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Research article

Almost multi-quadratic mappings in non-Archimedean spaces

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Abstract: In this article, we introduce the generalized multi-quadratic mappings and then describe them as a equation. As a special case of such mappings, we study the Hyers-Ulam stability of multi-quadratic mappings in non-Archimedean spaces by applying a fixed point theorem. Moreover, we prove that such mappings can be hyperstable.

Keywords: non-Archimedean space; Hyers-Ulam stability; multi-quadratic mapping; fixed point method
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1. Introduction

Throughout this paper, $\mathbb{N}$ and $\mathbb{Q}$ are the set of all positive integers and rational numbers, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{R}_+ := [0, \infty)$. Moreover, for the set X, we denote $\overbrace{X \times X \times \cdots \times X}^{n-\text{times}}$ by X^n. For any $l \in \mathbb{N}_0, m \in \mathbb{N}$, $t = (t_1, \ldots, t_m) \in \{-1, 1\}^m$ and $x = (x_1, \ldots, x_m) \in V^m$ we write $lx := (lx_1, \ldots, lx_m)$ and $tx := (t_1x_1, \ldots, t_mx_m)$, where ra stands, as usual, for the rth power of an element a of the commutative group V.

Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. Recall from [15] that a mapping $f : V^n \longrightarrow W$ is called *multi-additive* if it is additive (satisfies Cauchy's functional equation $A(x + y) = A(x) + A(y)$) in each variable. Some basic facts on such mappings can be found in [19] and many other sources, where their application to the representation of polynomial functions is also presented. Besides, f is said to be *multi-quadratic* if it is quadratic in each variable, i.e., it satisfies the quadratic equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \tag{1.1}$$

in each variable [14]. In [27], Zhao *et al.* proved that the mapping $f : V^n \longrightarrow W$ is multi-quadratic if and only if the following relation holds.

$$\sum_{t\in\{-1,1\}^n} f(x_1 + tx_2) = 2^n \sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f(x_{1j_1}, x_{2j_2}, \ldots, x_{nj_n}) \tag{1.2}$$

where $x_j = (x_{1j}, x_{2j}, \ldots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$.

The stability of a functional equation originated from a question raised by Ulam: "when is it true that the solution of an equation differing slightly from a given one must of necessity be close to the solution of the given equation?" (see [26]). The first answer (in the case of Cauchy's functional equation in Banach spaces) to Ulam's question was given by Hyers in [18]. Following his result, a great number of papers on the the stability problems of several functional equations have been extensively published as generalizing Ulam's problem and Hyers' theorem in various directions; see for instance [1,4,21,23,28], and the references given there.

It is worth mentioning that the fixed point theorems have been considered for various mappings, integral and fractional equations in [3,12,13]. Some investigations have been carried out on the stability of functional equations via fixed point theorems in [5–7, 11]. Moreover, the fixed point theorem were recently applied to obtain similar stability results in [9, 16, 22, 25].

In [14, 15], Ciepliński studied the generalized Hyers-Ulam stability of multi-additive and multi-quadratic mappings in Banach spaces, respectively (see also [27]). Next, the stability of multi-Cauchy-Jensen mappings in non-Archimedean spaces are studied in [2] by applying the fixed point method, which was proved and used for the first time to investigate the Hyers-Ulam stability of functional equations in [11]. For more information about multi-quadratic, multi-cubic and multi-quartic mappings, we refer to [8, 10, 20, 24].

In this paper, we define the generalized multi-quadratic mappings and present a characterization of such mappings. In other words, we reduce the system of n equations defining the generalized multi-quadratic mappings to obtain a single functional equation. Then, we prove the generalized Hyers-Ulam stability of multi-quadratic mapping (which was recently introduced by Salimi and Bodaghi in [24]) in non-Archimedean normed spaces by a fixed point method.

2. Characterization of generalized multi-quadratic mappings

From now on, let V and W be vector spaces over $\mathbb{Q}$, $n \in \mathbb{N}$ and $x_i^n = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. Let $l_j \in \{1, 2\}$. Put

$$M_i^n = \{x = (x_{l_1 1}, x_{l_2 2}, \ldots, x_{l_n n}) \in V^n | \operatorname{Card}\{l_j : l_j = 1\} = i\}. \tag{2.1}$$

We shall denote x_i^n and M_i^n by x_i and M_i, respectively if there is no risk of ambiguity.

