Let $(X, ||.||)$ be a normed linear space and $\beta > \alpha > 0$. We define $N : X \times \mathbb{R} \to [0, 1]$ by

$$N(x,t) = \begin{cases} 0, & t \leq \alpha||x||; \\ \frac{t}{t + (\beta-\alpha)||x||}, & \alpha||x|| < t \leq \beta||x||; \\ 1, & t > \beta||x||. \end{cases}$$

It is easy to check that N is fuzzy norm on X.

Example 2.4. Let $(X, \|.\|)$ be a normed linear space and $\alpha > 0$. Define $N : X \times \mathbb{R} \to [0, 1]$ by

$$N(x,t) = \begin{cases} 0, & t \leq 0; \\ \frac{t^\alpha}{t^\alpha + \|x\|^\alpha}, & t > 0. \end{cases}$$

It is easy to check that N is fuzzy norm on X.

Definition 2.5. [3] Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to *be convergent* if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}_{n=1}^{\infty}$ and we denote it by N-$\lim x_n = x$.

It is easy to see that the limit of the convergent sequence $\{x_n\}_{n=1}^{\infty}$ in a fuzzy normed space (X, N) is unique (see [14]).

Definition 2.6. [3] A sequence $\{x_n\}_{n=1}^{\infty}$ in a fuzzy normed space (X, N) is called a *Cauchy sequence* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

The property (N_4) implies that every convergent sequence in a fuzzy normed space is a Cauchy sequence. A fuzzy normed space (X, N) is called a *fuzzy Banach space* if each Cauchy sequence in X is convergent.

Proposition 2.7. *Let $(X, \|.\|)$ be a normed linear space and let $N : X \times \mathbb{R} \to [0, 1]$ be the fuzzy norm defined by*

$$N(x,t) = \begin{cases} \frac{t}{t + \|x\|}, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

Then (X, N) is a fuzzy Banach space if and only if $(X, \|.\|)$ is Banach.

Proof. Suppose that (X, N) is a fuzzy Banach space. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(X, \|.\|)$ and $0 < \varepsilon < 1$. Suppose that $t > 0$ and $\delta = \frac{\varepsilon}{1-\varepsilon}$. Then there exists an $M \in \mathbb{N}$ such that $\|x_{n+p} - x_n\| < t\delta$ for all $n \geq M$ and all $p > 0$. Therefore $\frac{t}{t + \|x_{n+p} - x_n\|} > 1 - \varepsilon$ for all $n \geq M$ and all $p > 0$. Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, N). Then $\lim_{n\to\infty} N(x_n - x, 1) = 1$ for some $x \in X$ and this shows $\lim_{n\to\infty} \|x_n - x\| = 0$.

Conversely, suppose that $(X, \|.\|)$ is a Banach space and $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, N). Let $0 < \delta < 1$ and $\varepsilon = \frac{\delta}{1-\delta}$. Then there exists an $M \in \mathbb{N}$ such that $\frac{1}{1 + \|x_{n+p} - x_n\|} > 1 - \delta$ for all $n \geq M$ and all $p > 0$. So $\|x_{n+p} - x_n\| < \varepsilon$ for all $n \geq M$ and all $p > 0$. Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(X, \|.\|)$. Let $x_n \to x_0 \in X$ (in $\|.\|$) as $n \to \infty$. Then $\lim_{n\to\infty} N(x_n - x_0, t) = 1$ for all $t > 0$. □

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X.

2.1. Generalized metric spaces

Let $\mathcal{X}$ be a set. A function $d : \mathcal{X} \times \mathcal{X} \to [0, +\infty]$ is called a generalized metric on $\mathcal{X}$ if d satisfies

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
3. $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$.

It should be noted that the only difference between the generalized metric and the metric is that the generalized metric accepts the infinity.

We will use the following fundamental result in fixed point theory.

Theorem 2.8. *[5] Let $(\mathcal{X}, d)$ be a generalized complete metric space and $\Lambda : \mathcal{X} \to \mathcal{X}$ be a strictly contractive function with the Lipschitz constant $L < 1$. Suppose that for a given element $a \in \mathcal{X}$ there exists a nonnegative integer k such that $d(\Lambda^{k+1}a, \Lambda^k a) < \infty$. Then*

(i) the sequence $\{\Lambda^n a\}_{n=1}^{\infty}$ converges to a fixed point $b \in \mathcal{X}$ of Λ;
(ii) b is the unique fixed point of Λ in the set $\mathcal{Y} = \{y \in \mathcal{X} : \ d(\Lambda^k a, y) < \infty\}$;
(iii) $d(y, b) \leqslant \frac{1}{1-L} d(y, \Lambda y)$ for all $y \in \mathcal{Y}$.

3. General solution for the equation (1.2)

In this segment, we achieve the general solution of the even-quadratic functional equation (1.2). For $m = 3$ the functional equation (1.2) is presented as follows:

$$\begin{aligned} \varphi(x + y + z) = \varphi(x + y) + \varphi(y + z) + \varphi(x + z) \\ - \frac{[\varphi(x) + \varphi(y) + \varphi(z)] + [\varphi(-x) + \varphi(-y) + \varphi(-z)]}{2}, \end{aligned} \tag{3.1}$$

where φ is a function between two linear spaces. Letting $x = y = z = 0$ in (3.1), we get $\varphi(0) = 0$. Setting $y = z = 0$ in (3.1), we infer φ is even. Substituting $z = -y$ in (3.1), we conclude φ satisfies (1.1).

