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# On Multipolar Intuitionistic Fuzzy $B$ -Algebras

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**Abstract:** In this paper, we discuss the notion of an  $m$ -polar fuzzy (normal) subalgebra in  $B$ -algebras and its related properties. We consider characterizations of an  $m$ -polar fuzzy (normal) subalgebra. We define the concept of an  $m$ -polar intuitionistic fuzzy (normal) subalgebra in a  $B$ -algebra, and we characterize the  $m$ -polar intuitionistic fuzzy (normal) subalgebra. Given an  $m$ -polar fuzzy set, we construct a simple  $m$ -polar fuzzy set and discuss on  $m$ -polar intuitionistic fuzzy subalgebras of  $B$ -algebras.

**Keywords:** (simple)  $m$ -polar fuzzy set (subalgebra);  $m$ -polar fuzzy normal (subalgebra);  $m$ -polar intuitionistic fuzzy set (subalgebra);  $m$ -polar fuzzy normal (subalgebra);  $B$ -algebra

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## 1. Introduction

Negggers and Kim [1] introduced the notion of a  $B$ -algebra, and investigated several properties. They established the fundamental homomorphism theorem of  $B$ -algebras in [2]. Jun et al. [3] defined the notion of fuzzy  $B$ -algebra. Kim et al. [4] constructed a quotient  $B$ -algebra by using a fuzzy normal  $B$ -algebra. Cho and Kim [1,5] discussed  $B$ -algebras related to quasigroups.

Fuzzy sets, which were introduced by Zadeh [6] introduced the notion of a fuzzy set, and as an extension of fuzzy set, Zhang [7] introduced the notion of bipolar fuzzy sets. Bipolar fuzzy information was applied to many (algebraic) structures, e.g.,  $\Gamma$ -semihypergroups (see [8]) and (ordered) semigroups (see [9–12]).

Chen et al. [13] introduced an  $m$ -polar fuzzy set which is an extension of bipolar fuzzy set. It has been applied to decision making problem (see [14]) and graph theory (see [15]). Takallo et al. [16] introduced the notion of multipolar fuzzy  $p$ -ideals of  $BCI$ -algebras, and discussed on several kinds of fuzzy  $p$ -ideals. Al-Masarwah and Ahmad [17] introduced the notion of  $m$ -polar  $(\alpha, \beta)$ -fuzzy ideals in  $BCK/BCI$ -algebras which is a generalization of fuzzy ideals discussed on  $BCK/BCI$ -algebras. Kang et al. [18] studied on  $k$ -polar intuitionistic fuzzy subalgebras and a (closed)  $k$ -polar intuitionistic fuzzy ideal in  $BCK/BCI$ -algebras.

In this paper, we introduce the notion of an  $m$ -polar fuzzy (normal) subalgebra in  $B$ -algebras, and we characterize  $m$ -polar fuzzy (normal) subalgebras of  $B$ -algebras. The concept of an  $m$ -polar intuitionistic fuzzy (normal) subalgebra in a  $B$ -algebra will be discussed. Given a  $m$ -polar fuzzy set, we construct a simple  $m$ -polar fuzzy set and obtain some properties on  $m$ -polar intuitionistic fuzzy subalgebras of  $B$ -algebras.

## 2. Preliminaries

A non-empty set  $U$  with a constant  $0$  is said to be a  $B$ -algebra [1] if there exists a binary operation “ $*$ ” such that

- (B1)  $x * x = 0$ ,
- (B2)  $x * 0 = x$ ,
- (B)  $(x * y) * z = x * (z * (0 * y))$

for any  $x, y, z$  in  $U$ . We define a binary relation “ $\leq$ ” on  $U$  by  $x \leq y$  if and only if  $x * y = 0$ .

**Proposition 1** ([1,5]). *If  $(U, *, 0)$  is a  $B$ -algebra, then*

- (i)  $x * y = x * z$  implies  $y = z$ ,
- (ii) if  $x * y = 0$ , then  $x = y$ ,
- (iii) if  $0 * x = 0 * y$ , then  $x = y$ ,
- (iv)  $0 * (0 * x) = x$ ,
- (v)  $x * (y * z) = (x * (0 * z)) * y$

for all  $x, y, z \in U$ .

Let  $U$  be a  $B$ -algebra. A non-empty subset  $S$  of  $U$  is said to be a *subalgebra* of  $U$  if  $x * y \in S$  for any  $x, y \in S$ . A non-empty subset  $N$  of  $U$  is said to be a *normal* if  $(x * a) * (y * b) \in N$  for any  $x * y, a * b \in N$ . It is known that any normal subset  $N$  of a  $B$ -algebra  $U$  is a subalgebra of  $U$ , but the converse need not be true in general [2].

By an  $m$ -polar fuzzy set on a set  $U$  (see [13]), we mean a function  $\hat{\ell} : U \rightarrow [0, 1]^m$ . The membership value of every element  $x \in U$  is denoted by

$$\hat{\ell}(x) = ((\pi_1 \circ \hat{\ell})(x), (\pi_2 \circ \hat{\ell})(x), \dots, (\pi_m \circ \hat{\ell})(x)),$$

where  $\pi_i : [0, 1]^m \rightarrow [0, 1]$  is the  $i$ -th projection for all  $i = 1, 2, \dots, m$ .

Given an  $m$ -polar fuzzy set on a set  $U$ , we consider the sets

$$U(\hat{\ell}; \hat{t}) := \{x \in U \mid \hat{\ell}(x) \geq \hat{t}\}, \text{ and } L(\hat{\ell}; \hat{r}) := \{x \in U \mid \hat{\ell}(x) \leq \hat{r}\}$$

where  $\hat{t} = (t_1, t_2, \dots, t_m), \hat{r} = (r_1, r_2, \dots, r_m) \in [0, 1]^m$ , that is,

$$U(\hat{\ell}; \hat{t}) := \{x \in U \mid (\pi_1 \circ \hat{\ell})(x) \geq t_1, (\pi_2 \circ \hat{\ell})(x) \geq t_2, \dots, (\pi_m \circ \hat{\ell})(x) \geq t_m\}, \text{ and}$$

$$L(\hat{\ell}; \hat{r}) := \{x \in U \mid (\pi_1 \circ \hat{\ell})(x) \leq r_1, (\pi_2 \circ \hat{\ell})(x) \leq r_2, \dots, (\pi_m \circ \hat{\ell})(x) \leq r_m\},$$

which are called an  $m$ -polar upper (resp., lower) set of  $\hat{\ell}$ .

### 3. $m$ -Polar Fuzzy (Normal) Subalgebras

Let  $U$  be a  $B$ -algebra unless otherwise specified.

**Definition 1.** *An  $m$ -polar fuzzy set  $\hat{\ell}$  on  $U$  is called an  $m$ -polar fuzzy subalgebra of  $U$  if it satisfies*

$$(\forall x, y \in U)(\hat{\ell}(x * y) \geq \inf\{\hat{\ell}(x), \hat{\ell}(y)\}), \tag{1}$$

$$\text{that is, } (\pi_i \circ \hat{\ell})(x * y) \geq \inf\{(\pi_i \circ \hat{\ell})(x), (\pi_i \circ \hat{\ell})(y)\}, \tag{2}$$

for all  $x, y \in U$  and  $i = 1, \dots, m$ .

**Example 1.** Let  $U := \{0, 1, 2, 3\}$  be a set  $B$ -algebra with the following Table 1.

**Table 1.** Cayley table for the binary operation “\*”.

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 1 | 2 | 0 |

Then  $(U, *, 0)$  is a  $B$ -algebra [2]. If  $\hat{\ell}$  is a 5-polar fuzzy set on  $U$  defined by

$$\hat{\ell} : U \rightarrow [0, 1]^5, x \mapsto \begin{cases} (0.7, 0.6, 0.8, 0.6, 0.9) & \text{if } x \in \{0, 3\}, \\ (0.3, 0.5, 0.6, 0.2, 0.5) & \text{if } x \in \{1, 2\}, \end{cases}$$

then it is easy to see that  $(\hat{\ell}, \hat{m})$  is a 5-polar fuzzy subalgebra of  $U$ .