A general form of (1.1), say the *generalized quadratic* functional equation is as follows:

$$\mathfrak{Q}(ax + by) + \mathfrak{Q}(ax - by) = 2a^2\mathfrak{Q}(x) + 2b^2\mathfrak{Q}(y) \tag{2.2}$$

where a, b are the fixed non-zero numbers in $\mathbb{Q}$. The mapping $f : V^n \longrightarrow W$ is said to be *generalized n-multi-quadratic* or *generalized multi-quadratic* if f is generalized quadratic in each variable.

Put $\mathbf{n} := \{1,\ldots,n\}$, $n \in \mathbb{N}$. For a subset $T = \{j_1,\ldots,j_i\}$ of $\mathbf{n}$ with $1 \le j_1 < \ldots < j_i \le n$ and $x = (x_1,\ldots,x_n) \in V^n$,

$$ {}_T x := (0,\ldots,0,x_{j_1},0,\ldots,0,x_{j_i},0,\ldots,0) \in V^n $$

denotes the vector which coincides with x in exactly those components, which are indexed by the elements of T and whose other components are set equal zero. Note that ${}_{\phi}x = 0$, ${}_{\mathbf{n}}x = x$. We use these notations in the proof of upcoming lemma.

Let $a \in \mathbb{Q}$ be as in (2.2). We say the mapping $f : V^n \longrightarrow W$ satisfies *the r-power condition* in the jth variable if

$$ f(z_1,\ldots,z_{j-1},az_j,z_{j+1},\ldots,z_n) = a^r f(z_1,\ldots,z_{j-1},z_j,z_{j+1},\ldots,z_n), $$

for all $(z_1,\ldots,z_n) \in V^n$. In the sequel, $\begin{pmatrix} n \\ k \end{pmatrix}$ is the binomial coefficient defined for all $n,k \in \mathbb{N}_0$ with $n \ge k$ by $n!/(k!(n-k)!)$. We shall to show that if a mapping $f : V^n \longrightarrow W$ satisfies the equation

$$ \sum_{q\in\{-1,1\}^n} f(ax_1 + qbx_2) = 2^n \sum_{i=0}^{n} a^{2i}b^{2(n-i)} \sum_{x\in M_i} f(x), \tag{2.3} $$

where a,b are the fixed non-zero in $\mathbb{Q}$ with $a+b \ne 1$, then it is generalized multi-quadratic quadratic. In order to do this, we need the next lemma.

Lemma 2.1. *If the mapping $f : V^n \longrightarrow W$ satisfies the Eq.* (2.3) *with 2-power condition in all variables, then $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.*

Proof. Putting $x_1 = x_2 = (0,\ldots,0)$ in (2.3), we get

$$ 2^n f(0,\ldots,0) = 2^n \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} a^{2i}b^{2(n-i)} f(0,\ldots,0) = 2^n(a+b)^{2n} f(0,\ldots,0). $$

Since $a+b \ne 1$, $f(0,\ldots,0) = 0$. Letting $x_{1k} = 0$ for all $k \in \{1,\ldots,n\}\backslash\{j\}$ and $x_{2k} = 0$ for $1 \le k \le n$ in (2.3) and using $f(0,\ldots,0) = 0$, we obtain

$$ \begin{aligned} 2^n a^2 f(0,\ldots,0,x_{1j},0,\cdots,0) &= 2^n f(0,\ldots,0,ax_{1j},0,\ldots,0) \\ &= 2^n a^2 \sum_{i=0}^{n-1} \begin{pmatrix} n-1 \\ i \end{pmatrix} a^{2i}b^{2(n-1-i)} f(0,\ldots,0,x_{1j},0,\ldots,0) \\ &= 2^n a^2 (a+b)^{2(n-1)} f(0,\ldots,0,x_{1j},0,\ldots,0). \end{aligned} $$