It is well known that a quadratic function can be represented as the diagonal of a symmetric biadditive map. In fact a function $\varphi : \mathcal{X} \to \mathcal{Y}$ between two linear spaces $\mathcal{X}$ and $\mathcal{Y}$ is quadratic if and only if there exists a symmetric biadditive map $B : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ such that $\varphi(x) = B(x, x)$ for all $x \in \mathcal{X}$. Therefore a function φ between two linear spaces is quadratic if and only if φ satisfies (3.1).

Theorem 3.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces. A function $\varphi : \mathcal{X} \to \mathcal{Y}$ fulfils the functional equation* (1.2) *if and only if φ is quadratic, i.e., φ fulfils the functional equation* (1.1).

Proof. Suppose that φ fulfils the functional equation (1.2). Substituting $(r_1, r_2, r_3, \ldots, r_m) = (0, 0, 0, \ldots, 0)$ in (1.2), we occur $\varphi(0) = 0$. Exchanging $(r_1, r_2, r_3, \ldots, r_m) = (r, 0, 0, \ldots, 0)$ in (1.2), we get

$$(m-2)\varphi(r) + (m-3)\varphi(r) + \cdots + \varphi(r) = (m-2)(m-1)\varphi(r) - \frac{(m-2)(m-1)}{4}[\varphi(r) + \varphi(-r)].$$

Then $\varphi(-r) = \varphi(r)$ for all $r \in \mathcal{X}$. Thus φ is an even function. Setting $(r_1, r_2, r_3, \ldots, r_m) = (x, y, -x, 0, \ldots, 0)$ and using $\varphi(0) = 0$ with the evenness of φ, a straightforward computation yields

$$\begin{aligned}
\sum_{1\le a<b<c\le m} \varphi\left(r_a+r_b+r_c\right) &= \sum_{c=3}^{m}\varphi(r_1+r_2+r_c)+\sum_{c=4}^{m}\varphi(r_1+r_3+r_c)+\cdots+\sum_{c=m}^{m}\varphi(r_1+r_{m-1}+r_c)\\
&\quad+\sum_{c=4}^{m}\varphi(r_2+r_3+r_c)+\sum_{c=5}^{m}\varphi(r_2+r_4+r_c)+\cdots+\sum_{c=m}^{m}\varphi(r_2+r_{m-1}+r_c)\\
&\quad+\sum_{c=5}^{m}\varphi(r_3+r_4+r_c)+\sum_{c=6}^{m}\varphi(r_3+r_5+r_c)+\cdots+\sum_{c=m}^{m}\varphi(r_3+r_{m-1}+r_c)\\
&=\varphi(y)+(m-3)\varphi(x+y)+(m-4)\varphi(x)+(m-5)\varphi(x)+\cdots+\varphi(x)\\
&\quad+(m-3)\varphi(x-y)+(m-4)\varphi(y)+(m-5)\varphi(y)+\cdots+\varphi(y)\\
&\quad+(m-4)\varphi(x)+(m-5)\varphi(x)+\cdots+\varphi(x)\\
&=(m-4)(m-3)\varphi(x)+\varphi(y)+\frac{(m-4)(m-3)}{2}\varphi(y)+(m-3)\left[\varphi(x+y)+\varphi(x-y)\right],
\end{aligned}$$

on the other hand

$$\begin{aligned}
\sum_{1\le a<b\le m}\varphi\left(r_a+r_b\right) &= \sum_{b=2}^{m}\varphi(r_1+r_b)+\sum_{b=3}^{m}\varphi(r_2+r_b)+\sum_{b=4}^{m}\varphi(r_3+r_b)\\
&=\varphi(x+y)+\varphi(x-y)+2(m-3)\varphi(x)+(m-3)\varphi(y).
\end{aligned}$$

Hence setting $(r_1, r_2, r_3, \ldots, r_m) = (x, y, -x, 0, \ldots, 0)$ in (1.2), we get

$$\begin{aligned}
&(m-4)(m-3)\varphi(x)+\varphi(y)+\frac{(m-4)(m-3)}{2}\varphi(y)+(m-3)\left[\varphi(x+y)+\varphi(x-y)\right]\\
&=(m-2)\left[\varphi(x+y)+\varphi(x-y)+2(m-3)\varphi(x)+(m-3)\varphi(y)\right]-\frac{m^2-3m+2}{2}[2\varphi(x)+\varphi(y)].
\end{aligned}$$

Then

$$\varphi(x+y)+\varphi(x-y)=2\varphi(x)+2\varphi(y).$$

Then φ fulfils the functional equation (1.1).