**Theorem 1.** Given an  $m$ -polar fuzzy set  $\hat{\ell}$  over  $U$ , the followings are equivalent:

- (i)  $\hat{\ell}$  is an  $m$ -polar fuzzy subalgebra of  $U$ ,
- (ii) The  $m$ -polar upper level set  $U(\hat{\ell}; \hat{t})$  of  $\hat{\ell}$  is a subalgebra of  $U$  for all  $\hat{t} \in [0, 1]^m$  with  $U(\hat{\ell}; \hat{t}) \neq \emptyset$ .

**Proof.** Let  $\hat{\ell}$  be an  $m$ -polar fuzzy subalgebra of  $U$ . If  $x, y \in U(\hat{\ell}; \hat{t})$  for every  $\hat{t} \in [0, 1]^m$  with  $U(\hat{\ell}; \hat{t}) \neq \emptyset$ , then  $(\pi_i \circ \hat{\ell})(x) \geq t_i$  and  $(\pi_i \circ \hat{\ell})(y) \geq t_i$ . It follows from (2) that

$$(\pi_i \circ \hat{\ell})(x * y) \geq \inf\{(\pi_i \circ \hat{\ell})(x), (\pi_i \circ \hat{\ell})(y)\} \geq t_i,$$

for all  $x, y \in U$  and  $i = 1, 2, \dots, m$ . This shows that  $x * y \in U(\hat{\ell}; \hat{t})$ . Therefore  $U(\hat{\ell}; \hat{t})$  is a subalgebra of  $U$ .

Conversely, assume that (ii) is true. Let  $a, b \in U$  such that  $\hat{\ell}(a * b) < \inf\{\hat{\ell}(a), \hat{\ell}(b)\}$ . If we take  $\hat{t} := \inf\{\hat{\ell}(a), \hat{\ell}(b)\}$ , then  $a, b \in U(\hat{\ell}; \hat{t})$ . Since  $U(\hat{\ell}; \hat{t})$  is a subalgebra of  $U$ , we have  $a * b \in U(\hat{\ell}; \hat{t})$ . It shows that  $\hat{\ell}(a * b) \geq \hat{t} = \inf\{\hat{\ell}(a), \hat{\ell}(b)\}$ , a contradiction. Thus  $\hat{\ell}(x * y) \geq \inf\{\hat{\ell}(x), \hat{\ell}(y)\}$  for all  $x, y \in U$ , i.e.,  $\hat{\ell}$  is an  $m$ -polar fuzzy subalgebra of  $U$ .  $\square$

For any  $m$ -polar fuzzy set  $\hat{\ell}$  over  $U$ , we define a fuzzy set  $\hat{\ell}^*$  of  $U$  by

$$\hat{\ell}^* : U \rightarrow [0, 1]^m, x \mapsto \begin{cases} \hat{\ell}(x) & \text{if } x \in U(\hat{\ell}; \hat{t}), \\ \hat{0} & \text{otherwise,} \end{cases}$$

where  $\hat{0} = (0, \dots, 0) \in [0, 1]^m$ . We call  $\hat{\ell}^*$  a simple  $m$ -polar fuzzy set of  $U$ .

**Theorem 2.** Let  $\hat{\ell}$  be an  $m$ -polar fuzzy subalgebra of  $U$ . Then the simple  $m$ -polar fuzzy set  $\hat{\ell}^*$  of  $U$  is also an  $m$ -polar fuzzy subalgebra of  $U$ .

**Proof.** Let  $\hat{\ell}$  be an  $m$ -polar fuzzy subalgebra of  $U$ . Then an  $m$ -polar upper level set  $U(\hat{\ell}; \hat{t})$  of  $\hat{\ell}$  is a subalgebra of  $U$  for any  $\hat{t} \in [0, 1]^m$  with  $U(\hat{\ell}; \hat{t}) \neq \emptyset$ , by Theorem 1. Given  $x, y \in U$ , if  $x, y \in U(\hat{\ell}; \hat{t})$ , then  $x * y \in U(\hat{\ell}; \hat{t})$ . Thus

$$\hat{\ell}^*(x * y) = \hat{\ell}(x * y) \geq \inf\{\hat{\ell}(x), \hat{\ell}(y)\} = \inf\{\hat{\ell}^*(x), \hat{\ell}^*(y)\}.$$

If  $x \notin U(\hat{\ell}; \hat{t})$  or  $y \notin U(\hat{\ell}; \hat{t})$ , then  $\hat{\ell}^*(x) = \hat{0}$  or  $\hat{\ell}^*(y) = \hat{0}$ . It follows that  $\hat{\ell}^*(x * y) \geq \inf\{\hat{\ell}^*(x), \hat{\ell}^*(y)\}$ . This shows that  $\hat{\ell}^*$  is an  $m$ -polar fuzzy subalgebra of  $U$ .  $\square$

**Definition 2.** An  $m$ -polar fuzzy set  $\hat{\ell}$  on  $U$  is said to be an  $m$ -polar fuzzy normal over  $U$  if it satisfies

$$(\forall x, y, a, b \in U)(\hat{\ell}((x * a) * (y * b))) \geq \inf\{\hat{\ell}(x * y), \hat{\ell}(a * b)\}, \tag{3}$$

$$\text{i.e. , } (\pi_i \circ \hat{\ell})((x * a) * (y * b)) \geq \inf\{(\pi_i \circ \hat{\ell})(x * y), (\pi_i \circ \hat{\ell})(a * b)\}, \tag{4}$$

for all  $x, y, a, b \in U$  and  $i = 1, \dots, m$ . An  $m$ -polar fuzzy set  $\hat{\ell}$  on  $U$  is called an  $m$ -polar fuzzy normal subalgebra of  $U$  if it satisfies (1) and (3).

**Proposition 2.** Every  $m$ -polar fuzzy normal over  $U$  is an  $m$ -polar fuzzy normal subalgebra of  $U$ .

**Proof.** Setting  $y := 0, b := 0$  and  $a := y$  in (4), we have  $(\pi_i \circ \hat{\ell})((x * y) * (0 * 0)) \geq \inf\{(\pi_i \circ \hat{\ell})(x * 0), (\pi_i \circ \hat{\ell})(y * 0)\}$  for all  $x, y \in U$  and  $i = 1, \dots, m$ . Using (B2) and (B1), we get  $(\pi_i \circ \hat{\ell})(x * y) \geq \inf\{(\pi_i \circ \hat{\ell})(x), (\pi_i \circ \hat{\ell})(y)\}$  for all  $x, y \in U$  and  $i = 1, \dots, m$ . Hence  $\hat{\ell}$  is an  $m$ -polar fuzzy subalgebra of  $U$ . Therefore  $\hat{\ell}$  is an  $m$ -polar fuzzy normal subalgebra of  $U$ .  $\square$

The converse of Proposition 2 may not be true in general (see the following example).