Hence, $f(0,\ldots,0,x_{1j},0,\ldots,0) = 0$. We now assume that $f({}_{k-1}x_1) = 0$ for $1 \le k \le n-1$. We are going to show that $f({}_k x_1) = 0$. By assumptions, the above process can be repeated to obtain

$$ 2^n f({}_k x_1) = 2^n a^{2k} \sum_{i=0}^{n-k} \begin{pmatrix} n-k \\ i \end{pmatrix} a^{2i}b^{2(n-k-i)} f({}_k x_1) = 2^n a^{2k}(a+b)^{2(n-k)} f({}_k x_1), \tag{2.4} $$

where $1 \le k \le n-1$ and so $f({}_k x_1) = 0$. This shows that $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero. □

Theorem 2.2. *Consider the mapping $f : V^n \longrightarrow W$. Then, the following assertions are equivalent:*

(i) f is generalized multi-quadratic;
(ii) f satisfies Eq. (2.3) *with* 2*-power condition in all variables.*

Proof. (i)⇒(ii) We firstly note that it is not hard to show that f satisfies 2-power condition in all variables. We now prove that f satisfies Eq. (2.3) by induction on n. For $n = 1$, it is trivial that f satisfies Eq. (2.2). Assume that (2.3) is valid for some positive integer $n > 1$. Then,

$$\begin{aligned}&\sum_{q\in\{-1,1\}^{n+1}} f\left(ax_1^{n+1}+qbx_2^{n+1}\right) = 2a^2\sum_{q\in\{-1,1\}^{n}} f\left(ax_1^{n}+qbx_2^{n}, x_{1n+1}\right)\\&+2b^2\sum_{q\in\{-1,1\}^{n}} f\left(ax_1^{n}+qbx_2^{n}, x_{2n+1}\right)\\&=2^{n+1}a^2\sum_{i=0}^{n}a^{2i}b^{2(n-i)}\sum_{x\in M_i^n} f\left(x, x_{1n+1}\right)+2^{n+1}b^2\sum_{i=0}^{n}a^{2i}b^{2(n-i)}\sum_{x\in M_i^n} f(x, x_{2n+1})\\&=2^{n+1}\sum_{i=0}^{n+1}a^{2i}b^{2(n+1-i)}\sum_{x\in M_i^{n+1}} f(x).\end{aligned}$$

This means that (2.3) holds for $n+1$.

(ii)⇒(i) Fix $j\in\{1,\cdots,n\}$, put $x_{2k}=0$ for all $k\in\{1,\cdots,n\}\backslash\{j\}$. Using Lemma 2.1, we obtain

$$\begin{aligned}2^{n-1}a^{2(n-1)}&[f(x_{11},\dots,x_{1j-1},ax_{1j}+bx_{2j},x_{1j+1},\dots,x_{1n})\\&+f(x_{11},\cdots,x_{1j-1},ax_{1j}-bx_{2j},x_{1j+1},\cdots,x_{1n})]\\&=2^{n-1}[f\left(ax_{11},\dots,ax_{1j-1},ax_{1j}+bx_{2j},ax_{1j+1},\dots,ax_{1n}\right)\\&+f\left(ax_{11},\dots,ax_{1j-1},ax_{1j}-bx_{2j},ax_{1j+1},\dots,ax_{1n}\right)]\\&=2^{n}a^{2(n-1)}[a^2f(x_{11},\dots,x_{1j-1},x_{1j},x_{1j+1},\dots,x_{1n})\\&+b^2f(x_{11},\dots,x_{1j-1},x_{2j},x_{1j+1},\dots,x_{1n})].\end{aligned}\tag{2.5}$$

It follows from relation (2.5) that f is quadratic in the jth variable. Since j is arbitrary, we obtain the desired result. □

3. Stability Results

An special case of (2.2) is the following quadratic functional equation when $a=b=\frac{1}{2}$.

$$2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)=Q(x)+Q(y).\tag{3.1}$$

A mapping $f: V^n \longrightarrow W$ is called *n-multi-quadratic* or *multi-quadratic* if f is quadratic in each variable (see Eq. (3.1)). It is shown in [24, Proposition 2.2] (without extra 2-power condition in each variable) that a mapping $f: V^n \longrightarrow W$ is multi-quadratic if and only if it satisfies the equation

$$2^n\sum_{q\in\{-1,1\}^n} f\left(\frac{x_1+qx_2}{2}\right)=\sum_{l_1,\dots,l_n\in\{1,2\}} f(x_{l_11},x_{l_22},\dots,x_{l_nn}).\tag{3.2}$$

In this section, we prove the generalized Hyers-Ulam stability of Eq. (3.2) in non-Archimedean spaces.