Conversely, suppose that φ is quadratic. Then φ is even and there exists a symmetric biadditive map $B : \mathcal{X}\times\mathcal{X}\to\mathcal{Y}$ such that $\varphi(x) = B(x, x)$ for all $x\in\mathcal{X}$. So it suffices to show that

$$\sum_{1\le a<b<c\le m}\varphi\left(r_a+r_b+r_c\right)=(m-2)\sum_{1\le a<b\le m}\varphi\left(r_a+r_b\right)-\left(\frac{m^2-3m+2}{2}\right)\sum_{a=1}^{m}\varphi\left(r_a\right). \tag{3.2}$$

To prove (3.2), a straightforward computation (by using B) yields

$$\begin{aligned}\sum_{1\le a<b<c\le m}\varphi\left(r_a+r_b+r_c\right) &= \sum_{c=3}^{m}\varphi(r_1+r_2+r_c)+\sum_{c=4}^{m}\varphi(r_1+r_3+r_c)+\cdots+\sum_{c=m}^{m}\varphi(r_1+r_{m-1}+r_c)\\ &\quad+\sum_{c=4}^{m}\varphi(r_2+r_3+r_c)+\sum_{c=5}^{m}\varphi(r_2+r_4+r_c)+\cdots+\sum_{c=m}^{m}\varphi(r_2+r_{m-1}+r_c)\\ &\quad+\cdots+\sum_{c=m}^{m}\varphi(r_{m-2}+r_{m-1}+r_c)\\ &= \frac{(m-2)(m-1)}{2}\sum_{c=1}^{m}\varphi(r_c)+2(m-2)\left[\sum_{c=2}^{m}B(r_1,r_c)+\cdots+\sum_{c=m}^{m}B(r_{m-1},r_c)\right]\\ &= \frac{(m-2)(m-1)}{2}\sum_{c=1}^{m}\varphi(r_c)+2(m-2)\sum_{1\le a<b\le m}B(r_a,r_b).\end{aligned}$$

On the other hand

$$\begin{aligned}\sum_{1\le a<b\le m}\varphi\left(r_a+r_b\right) &= \sum_{b=2}^{m}\varphi(r_1+r_b)+\sum_{b=3}^{m}\varphi(r_2+r_b)+\cdots+\sum_{b=m}^{m}\varphi(r_{m-1}+r_b)\\ &= (m-1)\sum_{b=1}^{m}\varphi(r_b)+2\sum_{b=2}^{m}B(r_1,r_b)+2\sum_{b=3}^{m}B(r_2,r_b)+\cdots+2\sum_{b=m}^{m}B(r_{m-1},r_b)\\ &= (m-1)\sum_{b=1}^{m}\varphi(r_b)+2\sum_{1\le a<b\le m}B(r_a,r_b).\end{aligned}$$

Then φ satisfies the functional equation (1.2). □

4. Stability results for the functional equation (1.2): Direct method

In the rest of this paper, we take X, (Y, N) and (Z, M) are linear space, fuzzy Banach space and fuzzy normed space, respectively. For notational convenience, we use the following abbreviation for a given mapping $\varphi : X \to Y$

$$\begin{aligned}D\varphi(r_1,r_2,\cdots,r_m) &= \sum_{1\le a<b<c\le m}\varphi\left(r_a+r_b+r_c\right)-(m-2)\sum_{1\le a<b\le m}\varphi\left(r_a+r_b\right)\\ &\quad+\left(\frac{m^2-3m+2}{2}\right)\sum_{a=1}^{m}\frac{\varphi\left(r_a\right)+\varphi\left(-r_a\right)}{2}\end{aligned}$$

for every $r_1, r_2, \cdots, r_m \in X$. In this segment, we examine a fuzzy version of the Hyers–Ulam stability for the functional equation (1.2) in fuzzy normed spaces by means of direct method.

Theorem 4.1. *Let $t \in \{-1, 1\}$ be fixed, also consider $\zeta : X^m \to Z$ be a mapping such that for some $\gamma > 0$ with $\left(\frac{\gamma}{4}\right)^t < 1$*

$$M\left(\zeta\left(2^t r,-2^t r,2^t r,0,\cdots,0\right),\delta\right)\geqslant M\left(\gamma^t\zeta(r,-r,r,0,\cdots,0),\delta\right), \tag{4.1}$$

including

$$\lim_{n\to\infty} M\left(\zeta(2^{tn}r_1, 2^{tn}r_2, \cdots, 2^{tn}r_m), 4^{tn}\delta\right) = 1$$

for all $r, r_1, r_2, \cdots, r_m \in X$ and $\delta > 0$. Suppose an even mapping $\varphi : X \to Y$ with $\varphi(0) = 0$ fulfils the inequality

$$N\left(D\varphi(r_1, r_2, \cdots, r_m), \delta\right) \geqslant M\left(\zeta(r_1, r_2, \cdots, r_m), \delta\right), \tag{4.2}$$

for all $r_1, r_2, \cdots, r_m \in X$ and $\delta > 0$. Then the limit

$$Q(r) = N - \lim_{n\to\infty} \frac{\varphi(2^{tn}r)}{4^{tn}}$$

exists for all $x \in X$ and the mapping $Q : X \to Y$ is a unique quadratic mapping such that

$$N\left(\varphi(r) - Q(r), \delta\right) \geqslant M\left(\zeta(r, -r, r, 0, \cdots, 0), \delta|4 - \gamma|\right), \tag{4.3}$$

for all $r \in X$ and $\delta > 0$.