**Example 2.** Let  $X = \{0, 1, 2, 3, 4, 5\}$  be a set with the following Table 2.

**Table 2.** Cayley table for the binary operation “\*”.

| * | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 3 | 4 | 5 |
| 1 | 1 | 0 | 2 | 4 | 5 | 3 |
| 2 | 2 | 1 | 0 | 5 | 3 | 4 |
| 3 | 3 | 4 | 5 | 0 | 2 | 1 |
| 4 | 4 | 5 | 3 | 1 | 0 | 2 |
| 5 | 5 | 3 | 4 | 2 | 1 | 0 |

Then  $(U, *, 0)$  is a  $B$ -algebra [2]. Let  $\hat{\ell}$  be a 5-polar fuzzy set on  $U$  defined by

$$\hat{\ell} : U \rightarrow [0, 1]^5, x \mapsto \begin{cases} (0.7, 0.6, 0.8, 0.6, 0.9) & \text{if } x = 0, \\ (0.4, 0.5, 0.7, 0.4, 0.6) & \text{if } x = 5, \\ (0.3, 0.3, 0.6, 0.2, 0.5) & \text{if } x \in \{1, 2, 3, 4\}. \end{cases}$$

It is easy to see that  $\hat{\ell}$  is a 5-polar fuzzy subalgebra of  $U$ , but not a 5-polar fuzzy normal over  $U$ , since  $0.3 = (\pi_2 \circ \hat{\ell})(1) = (\pi_2 \circ \hat{\ell})((1 * 3) * (4 * 2)) \not\geq \inf\{(\pi_2 \circ \hat{\ell})(1 * 4), (\pi_2 \circ \hat{\ell})(3 * 2)\} = \inf\{(\pi_2 \circ \hat{\ell})(5), (\pi_2 \circ \hat{\ell})(5)\} = \inf\{0.5, 0.5\} = 0.5$ .

Let  $\hat{\ell}_U$  and  $\hat{\ell}_V$  be  $m$ -polar fuzzy sets over  $B$ -algebras  $U$  and  $V$ , respectively. The  $\hat{\ell}_\wedge$ -product of  $\hat{\ell}_U$  and  $\hat{\ell}_V$  is defined to be an  $m$ -polar fuzzy set  $\hat{\ell}_{U \wedge V}$  on  $U \times V$  in which

$$\hat{\ell}_{U \wedge V} : U \times V \rightarrow [0, 1]^m, (x, y) \mapsto \min\{\hat{\ell}_U(x), \hat{\ell}_V(y)\} \text{ for any } (x, y) \in U \times V,$$

that is,

$$(\pi_i \circ \hat{\ell}_{U \wedge V})(x, y) = \min\{(\pi_i \circ \hat{\ell}_U)(x), (\pi_i \circ \hat{\ell}_V)(y)\},$$

for any  $(x, y) \in U \times V$  and  $i = 1, \dots, m$ .

**Theorem 3.** For any B-algebras  $U$  and  $V$ , let  $\hat{\ell}_U$  and  $\hat{\ell}_V$  be  $m$ -polar fuzzy subalgebras of  $U$  and  $V$ , respectively. Then the  $\hat{\ell}_\wedge$ -product  $\hat{\ell}_{U \wedge V}$  of  $\hat{\ell}_U$  and  $\hat{\ell}_V$  is also an  $m$ -polar fuzzy subalgebra of  $U \times V$ .

**Proof.** Since  $(U \times V, \otimes, (0, 0))$  is a B-algebra, for any  $(x, y), (a, b) \in U \times V$ , we have

$$\begin{aligned} \hat{\ell}_{U \wedge V}((x, y) \otimes (a, b)) &= \hat{\ell}_{U \wedge V}(x * a, y * b) \\ &= \min\{\hat{\ell}_U(x * a), \hat{\ell}_V(y * b)\} \geq \min\{\inf\{\hat{\ell}_U(x), \hat{\ell}_U(a)\}, \inf\{\hat{\ell}_V(y), \hat{\ell}_V(b)\}\} \\ &= \inf\{\min\{\hat{\ell}_U(x), \hat{\ell}_V(y)\}, \min\{\hat{\ell}_U(a), \hat{\ell}_V(b)\}\} \\ &= \inf\{\hat{\ell}_{U \wedge V}(x, y), \hat{\ell}_{U \wedge V}(a, b)\} \end{aligned}$$

This shows that  $\hat{\ell}_{U \wedge V}$  is an  $m$ -polar fuzzy subalgebra of  $U \times V$ .  $\square$

**Example 3.** Suppose that  $U := \{0, 1, 2, 3\}$  is a B-algebra discussed in Example 1. Let  $V := \{0, 4, 5, 6\}$  be a set with a binary operation “ $*$ ” as in Table 3.

**Table 3.** Cayley table for the binary operation “ $*$ ”.

| $*$ | 0 | 4 | 5 | 6 |
|-----|---|---|---|---|
| 0   | 0 | 6 | 5 | 4 |
| 4   | 4 | 0 | 6 | 5 |
| 5   | 5 | 1 | 0 | 6 |
| 6   | 6 | 5 | 4 | 0 |

Then it is easy to see that  $(V, *, 0)$  is a B-algebra, and  $U \times V = \{(0, 0), (0, 4), (0, 5), (0, 6), (1, 0), (1, 4), (1, 5), (1, 6), (2, 0), (1, 4), (2, 5), (2, 6), (3, 0), (3, 4), (3, 5), (3, 6)\}$  is also a B-algebra. Define 5-polar fuzzy sets  $\hat{\ell}_U$  and  $\hat{\ell}_V$  on  $U$  and  $V$ , respectively, as follows:

$$\hat{\ell}_U : U \rightarrow [0, 1]^5, y \mapsto \begin{cases} (0.7, 0.6, 0.8, 0.6, 0.9) & \text{if } x \in \{0, 3\} \\ (0.2, 0.4, 0.5, 0.1, 0.3) & \text{if } x \in \{1, 2\}, \end{cases}$$

$$\hat{\ell}_V : V \rightarrow [0, 1]^5, y \mapsto \begin{cases} (0.8, 0.7, 0.8, 0.7, 0.8) & \text{if } x \in \{0, 5\} \\ (0.3, 0.5, 0.6, 0.2, 0.5) & \text{if } x \in \{4, 6\}, \end{cases}$$

Then  $\hat{\ell}_U$  and  $\hat{\ell}_V$  are 5-polar fuzzy subalgebras of  $U$  and  $V$ , respectively. Moreover, the  $\hat{\ell}_\wedge$ -product  $\hat{\ell}_{U \wedge V}$  of  $\hat{\ell}_U$  and  $\hat{\ell}_V$  is a 5-polar fuzzy subalgebra of  $U \times V$ .

#### 4. $m$ -Polar Intuitionistic Fuzzy (Normal) Subalgebras

**Definition 3.** A multipolar intuitionistic fuzzy set with finite degree  $m$  (briefly,  $m$ -polar intuitionistic fuzzy set) over  $U$  is a mapping

$$(\hat{\ell}, \hat{m}) : U \rightarrow [0, 1]^m \times [0, 1]^m, x \mapsto (\hat{\ell}(x), \hat{m}(x))$$

where  $\hat{\ell} : U \rightarrow [0, 1]^m$  and  $\hat{m} : U \rightarrow [0, 1]^m$  are  $m$ -polar fuzzy sets over a universe set  $U$  satisfying the condition:

$$\hat{\ell}(x) + \hat{m}(x) \leq 1$$

for all  $x \in U$ , i.e.,

$$(\pi_i \circ \hat{\ell})(x) + (\pi_i \circ \hat{m})(x) \leq 1$$

for all  $x \in U$  and  $i = 1, \dots, m$ .

Given an  $m$ -polar intuitionistic fuzzy set  $(\hat{\ell}, \hat{m})$  over a universe set  $U$ , we consider the following two sets:

$$U(\hat{\ell}; \hat{t}) := \{x \in U \mid \hat{\ell}(x) \geq \hat{t}\} \text{ and } L(\hat{m}; \hat{r}) := \{x \in U \mid \hat{\ell}(x) \leq \hat{r}\}$$

where  $\hat{t} = (t_1, t_2, \dots, t_m), \hat{r} = (r_1, r_2, \dots, r_m) \in [0, 1]^m$ , that is,

$$U(\hat{\ell}; \hat{t}) := \{x \in U \mid (\pi_1 \circ \hat{\ell})(x) \geq t_1, (\pi_2 \circ \hat{\ell})(x) \geq t_2, \dots, (\pi_m \circ \hat{\ell})(x) \geq t_m\}, \text{ and}$$

$$L(\hat{m}; \hat{r}) := \{x \in U \mid (\pi_1 \circ \hat{m})(x) \leq r_1, (\pi_2 \circ \hat{m})(x) \leq r_2, \dots, (\pi_m \circ \hat{m})(x) \leq r_m\},$$

which are called an  $m$ -polar upper (resp. lower) set of  $(\hat{\ell}, \hat{m})$ . It is clear that  $U(\hat{\ell}; \hat{t}) = \bigcap_{i=1}^m U(\hat{\ell}; \hat{t})^i$  and  $L(\hat{m}; \hat{r}) = \bigcap_{i=1}^m L(\hat{m}; \hat{r})^i$  where  $U(\hat{\ell}; \hat{t})^i = \{x \in U \mid (\pi_i \circ \hat{\ell})(x) \geq t_i\}$  and  $L(\hat{m}; \hat{r})^i = \{x \in U \mid (\pi_i \circ \hat{m})(x) \leq r_i\}$  for all  $i = 1, \dots, m$ .