We recall some basic facts concerning non-Archimedean spaces and some preliminary results. By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let $\mathcal{X}$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : \mathcal{X} \longrightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\|, \quad (x \in \mathcal{X}, r \in \mathbb{K})$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \qquad (x, y \in \mathcal{X}).$$

Then $(\mathcal{X}, \|\cdot\|)$ is called a *non-Archimedean normed space*. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\|; m \leq j \leq n-1\} \quad (n \geq m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space $\mathcal{X}$. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

In [17], Hensel discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most interesting example of non-Archimedean normed spaces is p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$.

Let p be a prime number. For any non-zero rational number $x = p^r \frac{m}{n}$ in which m and n are coprime to the prime number p. Consider the p-adic absolute value $|x|_p = p^{-r}$ on $\mathbb{Q}$. It is easy to check that $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ which is denoted by $\mathbb{Q}_p$ is said to be the p-adic number field. One should remember that if $p > 2$, then $|2^n| = 1$ in for all integers n.

Throughout, for two sets A and B, the set of all mappings from A to B is denoted by B^A. The proof is based on a fixed point result that can be derived from [11, Theorem 1]. To present it, we introduce the following three hypotheses:

(H1) E is a nonempty set, Y is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2, $j \in \mathbb{N}$, $g_1, \ldots, g_j : E \longrightarrow E$ and $L_1, \ldots, L_j : E \longrightarrow \mathbb{R}_+$,
(H2) $\mathcal{T} : Y^E \longrightarrow Y^E$ is an operator satisfying the inequality

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \leq \max_{i\in\{1,\ldots,j\}} L_i(x) \, \|\lambda(g_i(x)) - \mu(g_i(x))\|, \quad \lambda, \mu \in Y^E, x \in E,$$

(H3) $\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ is an operator defined through

$$\Lambda\delta(x) := \max_{i\in\{1,\ldots,j\}} L_i(x)\delta(g_i(x)) \qquad \delta \in \mathbb{R}_+^E, x \in E.$$

Here, we highlight the following theorem which is a fundamental result in fixed point theory [11]. This result plays a key role in obtaining our goal in this paper.

Theorem 3.1. *Let hypotheses* (H1)-(H3) *hold and the function* $\epsilon : E \longrightarrow \mathbb{R}_+$ *and the mapping* $\varphi : E \longrightarrow Y$ *fulfill the following two conditions:*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \epsilon(x), \quad \lim_{l\to\infty} \Lambda^l \epsilon(x) = 0 \qquad (x \in E).$$

Then, for every $x \in E$, *the limit* $\lim_{l\to\infty} \mathcal{T}^l\varphi(x) =: \psi(x)$ *and the function* $\psi \in Y^E$, *defined in this way, is a fixed point of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \leq \sup_{l\in\mathbb{N}_0} \Lambda^l \epsilon(x) \qquad (x \in E).$$

Here and subsequently, given the mapping $f : V^n \longrightarrow W$, we consider the difference operator $\Gamma f : V^n \times V^n \longrightarrow W$ by

$$\Gamma f(x_1, x_2) = 2^n \sum_{q\in\{-1,1\}^n} f\left(\frac{x_1 + qx_2}{2}\right) - \sum_{l_1,\ldots,l_n\in\{1,2\}} f(x_{l_1 1}, x_{l_2 2}, \ldots, x_{l_n n}).$$

In the sequel, S stands for $\{0, 1\}^n$. With this notations, we have the upcoming result.

Theorem 3.2. *Let* V *be a linear space and* W *be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from* 2. *Suppose that* $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ *is a mapping satisfying the equality*

$$\lim_{l\to\infty} \left(\frac{1}{|2|^{2n}}\right)^l \max_{s\in S} \phi(2^l(sx_1, sx_2)) = 0 \tag{3.3}$$

for all $x_1, x_2 \in V^n$. *Assume also* $f : V^n \longrightarrow W$ *is a mapping satisfying the inequality*

$$\|\Gamma f(x_1, x_2)\| \leq \phi(x_1, x_2) \tag{3.4}$$

for all $x_1, x_2 \in V^n$. *Then, there exists a unique multi-quadratic mapping* $\mathcal{Q} : V^n \longrightarrow W$ *such that*

$$\|f(x) - \mathcal{Q}(x)\| \leq \sup_{l\in\mathbb{N}_0} \left(\frac{1}{|2|^{2n}}\right)^{l+1} \max_{s\in S} \phi(2^l sx, 0) \tag{3.5}$$

for all $x \in V^n$.