Proof. Initially we consider $t = 1$. Substituting $(r_1, r_2, \cdots, r_m)$ through $(r, -r, r, 0, \cdots, 0)$ in (4.2), we reach

$$N\left(\varphi(2r) - 4\varphi(r), \delta\right) \geqslant M\left(\zeta(r, -r, r, 0, \cdots, 0), \delta\right), \quad r \in X,\ \delta > 0.$$

Then we have

$$N\left(\frac{\varphi(2r)}{4} - \varphi(r), \frac{\delta}{4}\right) \geqslant M\left(\zeta(r, -r, r, 0, \cdots, 0), \delta\right), \quad r \in X,\ \delta > 0. \tag{4.4}$$

Exchanging r through $2^n r$ in (4.4), we acquire

$$N\left(\frac{\varphi(2^{n+1}r)}{4} - \varphi(2^n r), \frac{\delta}{4}\right) \geqslant M\left(\zeta(2^n r, -2^n r, 2^n r, 0, \cdots, 0), \delta\right), \quad r \in X,\ \delta > 0.$$

Utilizing (4.1) and (N_3) in the above inequality, we reach

$$N\left(\frac{\varphi(2^{n+1}r)}{4^{n+1}} - \frac{\varphi(2^n r)}{4^n}, \frac{\delta}{4^{n+1}}\right) \geqslant M\left(\zeta(r, -r, r, 0, \cdots, 0), \frac{\delta}{\gamma^n}\right), \quad r \in X,\ \delta > 0.$$

Switching δ through $\gamma^n\delta$ in the last inequality, we acquire

$$N\left(\frac{\varphi(2^{n+1}r)}{4^{n+1}} - \frac{\varphi(2^n r)}{4^n}, \frac{\gamma^n\delta}{4^{n+1}}\right) \geqslant M\left(\zeta(r, -r, r, 0, \cdots, 0), \delta\right), \quad r \in X,\ \delta > 0. \tag{4.5}$$

From (4.5), we obtain

$$\begin{aligned} N\left(\frac{\varphi(2^n r)}{4^n} - \varphi(r), \sum_{a=0}^{n-1} \frac{\delta\gamma^a}{4^{a+1}}\right) &= N\left(\sum_{a=0}^{n-1}\left[\frac{\varphi(2^{a+1}r)}{4^{a+1}} - \frac{\varphi(2^a r)}{4^a}\right], \sum_{a=0}^{n-1} \frac{\delta\gamma^a}{4^{a+1}}\right) \\ &\geqslant \min_{0\leqslant a\leqslant n-1} N\left(\frac{\varphi(2^{a+1}r)}{4^{a+1}} - \frac{\varphi(2^a r)}{4^a}, \frac{\delta\gamma^a}{4^{a+1}}\right) \\ &\geqslant M\left(\zeta\left(r, -r, r, 0, \cdots, 0\right), \delta\right), \end{aligned} \tag{4.6}$$

for all $r \in X$, $\delta > 0$ and all $n \in \mathbb{N}$. Substituting r by $2^v r$ in (4.6) and utilizing (4.1) with (N_3), we acquire

$$\begin{aligned} N\left(\frac{\varphi(2^{n+v}r)}{4^{n+v}} - \frac{\varphi(2^v r)}{4^v}, \sum_{a=0}^{n-1} \frac{\delta\gamma^a}{4^{a+v+1}}\right) &\geqslant M\left(\zeta(2^v r, -2^v r, 2^v r, 0, \cdots, 0), \delta\right) \\ &\geqslant M\left(\zeta(r, -r, r, 0, \cdots, 0), \frac{\delta}{\gamma^v}\right), \end{aligned}$$

and so

$$N\left(\frac{\varphi(2^{n+v}r)}{4^{n+v}} - \frac{\varphi(2^v r)}{4^v}, \sum_{a=v}^{n+v-1} \frac{\delta\gamma^a}{4^{a+1}}\right) \geqslant M\left(\zeta(r, -r, r, 0, \cdots, 0), \delta\right)$$

for all $r \in X$, $\delta > 0$ and all integers $v, n \geqslant 0$. Exchanging δ through $\frac{\delta}{\sum_{a=v}^{n+v-1} \frac{\gamma^a}{4^{a+1}}}$ in the last inequality, we obtain

$$N\left(\frac{\varphi(2^{n+v}r)}{4^{n+v}} - \frac{\varphi(2^v r)}{4^v}, \delta\right) \geqslant M\left(\zeta(r, -r, r, 0, \cdots, 0), \frac{\delta}{\sum_{a=v}^{n+v-1} \frac{\gamma^a}{4^{a+1}}}\right), \tag{4.7}$$

for all $r \in X$, $\delta > 0$ and all integers $v, n \geqslant 0$. Since $\sum_{a=0}^{\infty} \left(\frac{\gamma}{4}\right)^a < \infty$, it follows from (4.7) and (N_5) that $\{\frac{\varphi(2^n r)}{4^n}\}_{n=1}^{\infty}$ is a Cauchy sequence in (Y, N) for each $r \in X$. Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $Q(r) \in Y$ for each $r \in X$. So one can define the mapping $Q : X \to Y$ by