**Definition 4.** An  $m$ -polar intuitionistic fuzzy set  $(\hat{\ell}, \hat{m})$  over  $U$  is said to be an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$  if it satisfies

$$(\forall x, y \in U) \left( \hat{\ell}(x * y) \geq \inf\{\hat{\ell}(x), \hat{\ell}(y)\}, \hat{m}(x * y) \leq \sup\{\hat{m}(x), \hat{m}(y)\} \right), \tag{5}$$

that is,

$$(\pi_i \circ \hat{\ell})(x * y) \geq \inf\{(\pi_i \circ \hat{\ell})(x), (\pi_i \circ \hat{\ell})(y)\}, (\pi_i \circ \hat{m})(x * y) \leq \sup\{(\pi_i \circ \hat{m})(x), (\pi_i \circ \hat{m})(y)\} \tag{6}$$

for all  $x, y \in U$  and  $i = 1, 2, \dots, m$ .

**Proposition 3.** Every  $m$ -polar intuitionistic fuzzy subalgebra  $(\hat{\ell}, \hat{m})$  of  $U$  satisfies the following conditions:

$$(\forall x \in U) (\hat{\ell}(0) \geq \hat{\ell}(x), \hat{m}(0) \leq \hat{m}(x)). \tag{7}$$

**Proof.** It can be easily proved if we put  $y := x$  in (5).  $\square$

**Example 4.** Let  $U = \{0, 1, 2, 3\}$  be a  $B$ -algebra as in Example 1. Let  $(\hat{\ell}, \hat{m})$  be a 5-polar intuitionistic fuzzy set on  $U$  defined by

$$(\hat{\ell}, \hat{m}) : U \rightarrow [0, 1]^5 \times [0, 1]^5,$$

$$x \mapsto \begin{cases} ((0.6, 0.3), (0.67, 0.25), (0.7, 0.15), (0.63, 0.2), (0.8, 0.18)) & \text{if } x = 0, \\ ((0.4, 0.5), (0.5, 0.4), (0.6, 0.3), (0.3, 0.4), (0.7, 0.29)) & \text{if } x = 3, \\ ((0.3, 0.6), (0.4, 0.5), (0.5, 0.4), (0.2, 0.7), (0.4, 0.5)) & \text{if } x \in \{1, 2\}. \end{cases}$$

Then we can see that  $(\hat{\ell}, \hat{m})$  is a 5-polar intuitionistic fuzzy subalgebra of  $U$ .

**Proposition 4.** Given an  $m$ -polar intuitionistic fuzzy subalgebra  $(\hat{\ell}, \hat{m})$  of  $U$ , the followings are equivalent:

- (i)  $(\forall x \in U) (\hat{\ell}(x) = \hat{\ell}(0), \hat{m}(x) = \hat{m}(0))$ ,
- (ii)  $(\forall x, y \in U) (\hat{\ell}(y) \leq \hat{\ell}(x * y), \hat{m}(y) \geq \hat{m}(x * y))$ ,

that is,  $(\pi_i \circ \hat{\ell})(y) \leq (\pi_i \circ \hat{\ell})(x * y), (\pi_i \circ \hat{m})(y) \geq (\pi_i \circ \hat{m})(x * y)$  for all  $x, y \in U$  and  $i = 1, \dots, m$ .

**Proof.** Assume that  $\hat{\ell}(x) = \hat{\ell}(0)$  and  $\hat{m}(x) = \hat{m}(0)$  for any  $x \in U$ . Using (7) and (6), we have

$$(\pi_i \circ \hat{\ell})(y) = \inf\{(\pi_i \circ \hat{\ell})(0), (\pi_i \circ \hat{\ell})(y)\} = \inf\{(\pi_i \circ \hat{\ell})(x), (\pi_i \circ \hat{\ell})(y)\} \leq (\pi_i \circ \hat{\ell})(x * y) \text{ and}$$

$$(\pi_i \circ \hat{m})(y) = \sup\{(\pi_i \circ \hat{m})(0), (\pi_i \circ \hat{m})(y)\} = \sup\{(\pi_i \circ \hat{m})(x), (\pi_i \circ \hat{m})(y)\} \geq (\pi_i \circ \hat{m})(x * y)$$

for all  $x, y \in U$  and  $i = 1, \dots, m$ .

Conversely, assume that (ii) is valid. Taking  $y := 0$  in (ii), and by using (B2), we obtain  $(\pi_i \circ \hat{\ell})(0) \leq (\pi_i \circ \hat{\ell})(x * 0) = (\pi_i \circ \hat{\ell})(x)$  and  $\hat{m}(0) \geq \hat{m}(x * 0) = \hat{m}(x)$ . It follows from (7) that  $(\pi_i \circ \hat{\ell})(x) = (\pi_i \circ \hat{\ell})(0)$  and  $(\pi_i \circ \hat{m})(x) = (\pi_i \circ \hat{m})(0)$  for all  $i = 1, \dots, m$ , i.e.,  $\hat{\ell}(x) = \hat{\ell}(0)$  and  $\hat{m}(x) = \hat{0}$ . This completes the proof.  $\square$

Given an  $m$ -polar intuitionistic fuzzy set  $(\hat{\ell}, \hat{m})$  over  $U$  and  $(\hat{t}, \hat{r}) \in (0, 1]^m \times [0, 1)^m$ , we consider the sets  $P_{(\hat{\ell}, \hat{t})}(U) := \{x \in U | \hat{\ell}(x) + \hat{t} > 1\}$  and  $P_{(\hat{m}, \hat{r})}(U) := \{x \in U | \hat{m}(x) + \hat{r} < 1\}$ . Then  $P_{(\hat{\ell}, \hat{t})}(U) = \bigcap_{i=1}^m P_{(\hat{\ell}, \hat{t})}^i(U)$  and  $P_{(\hat{m}, \hat{r})}(U) = \bigcap_{i=1}^m P_{(\hat{m}, \hat{r})}^i(U)$ , where  $P_{(\hat{\ell}, \hat{t})}^i(U) := \{x \in U | (\pi_i \circ \hat{\ell})(x) + \hat{t}_i > 1\}$  and  $P_{(\hat{m}, \hat{r})}^i(U) := \{x \in U | (\pi_i \circ \hat{m})(x) + \hat{r}_i < 1\}$  for all  $i = 1, \dots, m$ .

**Theorem 4.** Let  $(\hat{\ell}, \hat{m})$  be an  $m$ -polar intuitionistic fuzzy set over  $U$ . Then the followings are equivalent:

- (i)  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$ ,
- (ii) The sets  $P_{(\hat{\ell}, \hat{t})}(U)$  and  $P_{(\hat{m}, \hat{r})}(U)$  are subalgebras of  $U$  for all  $(\hat{t}, \hat{r}) \in (0, 1]^m \times [0, 1)^m$  with  $P_{(\hat{\ell}, \hat{t})}(U) \neq \emptyset \neq P_{(\hat{m}, \hat{r})}(U)$ .

**Proof.** Let  $x, y \in P_{(\hat{\ell}, \hat{t})}(U) = \bigcap_{i=1}^m P_{(\hat{\ell}, \hat{t})}^i(U)$  and let  $a, b \in P_{(\hat{m}, \hat{r})}(U) = \bigcap_{i=1}^m P_{(\hat{m}, \hat{r})}^i(U)$ . Then  $(\pi_i \circ \hat{\ell})(x) + \hat{t}_i > 1, (\pi_i \circ \hat{\ell})(y) + \hat{t}_i > 1, (\pi_i \circ \hat{m})(a) + \hat{r}_i < 1$  and  $(\pi_i \circ \hat{m})(b) + \hat{r}_i < 1$  for all  $i = 1, \dots, m$ . It follows from (6) that

$$\begin{aligned} (\pi_i \circ \hat{\ell})(x * y) + \hat{t}_i &\geq \inf\{(\pi_i \circ \hat{\ell})(x), (\pi_i \circ \hat{\ell})(y)\} + \hat{t}_i \\ &= \inf\{(\pi_i \circ \hat{\ell})(x) + \hat{t}_i, (\pi_i \circ \hat{\ell})(y) + \hat{t}_i\} > 1, \text{ and} \\ (\pi_i \circ \hat{m})(a * b) + \hat{r}_i &\leq \sup\{(\pi_i \circ \hat{m})(a), (\pi_i \circ \hat{m})(b)\} + \hat{r}_i \\ &= \sup\{(\pi_i \circ \hat{m})(a) + \hat{r}_i, (\pi_i \circ \hat{m})(b) + \hat{r}_i\} < 1 \end{aligned}$$

for all  $i = 1, \dots, m$ . Hence  $x * y \in P_{(\hat{\ell}, \hat{t})}(U) = \bigcap_{i=1}^m P_{(\hat{\ell}, \hat{t})}^i(U)$  and  $a * b \in P_{(\hat{m}, \hat{r})}(U) = \bigcap_{i=1}^m P_{(\hat{m}, \hat{r})}^i(U)$ . Therefore  $P_{(\hat{\ell}, \hat{t})}(U)$  and  $P_{(\hat{m}, \hat{r})}(U)$  are subalgebras of  $U$  for all  $(\hat{t}, \hat{r}) \in (0, 1]^m \times [0, 1)^m$  with  $P_{(\hat{\ell}, \hat{t})}(U) \neq \emptyset \neq P_{(\hat{m}, \hat{r})}(U)$ .

Conversely, let  $a, b \in U$  be such that  $\hat{\ell}(a * b) < \inf\{\hat{\ell}(a), \hat{\ell}(b)\}$ . Then  $\hat{\ell}(a * b) + \hat{t} \leq 1 < \inf\{\hat{\ell}(a), \hat{\ell}(b)\} + \hat{t}$  for some  $\hat{t} \in (0, 1]^m$ . It follows that  $a, b \in P_{(\hat{\ell}, \hat{t})}(U)$ , which implies  $a * b \in P_{(\hat{\ell}, \hat{t})}(U)$ , since  $P_{(\hat{\ell}, \hat{t})}(U)$  is a subalgebra of  $U$ . Hence  $\hat{\ell}(a * b) + \hat{t} > 1$ , which is a contradiction. If  $\hat{m}(c * d) > \sup\{\hat{m}(c), \hat{m}(d)\}$  for some  $c, d \in U$ , then  $\hat{m}(c * d) + \hat{r} \geq \hat{1} > \sup\{\hat{m}(c), \hat{m}(d)\} + \hat{r}$  for some  $\hat{r} \in [0, 1)^m$ . Hence  $c, d \in P_{(\hat{m}, \hat{r})}(U)$ . Since  $P_{(\hat{m}, \hat{r})}(U)$  is a subalgebra of  $U$ , we obtain  $c * d \in P_{(\hat{m}, \hat{r})}(U)$ , i.e.,  $\hat{m}(c * d) + \hat{r} < 1$ . This is a contradiction. This shows that  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$ .  $\square$

**Theorem 5.** An  $m$ -polar intuitionistic fuzzy set  $(\hat{\ell}, \hat{m})$  over  $U$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$  if and only if  $\hat{\ell}$  and  $\hat{m}^c$  are  $m$ -polar fuzzy subalgebras of  $U$ , where  $\hat{m}^c = 1 - \hat{m}$ , i.e.,  $(\pi_i \circ \hat{m})^c = 1 - (\pi_i \circ \hat{m})$  for all  $i = 1, \dots, m$ .

**Proof.** Assume that  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$ . Then  $\hat{\ell}$  is an  $m$ -polar fuzzy subalgebra of  $U$ . If  $x, y \in U$ , then

$$\begin{aligned} (\pi_i \circ \hat{m})^c(x * y) &= 1 - (\pi_i \circ \hat{m})(x * y) \\ &\geq 1 - \sup\{(\pi_i \circ \hat{m})(x), (\pi_i \circ \hat{m})(y)\} \\ &= \inf\{1 - (\pi_i \circ \hat{m})(x), 1 - (\pi_i \circ \hat{m})(y)\} \\ &= \inf\{(\pi_i \circ \hat{m})^c(x), (\pi_i \circ \hat{m})^c(y)\}, \text{ for all } i = 1, \dots, m. \end{aligned}$$

Thus  $\hat{m}^c$  is an  $m$ -polar fuzzy subalgebra of  $U$ .

Conversely, assume that  $\hat{\ell}$  and  $\hat{m}^c$  are  $m$ -polar fuzzy subalgebras of  $U$ . For any  $a, b \in U$ , we have

$$\begin{aligned} 1 - (\pi_i \circ \hat{m})(a * b) &= (\pi_i \circ \hat{m})^c(a * b) \\ &\geq \inf\{(\pi_i \circ \hat{m})^c(a), (\pi_i \circ \hat{m})^c(b)\} \\ &= \inf\{1 - (\pi_i \circ \hat{m})(a), 1 - (\pi_i \circ \hat{m})(b)\} \\ &= 1 - \sup\{(\pi_i \circ \hat{m})(a), (\pi_i \circ \hat{m})(b)\}, \end{aligned}$$

that is,  $(\pi_i \circ \hat{m})(a * b) \leq \sup\{(\pi_i \circ \hat{m})(a), (\pi_i \circ \hat{m})(b)\}$  for all  $i = 1, \dots, m$ . Hence  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$ .  $\square$

**Corollary 1.** Let  $(\hat{\ell}, \hat{m})$  be an  $m$ -polar intuitionistic fuzzy set over  $U$ . Then  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$  if and only if the necessary operator  $\square(\hat{\ell}, \hat{m}) = (\hat{\ell}, \hat{\ell}^c)$  and the possibility operator  $\diamond(\hat{\ell}, \hat{m}) = (\hat{m}^c, \hat{m})$  are  $m$ -polar intuitionistic fuzzy subalgebras of  $U$ .

**Theorem 6.** Let  $I$  be a subset of  $U$  and let  $(\hat{\ell}_I, \hat{m}_I)$  be an  $m$ -polar intuitionistic fuzzy set on  $U$  defined by

$$\begin{aligned} \hat{\ell}_I : U \rightarrow [0, 1]^m, x \mapsto &\begin{cases} \hat{1} & \text{if } x \in I \\ \hat{0} & \text{otherwise,} \end{cases} \\ \hat{m}_I : U \rightarrow [0, 1]^m, x \mapsto &\begin{cases} \hat{0} & \text{if } x \in I \\ \hat{1} & \text{otherwise.} \end{cases} \end{aligned}$$

Then  $(\hat{\ell}_I, \hat{m}_I)$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$  if and only if  $I$  is a subalgebra of  $U$ .

**Proof.** Straightforward.  $\square$

**Theorem 7.** If  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy set over  $U$ , then the followings are equivalent:

- (i)  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$ ,
- (ii) the  $m$ -polar upper and lower level sets  $U(\hat{\ell}; \hat{t})$  and  $L(\hat{m}; \hat{r})$  of  $(\hat{\ell}, \hat{m})$  are subalgebras of  $U$  for all  $(\hat{t}, \hat{r}) \in [0, 1]^m \times [0, 1]^m$  with  $U(\hat{\ell}; \hat{t}) \neq \emptyset \neq L(\hat{m}; \hat{r})$ .