$$Q(r) := N - \lim_{n\to\infty} \frac{\varphi(2^n r)}{4^n}, \quad r \in X.$$

Since φ is even, Q is even. Letting $v = 0$ in (4.7), we obtain

$$N\left(\frac{\varphi(2^n r)}{4^n} - \varphi(r), \delta\right) \geqslant M\left(\zeta(r, -r, r, 0, \cdots, 0), \frac{\delta}{\sum_{a=0}^{n-1} \frac{\gamma^a}{4^{a+1}}}\right), \tag{4.8}$$

for all $r \in X$, $\delta > 0$ and all integer $n \geqslant 1$. Then

$$\begin{aligned} N\left(\varphi(r) - Q(r), \delta + \varepsilon\right) &\geqslant \min\left\{N\left(\frac{\varphi(2^n r)}{4^n} - \varphi(r), \delta\right), N\left(\frac{\varphi(2^n r)}{4^n} - Q(r), \varepsilon\right)\right\} \\ &\geqslant \min\left\{M\left(\zeta(r, -r, r, 0, \cdots, 0), \frac{\delta}{\sum_{a=0}^{n-1} \frac{\gamma^a}{4^{a+1}}}\right), N\left(\frac{\varphi(2^n r)}{4^n} - Q(r), \varepsilon\right)\right\}, \end{aligned}$$

for all $r \in X$, $\delta, \varepsilon > 0$ and all integer $n \geqslant 1$. Hence taking the limit as $n \to \infty$ in the last inequality and using (N_6), we get

$$N\left(\varphi(r) - Q(r), \delta + \varepsilon\right) \geqslant M\Big(\zeta(r, -r, r, 0, \cdots, 0), (4 - \gamma)\delta\Big), \quad r \in X,\ \delta, \varepsilon > 0.$$

Taking the limit as $\varepsilon \to 0$, we get (4.3).

Now, we assert that Q is quadratic. It is clear that

$$\begin{aligned} N\left(DQ(r_1, r_2, \cdots, r_m), 2\delta\right) &\geqslant \min\Big\{N\Big(DQ(r_1, r_2, \cdots, r_m) - \frac{1}{4^n} D\varphi(2^n r_1, 2^n r_2, \cdots, 2^n r_m), \delta\Big), \\ &\qquad N\Big(\frac{1}{4^n} D\varphi(2^n r_1, 2^n r_2, \cdots, 2^n r_m), \delta\Big)\Big\} \\ \text{by (4.2)} &\geqslant \min\Big\{N\Big(DQ(r_1, r_2, \cdots, r_m) - \frac{1}{4^n} D\varphi(2^n r_1, 2^n r_2, \cdots, 2^n r_m), \delta\Big), \\ &\qquad M\left(\zeta(2^n r_1, 2^n r_2, \cdots, 2^n r_m), 4^n\delta\right)\Big\}, \quad r \in X,\ \delta > 0. \end{aligned}$$

Since

$$\lim_{n\to\infty} N\Big(DQ(r_1, r_2, \cdots, r_m) - \frac{1}{4^n} D\varphi(2^n r_1, 2^n r_2, \cdots, 2^n r_m), \delta\Big) = 1,$$
$$\lim_{n\to\infty} M\left(\zeta(2^n r_1, 2^n r_2, \cdots, 2^n r_m), 4^n \delta\right) = 1$$

we infer $N\left(DQ(r_1, r_2, \cdots, r_m), 2\delta\right) = 1$ for all $r_1, r_2, \cdots, r_m \in X$ and all $\delta > 0$. Then (N_2) implies $DQ(r_1, r_2, \cdots, r_m) = 0$ for all $r_1, r_2, \cdots, r_m \in X$. Therefore $Q : X \to Y$ is quadratic by Theorem 3.1. To show the uniqueness of Q, let $T : X \to Y$ be another quadratic mapping fulfilling (4.3). Since $Q(2^n r) = 4^n Q(r)$ and $T(2^n r) = 4^n T(r)$ for all $r \in P$ and all $n \in \mathbb{N}$, it follows from (4.3) that

$$\begin{aligned}
N(Q(r) - T(r), \delta) &= N\left(\frac{Q(2^n r)}{4^n} - \frac{T(2^n r)}{4^n}, \delta\right) \\
&\geqslant \min\left\{N\left(\frac{Q(2^n r)}{4^n} - \frac{\varphi(2^n r)}{4^n}, \frac{\delta}{2}\right), N\left(\frac{\varphi(2^n r)}{4^n} - \frac{T(2^n r)}{4^n}, \frac{\delta}{2}\right)\right\} \\
&\geqslant M\left(\zeta(2^n r, -2^n r, 2^n r, 0, \cdots, 0), \frac{4^n(4-\gamma)\delta}{2}\right) \\
&\geqslant M\left(\zeta(r, -r, r, 0, \cdots, 0), \frac{4^n(4-\gamma)\delta}{2\gamma^n}\right)
\end{aligned}$$

for all $r \in X$, $\delta > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{(4^n)(4-\gamma)\delta}{2\gamma^n} = \infty$, we have

$$\lim_{n\to\infty} M\left(\zeta(r, -r, r, 0, \cdots, 0), \frac{4^n(4-\gamma)\delta}{2\gamma^n}\right) = 1.$$