**Proof.** Assume  $(\hat{\ell}; \hat{m})$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$ . Let  $x, y, a, b \in U$  be such that  $x, y \in U(\hat{\ell}; \hat{t})$  and  $a, b \in L(\hat{m}; \hat{r})$  for all  $(\hat{t}, \hat{r}) \in [0, 1]^m \times [0, 1]^m$ . Then  $(\pi_i \circ \hat{\ell})(x) \geq t_i, (\pi_i \circ \hat{\ell})(y) \geq t_i, (\pi_i \circ \hat{m})(a) \leq r_i$  and  $(\pi_i \circ \hat{m})(b) \leq r_i$  for all  $i = 1, \dots, m$ . It follows from (6) that  $(\pi_i \circ \hat{\ell})(x * y) \geq \inf\{(\pi_i \circ \hat{\ell})(x), (\pi_i \circ \hat{\ell})(y)\} \geq t_i$  and  $(\pi_i \circ \hat{m})(a * b) \leq \sup\{(\pi_i \circ \hat{m})(a), (\pi_i \circ \hat{m})(b)\} \leq r_i$  for all  $i = 1, \dots, m$ . Hence  $x * y \in U(\hat{\ell}; \hat{t})$  and  $a * b \in L(\hat{m}; \hat{r})$ . Therefore  $U(\hat{\ell}; \hat{t})$  and  $L(\hat{m}; \hat{r})$  are subalgebras of  $U$ .

Conversely, suppose that (ii) is valid. Let  $a, b \in X$  be such that  $\hat{\ell}(a * b) < \inf\{\hat{\ell}(a), \hat{\ell}(b)\}$  or  $\hat{m}(a * b) > \sup\{\hat{m}(a), \hat{m}(b)\}$ . We take  $\hat{t} := \inf\{\hat{\ell}(a), \hat{\ell}(b)\}$  and  $\hat{r} := \sup\{\hat{m}(a), \hat{m}(b)\}$ . Then  $a, b \in U(\hat{\ell}; \hat{t})$  or  $a, b \in L(\hat{m}; \hat{r})$ . Since  $U(\hat{\ell}; \hat{t})$  and  $L(\hat{m}; \hat{r})$  are subalgebras of  $U$ , we obtain that either  $a * b \in U(\hat{\ell}; \hat{t})$  or  $a * b \in L(\hat{m}; \hat{r})$ . Hence either  $\hat{\ell}(a * b) \geq \hat{t} = \inf\{\hat{\ell}(a), \hat{\ell}(b)\}$  or  $\hat{m}(a * b) \leq \hat{r} = \sup\{\hat{m}(a), \hat{m}(b)\}$ , which lead to a contradiction. Thus  $\hat{\ell}(x * y) \geq \inf\{\hat{\ell}(x), \hat{\ell}(y)\}$  and  $\hat{m}(x * y) \leq \sup\{\hat{m}(x), \hat{m}(y)\}$  for all  $x, y \in U$ . Therefore  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$ .  $\square$



Given an  $m$ -polar intuitionistic fuzzy set  $(\hat{\ell}, \hat{m})$  over  $U$ , we define an  $m$ -polar intuitionistic fuzzy set  $(\hat{\ell}^*, \hat{m}^*)$  over  $U$  by

$$\hat{\ell}^* : U \rightarrow [0, 1]^m, x \mapsto \begin{cases} \hat{1} & \text{if } x \in U(\hat{\ell}; \hat{t}), \\ \hat{0} & \text{otherwise,} \end{cases}$$

$$\hat{m}^* : U \rightarrow [0, 1]^m, x \mapsto \begin{cases} \hat{0} & \text{if } x \in L(\hat{m}; \hat{r}), \\ \hat{1} & \text{otherwise,} \end{cases}$$

where  $\hat{0} = (0, \dots, 0)$  and  $\hat{1} = (1, \dots, 1) \in [0, 1]^m$ .

**Theorem 8.** *If  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy subalgebra of a B-algebra  $U$ , then so is  $(\hat{\ell}^*, \hat{m}^*)$ .*

**Proof.** If  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$ , then  $m$ -polar upper and lower level sets  $U(\hat{\ell}; \hat{t})$  and  $L(\hat{m}; \hat{r})$  of  $(\hat{\ell}, \hat{m})$  are subalgebras of  $U$  for all  $\hat{t}, \hat{r} \in [0, 1]^m$  with  $U(\hat{\ell}; \hat{t}) \neq \emptyset \neq L(\hat{m}; \hat{r})$  by applying Theorem 7. Let  $x, y \in X$ . If  $x, y \in U(\hat{\ell}; \hat{t})$ , then  $x * y \in U(\hat{\ell}; \hat{t})$ . Thus

$$\hat{\ell}^*(x * y) = \hat{1} \geq \inf\{\hat{\ell}(x), \hat{\ell}(y)\} = \inf\{\hat{\ell}^*(x), \hat{\ell}^*(y)\}.$$

If  $x \notin U(\hat{\ell}; \hat{t})$  or  $y \notin U(\hat{\ell}; \hat{t})$ , then  $\hat{\ell}^*(x) = \hat{0}$  or  $\hat{\ell}^*(y) = \hat{0}$ . Hence  $\hat{\ell}^*(x * y) \geq \inf\{\hat{\ell}^*(x), \hat{\ell}^*(y)\}$ . Now, if  $x, y \in L(\hat{m}; \hat{r})$ , then  $x * y \in L(\hat{m}; \hat{r})$ . Thus

$$\hat{m}^*(x * y) = \hat{0} \leq \sup\{\hat{m}(x), \hat{m}(y)\} = \sup\{\hat{m}^*(x), \hat{m}^*(y)\}.$$

If  $x \notin L(\hat{m}; \hat{r})$  or  $y \notin L(\hat{m}; \hat{r})$ , then  $\hat{m}^*(x) = \hat{1}$  or  $\hat{m}^*(y) = \hat{1}$ . Hence we have

$$\hat{m}^*(x * y) \leq \hat{1} = \sup\{\hat{m}^*(x), \hat{m}^*(y)\}.$$

Therefore  $(\hat{\ell}^*, \hat{m}^*)$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$ . Also  $\hat{\ell}^*(x) + \hat{m}^*(x) \leq \hat{1}$  for all  $x \in U$ , i.e.,  $(\pi_i \circ \hat{\ell}^*)(x) + (\pi_i \circ \hat{m}^*)(x) \leq \hat{1}$  for all  $x \in U$  and  $i = 1, \dots, m$ . We proved the theorem.  $\square$

**Definition 5.** *An  $m$ -polar intuitionistic fuzzy set  $(\hat{\ell}, \hat{m})$  over  $U$  is said to be an  $m$ -polar intuitionistic fuzzy normal over  $U$  if it satisfies: for any  $x, y, a, b \in U$ ,*

$$\hat{\ell}((x * a) * (y * b)) \geq \inf\{\hat{\ell}(x * y), \hat{\ell}(a * b)\}, \hat{m}((x * a) * (y * b)) \leq \sup\{\hat{m}(x * y), \hat{m}(a * b)\}, \quad (8)$$

that is,

$$\begin{aligned} (\pi_i \circ \hat{\ell})((x * a) * (y * b)) &\geq \inf\{(\pi_i \circ \hat{\ell})(x * y), (\pi_i \circ \hat{\ell})(a * b)\} \text{ and} \\ (\pi_i \circ \hat{m})((x * a) * (y * b)) &\leq \sup\{(\pi_i \circ \hat{m})(x * y), (\pi_i \circ \hat{m})(a * b)\} \end{aligned} \quad (9)$$

for all  $x, y, a, b \in U$  and  $i = 1, 2, \dots, m$ . An  $m$ -polar intuitionistic fuzzy set  $(\hat{\ell}, \hat{m})$  over  $U$  is called an  $m$ -polar intuitionistic fuzzy normal subalgebra of  $X$  if it satisfies conditions (5) and (8).