Consequently, $N(Q(r) - T(r), \delta) = 1$ for all $r \in X$ and all $\delta > 0$. So $Q(r) = T(r)$ for all $r \in X$. For $t = -1$, we can demonstrate the consequence through homogeneous procedure. The proof of the theorem is now complete. □

5. Stability results for the functional equation (1.2): A fixed point method

Based on the fixed point alternative, Radu [20] proposed a new method to investigate the stability problem of functional equations. This method has recently been used by many authors (see, e.g., [6,16,19,23,24]). In this segment, we scrutinize the generalized Ulam–Hyers stability of the functional equation (1.2) in fuzzy normed spaces through the fixed point method. First, we define ξ_a as a constant such that

$$\xi_a = \begin{cases} 2 & if \;\; a = 0 \\ \frac{1}{2} & if \;\; a = 1 \end{cases}$$

and we consider $\Lambda = \{g : X \to Y : \; g(0) = 0\}$.

Theorem 5.1. *Let $\varphi : X \to Y$ be an even mapping with $\varphi(0) = 0$ for which there exists a function $\zeta : X^m \to Z$ with condition*

$$\lim_{n\to\infty} M\left(\zeta(\xi_a^n r_1, \xi_a^n r_2, \cdots, \xi_a^n r_m), \xi_a^{2n}\delta\right) = 1, \quad r_1, r_2, \cdots, r_m \in X, \; \delta > 0, \tag{5.1}$$

and satisfying the inequality

$$N(D\varphi(r_1, r_2, \cdots, r_m), \delta) \geqslant M(\zeta(r_1, r_2, \cdots, r_m), \delta), \quad r_1, r_2, \cdots, r_m \in X, \ \delta > 0. \tag{5.2}$$

Let $\phi(r) = \zeta\left(\frac{r}{2}, -\frac{r}{2}, \frac{r}{2}, 0, \cdots, 0\right)$ *for all* $r \in X$*. If there exist* $L = L_a \in (0, 1)$ *such that*

$$M\left(\frac{1}{\xi_a^2}\phi(\xi_a r), \delta\right) \geqslant M\left(L\phi(r), \delta\right), \quad r \in X, \ \delta > 0, \tag{5.3}$$

then there exist a unique quadratic function $Q : X \to Y$ *fulfilling*

$$N(\varphi(r) - Q(r), \delta) \geqslant M\left(\frac{L^{1-a}}{1-L}\phi(r), \delta\right), \quad r \in X, \ \delta > 0. \tag{5.4}$$

Proof. Let γ be the generalized metric on Λ:

$$\gamma(g, h) = \inf\left\{w \in (0, \infty) : \ N(g(r) - h(r), \delta) \geqslant M(w\phi(r), \delta), \ r \in X, \delta > 0\right\},$$

and we take, as usual, $\inf\emptyset = +\infty$. A similar argument provided in [[12], Lemma 2.1] shows that (Λ, γ) is a complete generalized metric space. Define $\Psi_a : \Lambda \longrightarrow \Lambda$ by $\Psi_a g(r) = \frac{1}{\xi_a^2} g(\xi_a r)$ for all $r \in X$. Let g, h in Λ be given such that $\gamma(g, h) \leq \varepsilon$. Then

$$N\left(g(r) - h(r), \delta\right) \geqslant M\left(\varepsilon\phi(r), \delta\right), \quad r \in X, \ \delta > 0,$$

whence

$$N(\Psi_a g(r) - \Psi_a h(r), \delta) \geqslant M\left(\frac{\varepsilon}{\xi_a^2}\phi(\xi_a r), \delta\right), \quad r \in X, \ \delta > 0.$$

It follows from (5.3) that

$$N(\Psi_a g(r) - \Psi_a h(r), \delta) \geqslant M(\varepsilon L\phi(r), \delta), \quad r \in X, \ \delta > 0.$$

Hence, we have $\gamma(\Psi_a g, \Psi_a h) \leq \varepsilon L$. This shows $\gamma(\Psi_a g, \Psi_a h) \leq L\gamma(g, h)$, i.e., Ψ_a is strictly contractive mapping on Λ with the Lipschitz constant L. Substituting $(r_1, r_2, \cdots, r_m)$ by $(r, -r, r, 0, \cdots, 0)$ in (5.2) and utilizing (N_3), we get

$$N\left(\frac{\varphi(2r)}{4} - \varphi(r), \delta\right) \geqslant M\left(\frac{\zeta(r, -r, r, 0, \cdots, 0)}{4}, \delta\right), \quad r \in X, \ \delta > 0. \tag{5.5}$$

Using (5.3) when $a = 0$, it follows from (5.5) that

$$N\left(\frac{\varphi(2r)}{4} - \varphi(r), \delta\right) \geqslant M(L\phi(r), \delta), \quad r \in X, \ \delta > 0.$$