**Example 5.** Consider a B-algebra  $U = \{0, 1, 2, 3\}$  as in Example 1. Let  $(\hat{\ell}, \hat{m})$  be a 5-polar intuitionistic fuzzy set on  $U$  given by

$$\begin{aligned} (\hat{\ell}, \hat{m}) : U &\rightarrow [0, 1]^5 \times [0, 1]^5, \\ x &\mapsto \begin{cases} ((0.7, 0.2), (0.6, 0.3), (0.8, 0.1), (0.5, 0.3), (0.4, 0.4)) & \text{if } x \in \{0, 3\}, \\ ((0.3, 0.6), (0.4, 0.5), (0.6, 0.3), (0.2, 0.4), (0.3, 0.6)) & \text{if } x \in \{1, 2\}. \end{cases} \end{aligned}$$

It is easy to see that  $(\hat{\ell}, \hat{m})$  is a 5-polar intuitionistic fuzzy normal over  $U$ .

**Proposition 5.** Every  $m$ -polar intuitionistic fuzzy normal over  $U$  is an  $m$ -polar intuitionistic fuzzy normal subalgebra of  $U$ .

**Proof.** If we let  $y := 0, b := 0$  and  $a := y$  in (9), then we have  $(\pi_i \circ \hat{\ell})((x * y) * (0 * 0)) \geq \inf\{(\pi_i \circ \hat{\ell})(x * 0), (\pi_i \circ \hat{\ell})(y * 0)\}$  and  $(\pi_i \circ \hat{m})((x * y) * (0 * 0)) \leq \sup\{(\pi_i \circ \hat{m})(x * 0), (\pi_i \circ \hat{m})(y * 0)\}$  for all  $x, y \in U$  and  $i = 1, \dots, m$ . By applying (B2) and (B1), we obtain  $(\pi_i \circ \hat{\ell})(x * y) \geq \inf\{(\pi_i \circ \hat{\ell})(x), (\pi_i \circ \hat{\ell})(y)\}$  and  $(\pi_i \circ \hat{m})(x * y) \leq \sup\{(\pi_i \circ \hat{m})(x), (\pi_i \circ \hat{m})(y)\}$  for all  $x, y \in U$  and  $i = 1, \dots, m$ . It follows that  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$ . Therefore  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy normal subalgebra of  $U$ .  $\square$

The following example shows that the converse of Proposition 5 need not be true in general.

**Example 6.** Consider a  $B$ -algebra  $U := \{0, 1, 2, 3, 4, 5\}$  as in Example 2. Let  $(\hat{\ell}, \hat{m})$  be a 5-polar intuitionistic fuzzy set on  $U$  given by

$$(\hat{\ell}, \hat{m}) : U \rightarrow [0, 1]^5 \times [0, 1]^5,$$

$$x \mapsto \begin{cases} ((0.6, 0.2), (0.6, 0.2), (0.7, 0.25), (0.7, 0.1), (0.7, 0.28)) & \text{if } x = 0, \\ ((0.4, 0.5), (0.5, 0.3), (0.6, 0.3), (0.3, 0.4), (0.6, 0.39)) & \text{if } x = 5, \\ ((0.3, 0.6), (0.4, 0.5), (0.5, 0.4), (0.2, 0.7), (0.4, 0.5)) & \text{if } x \in \{1, 2, 3, 4\}. \end{cases}$$

It is easy to check that  $(\hat{\ell}, \hat{m})$  is a 5-polar intuitionistic fuzzy subalgebra of  $U$ , but not an  $m$ -polar fuzzy normal over  $U$ , since  $0.4 = (\pi_2 \circ \hat{\ell})(1) = (\pi_2 \circ \hat{\ell})((1 * 3) * (4 * 2)) \not\geq \inf\{(\pi_2 \circ \hat{\ell})(1 * 4), (\pi_2 \circ \hat{\ell})(3 * 2)\} = \inf\{(\pi_2 \circ \hat{\ell})(5), (\pi_2 \circ \hat{\ell})(5)\} = \inf\{0.5, 0.5\} = 0.5$  and/or  $0.5 = (\pi_2 \circ \hat{m})(1) = (\pi_2 \circ \hat{m})((1 * 3) * (4 * 2)) \not\leq \sup\{(\pi_2 \circ \hat{m})(1 * 4), (\pi_2 \circ \hat{m})(3 * 2)\} = \sup\{(\pi_2 \circ \hat{m})(5), (\pi_2 \circ \hat{m})(5)\} = \sup\{0.3, 0.3\} = 0.3$ .

**Theorem 9.** If  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy set over  $U$ , then the followings are equivalent:

- (i)  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy normal subalgebra of  $U$ ,
- (ii) the  $m$ -polar upper and lower level sets  $U(\hat{\ell}; \hat{t})$  and  $L(\hat{m}; \hat{r})$  of  $(\hat{\ell}, \hat{m})$  are normals of  $U$  for all  $(\hat{t}, \hat{r}) \in [0, 1]^m \times [0, 1]^m$  with  $U(\hat{\ell}; \hat{t}) \neq \emptyset \neq L(\hat{m}; \hat{r})$ .

**Proof.** Suppose that  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy normal subalgebra of  $U$ . Let  $x, y, a, b \in X$  be such that  $x * y, a * b \in U(\hat{\ell}; \hat{t})$  and  $x * y, a * b \in L(\hat{m}; \hat{r})$  for all  $(\hat{t}; \hat{r}) \in [0, 1]^m \times [0, 1]^m$  with  $U(\hat{\ell}; \hat{t}) \neq \emptyset \neq L(\hat{m}; \hat{r})$ . Then  $(\pi_i \circ \hat{\ell})(x * y) \geq t_i, (\pi_i \circ \hat{\ell})(a * b) \geq t_i, (\pi_i \circ \hat{m})(x * y) \leq r_i$  and  $(\pi_i \circ \hat{m})(a * b) \leq r_i$  for all  $i = 1, \dots, m$ . It follows from (9) that

$$(\pi_i \circ \hat{\ell})((x * a) * (y * b)) \geq \inf\{(\pi_i \circ \hat{\ell})(x * y), (\pi_i \circ \hat{\ell})(a * b)\} \geq t_i,$$

$$\hat{m}((x * a) * (y * b)) \leq \sup\{\pi_i \circ \hat{m}(x * y), (\pi_i \circ \hat{m})(a * b)\} \leq r_i$$

for all  $i = 1, 2, \dots, m$ . Hence  $(x * a) * (y * b) \in U(\hat{\ell}; \hat{t})$  and  $(x * a) * (y * b) \in L(\hat{m}; \hat{r})$ . Therefore,  $U(\hat{\ell}; \hat{t})$  and  $L(\hat{m}; \hat{r})$  are normal over  $U$ . Thus  $U(\hat{\ell}; \hat{t})$  and  $L(\hat{m}; \hat{r})$  are normals of  $U$ .