Therefore

$$\gamma(\Psi_0\varphi, \varphi) \leqslant L = L^{1-a}. \tag{5.6}$$

Exchanging r through $\frac{r}{2}$ in (5.5), we obtain

$$N\left(\varphi(r)-4\varphi\left(\frac{r}{2}\right),4\delta\right)\geqslant M\left(\zeta\left(\frac{r}{2},-\frac{r}{2},\frac{r}{2},0,\cdots,0\right),4\delta\right)$$
$$=M(\phi(r),4\delta),\quad r\in X,\ \delta>0.$$

Therefore

$$\gamma(\Psi_1\varphi,\varphi)\leqslant 1=L^{1-a}. \tag{5.7}$$

Then from (5.6) and (5.7), we conclude $\gamma(\Psi_a\varphi,\varphi)\leqslant L^{1-a}<\infty$. Now from the fixed point alternative Theorem 2.8, it follows that there exists a fixed point Q of Ψ_a in Λ such that

(i) $\Psi_a Q=Q$ and $\lim_{n\to\infty}\gamma(\Psi_a^n\varphi,Q)=0$;
(ii) Q is the unique fixed point of Ψ in the set $\mathcal{E}=\{g\in\Lambda:\ d(\varphi,g)<\infty\}$;
(iii) $\gamma(\varphi,Q)\leqslant\frac{1}{1-L}\gamma(\varphi,\Psi_a\varphi)$.

Letting $\gamma(\Psi_a^n\varphi,Q)=\varepsilon_n$, we get $N(\Psi_a^n\varphi(r)-Q(r),\delta)\geqslant M(\varepsilon_n\phi(r),\delta)$ for all $r\in X$ and all $\delta>0$. Since $\lim_{n\to\infty}\varepsilon_n=0$, we infer

$$Q(r)=N-\lim_{n\to\infty}\frac{\varphi(\xi_a^n r)}{\xi_a^{2n}},\quad r\in X.$$

Switching $(r_1,r_2,\cdots,r_m)$ by $(\xi_a^n r_1,\xi_a^n r_2,\cdots,\xi_a^n r_m)$ in (5.2), we obtain

$$N\left(\frac{1}{\xi_a^{2n}}D\varphi(\xi_a^n r_1,\xi_a^n r_2,\cdots,\xi_a^n r_m),\delta\right)\geqslant M(\zeta(\xi_a^n r_1,\xi_a^n r_2,\cdots,\xi_a^n r_m),\xi_a^{2n}\delta),$$

for all $\delta>0$ and all $r_1,r_2,\cdots,r_m\in X$. Using the same argument as in the proof of Theorem 4.1, we can prove the function $Q:X\to Y$ is quadratic. Since $\gamma(\Psi_a\varphi,\varphi)\leqslant L^{1-a}$, it follows from ($iii$) that $\gamma(\varphi,Q)\leqslant\frac{L^{1-a}}{1-L}$ which means (5.4). To prove the uniqueness of Q, let $T:X\to Y$ be another quadratic mapping fulfilling (5.4). Since $Q(2^n r)=4^n Q(r)$ and $T(2^n r)=4^n T(r)$ for all $r\in P$ and all $n\in\mathbb{N}$, we have

$$\begin{aligned}N(Q(r)-T(r),\delta)&=N\left(\frac{Q(2^n r)}{4^n}-\frac{T(2^n r)}{4^n},\delta\right)\\&\geqslant\min\left\{N\left(\frac{Q(2^n r)}{4^n}-\frac{\varphi(2^n r)}{4^n},\frac{\delta}{2}\right),N\left(\frac{\varphi(2^n r)}{4^n}-\frac{T(2^n r)}{4^n},\frac{\delta}{2}\right)\right\}\\&\geqslant M\left(\frac{L^{1-a}}{1-L}\phi(2^n r),\frac{4^n\delta}{2}\right).\end{aligned}$$

By (5.1), we have

$$\lim_{n\to\infty}M\left(\frac{L^{1-a}}{1-L}\phi(2^n r),\frac{4^n\delta}{2}\right)=1.$$

Consequently, $N(Q(r)-T(r),\delta)=1$ for all $r\in X$ and all $\delta>0$. So $Q(r)=T(r)$ for all $r\in X$, which ends the proof. □

The upcoming corollaries are instantaneous outcome of Theorems 4.1 and 5.1, regarding the stability for the Eq (1.2). In the following results, we assume that X, (Y,N) and $(\mathbb{R},M)$ are a linear space, a fuzzy Banach space and a fuzzy normed space, respectively.

Corollary 5.2. *Suppose an even function $\varphi : X \to Y$ fulfils $\varphi(0) = 0$ and the inequality*

$$N(D\varphi(r_1, r_2, \cdots, r_m), \delta) \geqslant M\Big(\tau + \theta \prod_{a=1}^{m} \|r_a\|^q, \delta\Big),$$

for all $r_1, r_2, \cdots, r_m \in X$ and all $\delta > 0$, where τ, θ, q are real constants with $mq \in (0, 2)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N(\varphi(r) - Q(r), \delta) \geqslant M(\tau, 3\delta), \quad r \in X,\ \delta > 0.$$