Conversely, suppose that the  $m$ -polar upper and lower level sets  $U(\hat{\ell}; \hat{t})$  and  $L(\hat{m}; \hat{r})$  of  $(\hat{\ell}, \hat{m})$  are normals of  $U$  for all  $(\hat{t}, \hat{r}) \in [0, 1]^m \times [0, 1]^m$  with  $U(\hat{\ell}; \hat{t}) \neq \emptyset \neq L(\hat{m}; \hat{r})$ . Now, assume that there exist  $x, y, a, b \in X$  such that  $\hat{\ell}((x * a) * (y * b)) < \inf\{\hat{\ell}(x * y), \hat{\ell}(a * b)\}$  or  $\hat{m}((x * a) * (y * b)) > \sup\{\hat{m}(x * y), \hat{m}(a * b)\}$ . If we take  $\hat{t} := \inf\{\hat{\ell}(x * y), \hat{\ell}(a * b)\}$  and  $\hat{r} := \sup\{\hat{m}(x * y), \hat{m}(a * b)\}$ , then  $x * y, a * b \in U(\hat{\ell}; \hat{t})$  or  $x * y, a * b \in L(\hat{m}; \hat{r})$ . Since  $U(\hat{\ell}; \hat{t})$  and  $L(\hat{m}; \hat{r})$  are normals of  $U$ , we have  $(x * a) * (y * b) \in U(\hat{\ell}; \hat{t})$  or  $(x * a) * (y * b) \in L(\hat{m}; \hat{r})$ . Hence  $\hat{\ell}((x * a) * (y * b)) \geq \inf\{\hat{\ell}(x * y), \hat{\ell}(a * b)\}$  or  $\hat{m}((x * a) * (y * b)) \leq \sup\{\hat{m}(x * y), \hat{m}(a * b)\}$ , which lead to a contradiction. Thus  $\hat{\ell}((x * a) * (y * b)) \leq \inf\{\hat{\ell}(x * y), \hat{\ell}(a * b)\}$  and  $\hat{m}((x * a) * (y * b)) \geq \sup\{\hat{m}(x * y), \hat{m}(a * b)\}$  for any  $x, y, a, b \in U$ .

Therefore  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar fuzzy normal over  $U$ . By Proposition 5,  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy normal subalgebra of  $U$ .  $\square$

**Proposition 6.** Let an  $m$ -polar intuitionistic fuzzy set  $(\hat{\ell}, \hat{m})$  over  $U$  be an  $m$ -polar fuzzy normal subalgebra over  $U$ . Then  $\hat{\ell}(x * y) = \hat{\ell}(y * x)$  and  $\hat{m}(x * y) = \hat{m}(y * x)$  for all  $x, y \in U$ , i.e.,  $(\pi_i \circ \hat{\ell})(x * y) = (\pi_i \circ \hat{\ell})(y * x)$  and  $(\pi_i \circ \hat{m})(x * y) = (\pi_i \circ \hat{m})(y * x)$  for all  $x, y \in U$  and  $i = 1, \dots, m$ .

**Proof.** Let  $x, y \in U$ . By (B1) and (B2), we have

$$\begin{aligned} (\pi_i \circ \hat{\ell})(x * y) &= (\pi_i \circ \hat{\ell})((x * y) * (x * x)) \\ &\geq \inf\{(\pi_i \circ \hat{\ell})(x * x), (\pi_i \circ \hat{\ell})(y * x)\} \\ &= \inf\{(\pi_i \circ \hat{\ell})(0), (\pi_i \circ \hat{\ell})(y * x)\} \\ &= (\pi_i \circ \hat{\ell})(y * x) \end{aligned}$$

for all  $i = 1, \dots, m$ . Interchanging  $x$  with  $y$ , we obtain  $(\pi_i \circ \hat{\ell})(y * x) = (\pi_i \circ \hat{\ell})(x * y)$  for all  $i = 1, \dots, m$ .

By (B1) and (B2), we have

$$\begin{aligned} (\pi_i \circ \hat{m})(x * y) &= (\pi_i \circ \hat{m})((x * y) * (x * x)) \\ &\leq \sup\{(\pi_i \circ \hat{m})(x * x), (\pi_i \circ \hat{m})(y * x)\} \\ &= \sup\{(\pi_i \circ \hat{m})(0), (\pi_i \circ \hat{m})(y * x)\} \\ &= (\pi_i \circ \hat{m})(y * x) \end{aligned}$$

for all  $i = 1, \dots, m$ . Interchanging  $x$  with  $y$ , we obtain  $(\pi_i \circ \hat{m})(y * x) = (\pi_i \circ \hat{m})(x * y)$  for all  $i = 1, \dots, m$ .  $\square$

**Proposition 7.** Let  $(\hat{\ell}, \hat{m})$  be an  $m$ -polar intuitionistic fuzzy normal subalgebra of a  $B$ -algebra  $U$ . Then the set  $U_{(\hat{\ell}, \hat{m})} := \{x \in U \mid \hat{\ell}(x) = \hat{\ell}(0), \hat{m}(x) = \hat{m}(0)\}$  is a normal subalgebra of  $U$ .

**Proof.** We show that  $U_{(\hat{\ell}, \hat{m})}$  is a normal over  $U$ . Let  $a, b, x, y \in U$  be such that  $x * y \in U_{(\hat{\ell}, \hat{m})}$  and  $a * b \in U_{(\hat{\ell}, \hat{m})}$ . Then  $(\pi_i \circ \hat{\ell})(x * y) = (\pi_i \circ \hat{\ell})(0) = (\pi_i \circ \hat{\ell})(a * b)$  and  $(\pi_i \circ \hat{m})(x * y) = (\pi_i \circ \hat{m})(0) = (\pi_i \circ \hat{m})(a * b)$  for all  $i = 1, \dots, m$ . Since  $(\hat{\ell}, \hat{m})$  is an  $m$ -polar intuitionistic fuzzy normal subalgebra of  $U$ , we have  $(\pi_i \circ \hat{\ell})((x * a) * (y * b)) \geq \inf\{(\pi_i \circ \hat{\ell})(x * y), (\pi_i \circ \hat{\ell})(a * b)\} = (\pi_i \circ \hat{\ell})(0)$  and  $(\pi_i \circ \hat{m})((x * a) * (y * b)) \leq \sup\{(\pi_i \circ \hat{m})(x * y), (\pi_i \circ \hat{m})(a * b)\} = (\pi_i \circ \hat{m})(0)$  for all  $i = 1, \dots, m$ . By using (7), we obtain that  $(\pi_i \circ \hat{\ell})((x * a) * (y * b)) = (\pi_i \circ \hat{\ell})(0)$  and  $(\pi_i \circ \hat{m})((x * a) * (y * b)) = (\pi_i \circ \hat{m})(0)$  for all  $i = 1, \dots, m$ . Hence  $(x * a) * (y * b) \in U_{(\hat{\ell}, \hat{m})}$ . By Proposition 5,  $U_{(\hat{\ell}, \hat{m})}$  is a normal subalgebra of  $U$ . This completes the proof.  $\square$

### 5. Conclusions

In this paper, we applied the notions of an  $m$ -polar fuzzy set and an  $m$ -polar intuitionistic fuzzy set to subalgebras and normals in  $B$ -algebras. We introduced the notion of an  $m$ -polar fuzzy (normal) subalgebra in a  $B$ -algebra and investigated some related some properties. We obtained an equivalent condition for an  $m$ -polar fuzzy set  $\hat{\ell}$  over  $U$  to be an  $m$ -polar fuzzy subalgebra of  $U$ , and we showed that the  $\hat{\ell}_\wedge$ -product  $\hat{\ell}_{U \wedge V}$  of  $m$ -polar fuzzy subalgebras  $\hat{\ell}_U$  and  $\hat{\ell}_V$  is also an  $m$ -polar fuzzy subalgebra of  $U \times V$ . We found equivalent conditions for an  $m$ -polar intuitionistic fuzzy set  $(\hat{\ell}, \hat{m})$  over  $U$  to be an  $m$ -polar intuitionistic fuzzy subalgebra of  $U$ . Moreover, we showed that an  $m$ -polar intuitionistic fuzzy set  $(\hat{\ell}, \hat{m})$  over  $U$  to be an  $m$ -polar intuitionistic fuzzy (normal) subalgebra of  $U$ . Given an  $m$ -polar fuzzy set, we constructed a simple  $m$ -polar fuzzy set and discussed on  $m$ -polar intuitionistic fuzzy

subalgebras of  $B$ -algebras. The purpose of our research in future is to continue to think about these things and construct new concepts in several general algebraic structures.

The results of this research will be expanded to several algebraic structures, such as groups,  $BE$ -algebras,  $BF$ -algebras, and  $d$ -algebras.

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