Corollary 5.3. *Suppose an even function $\varphi : X \to Y$ fulfils $\varphi(0) = 0$ and the inequality*

$$N(D\varphi(r_1, r_2, \cdots, r_m), \delta) \geqslant M\Big(\varepsilon \sum_{a=1}^{m} \|r_a\|^p + \theta \prod_{a=1}^{m} \|r_a\|^q, \delta\Big),$$

for all $r_1, r_2, \cdots, r_m \in X$ and all $\delta > 0$, where ε, θ, p and q are real constants with $p, mq \in (0, 2) \cup (2, +\infty)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N(\varphi(r) - Q(r), \delta) \geqslant M(3\varepsilon\|r\|^p, |4 - 2^p|\delta), \quad r \in X,\ \delta > 0.$$

Corollary 5.4. *Suppose an even function $\varphi : X \to Y$ fulfils $\varphi(0) = 0$ and the inequality*

$$N(D\varphi(r_1, r_2, \cdots, r_m), \delta) \geqslant M\Big(\theta \prod_{a=1}^{m} \|r_a\|^q, \delta\Big),$$

for all $r_1, r_2, \cdots, r_m \in X$ and all $\delta > 0$, where θ and q are real constants with $0 < mq \neq 2$. Then φ is quadratic.

6. Conclusions

Using the direct method and the fixed point method, we have obtained the general solution and have proved the Hyers-Ulam stability of the following generalized quadratic functional equation

$$\sum_{1 \leq a < b < c \leq m} \varphi(r_a + r_b + r_c) = (m - 2) \sum_{1 \leq a < b \leq m} \varphi(r_a + r_b) - \left(\frac{m^2 - 3m + 2}{2}\right) \sum_{a=1}^{m} \frac{\varphi(r_a) + \varphi(-r_a)}{2}$$

where $m \geqslant 3$ is an integer.

Conflict of interest

The authors declare that they have no competing interests.

References

1. D. Amir, *Characterizations of Inner Product spaces*, Birkhäuser, Basel, 1986.
2. T. Aoki, *On the stability of the linear transformation in Banach spaces,* J. Math. Soc. Japan, **2** (1950), 64–66.
3. T. Bag, S. K. Samanta, *Finite dimensional fuzzy normed linear spaces,* J. Fuzzy Math., **11** (2003), 687–706.
4. S. Czerwik, *On the stability of the quadratic mapping in normed spaces,* Abh. Math. Sem. Hamburg, **62** (1992), 59–64.
5. J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space,* Bull. Amer. Math. Soc., **74** (1968), 305–309.
6. M. E. Gordji, A. Najati, *Approximately J^*-homomorphisms: A fixed point approach,* J. Geom. Phys., **60** (2010), 809–814.
7. P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings,* J. Math. Anal. Appl., **184** (1994), 431–436.
8. D. H. Hyers, *On the stability of the linear functional equation,* Proc. Natl. Acad. Sci. USA, **27** (1941), 222–224.
9. P. Jordan, J. Neumann, *On inner products in linear metric spaces,* Ann. Math., **36** (1935), 719–723.
10. S. M. Jung, P. K. Sahoo, *Hyers-Ulam stability of the quadratic equation of Pexider type,* J. Korean Math. Soc., **38** (2001), 645–656.
11. P. Kannappan, *Quadratic functional equation and inner product spaces,* Results Math., **27** (1995), 368–372.
12. D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces,* J. Math. Anal. Appl., **343** (2008), 567–572.
13. A. K. Mirmostafaee, M. S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem,* Fuzzy Set. Syst., **159** (2008), 720–729.
14. A. Najati, *Fuzzy stability of a generalized quadratic functional equation,* Commun. Korean Math. Soc., **25** (2010), 405–417.
15. A. Najati, M. B. Moghimi, *Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces,* J. Math. Anal. Appl., **337** (2008), 399–415.
16. A. Najati, C. Park, *Fixed points and stability of a generalized quadratic functional equation,* J. Inequal. Appl., **2009** (2009), 193035.
17. A. Najati, T. M. Rassias, *Stability of a mixed functional equation in several variables on Banach modules,* Nonlinear Anal., **72** (2010), 1755–1767.
18. C. Park, *Fuzzy stability of a functional equation associated with inner product spaces,* Fuzzy Set. Syst., **160** (2009), 1632–1642.
19. C. Park, J. R. Lee, X. Zhang, *Additive s-functional inequality and hom-derivations in Banach algebras,* J. Fix. Point Theory Appl., **21** (2019), 18.

20. V. Radu, *The fixed point alternative and the stability of functional equations,* Sem. Fix. Point Theory, **4** (2003), 91–96.
21. T. M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc., **72** (1978), 297–300.
22. T. M. Rassias, *New characterization of inner product spaces,* Bull. Sci. Math., **108** (1984), 95–99.
23. S. Pinelas, V. Govindan, K. Tamilvanan, *Stability of a quartic functional equation,* J. Fix. Point Theory Appl., **20** (2018), 148.
24. S. Pinelas, V. Govindan, K. Tamilvanan, *Solution and stability of an n-dimensional functional equation,* Analysis (Berlin), **39** (2019), 107–115.
25. F. Skof, *Proprietá locali e approssimazione di operatori,* Rend. Sem. Mat. Fis. Milano, **53** (1983), 113–129.
26. S. M. Ulam, *A collection of the mathematical problems*, Interscience Publ., New York, 1960.

AIMS Press

© 2020 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)