Proof. Replacing $x = x_1 = (x_{11}, \ldots, x_{1n}), x_2 = (x_{21}, \ldots, x_{2n})$ by $2x_1, (0, \ldots, 0)$ in (3.4), respectively, we have

$$\|2^{2n} f(x) - \sum_{s\in S} f(2sx)\| \leq \phi(2x, 0) \tag{3.6}$$

for all $x \in V^n$. Inequality (3.6) implies that

$$\|f(x) - \mathcal{T}f(x)\| \leq \xi(x) \tag{3.7}$$

for all $x \in V^n$, where $\xi(x) := \frac{1}{|2|^{2n}}\phi(2x, 0)$ and $\mathcal{T}f(x) := \frac{1}{2^{2n}} \sum_{s\in S} f(2sx)$. Define $\Lambda\eta(x) := \max_{s\in S} \frac{1}{|2|^{2n}}\eta(2sx)$ for all $\eta \in \mathbb{R}_+^{V^n}, x \in V^n$. It is easy to see that Λ has the form described in

(H3) with $E = V^n$, $g_i(x) := g_s(x) = 2sx$ for all $x \in V^n$ and $L_i(x) = \frac{1}{|2|^{2n}}$ for any i. Moreover, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we get

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \le \max_{s\in S} \frac{1}{|2|^{2n}} \|\lambda(2sx) - \mu(2sx)\|.$$

The above inequality shows that the hypothesis (H2) holds. By induction on l, one can check that for any $l \in \mathbb{N}$ and $x \in V^n$ that

$$\Lambda^l \xi(x) := \left(\frac{1}{|2|^{2n}}\right)^l \max_{s\in S} \xi(2^l sx) \tag{3.8}$$

for all $x \in V^n$. Indeed, by definition of Λ, equality (3.8) is true for $l = 1$. If (3.8) holds for $l \in \mathbb{N}$, then

$$\begin{aligned}\Lambda^{l+1}\xi(x) &= \Lambda(\Lambda^l \xi(x)) = \Lambda\left(\left(\frac{1}{|2|^{2n}}\right)^l \max_{s\in S} \xi(2^l sx)\right)\\ &= \left(\frac{1}{|2|^{2n}}\right)^l \max_{s\in S} \Lambda\left(\xi(2^l sx)\right) = \left(\frac{1}{|2|^{2n}}\right)^{l+1} \max_{s\in S} \xi(2^{l+1} sx)\end{aligned}$$

for all $x \in V^n$. Relations (3.7) and (3.8) necessitate that all assumptions of Theorem 3.1 are satisfied. Hence, there exists a unique mapping $Q : V^n \longrightarrow W$ such that $Q(x) = \lim_{l\to\infty}(\mathcal{T}^l f)(x)$ for all $x \in V^n$, and also (3.5) holds. We are going to show that

$$\|\Gamma(\mathcal{T}^l f)(x_1, x_2)\| \le \left(\frac{1}{|2|^{2n}}\right)^l \max_{s\in S} \phi(2^l sx_1, 2^l sx_2) \tag{3.9}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}$. We argue by induction on l. For $l = 1$ and for all $x_1, x_2 \in V^n$, we have

$$\begin{aligned}&\|\Gamma(\mathcal{T} f)(x_1, x_2)\|\\ &= \left\| 2^n \sum_{q\in\{-1,1\}^n} (\mathcal{T} f)\left(\frac{x_1 + qx_2}{2}\right) - \sum_{l_1,\ldots,l_n\in\{1,2\}} (\mathcal{T} f)(x_{l_1 1}, x_{l_2 2}, \ldots, x_{l_n n}) \right\|\\ &= \left\| \frac{1}{2^n} \sum_{q\in\{-1,1\}^n} \sum_{s\in S} f(sx_1 + sqx_2) - \frac{1}{2^{2n}} \sum_{l_1,\ldots,l_n\in\{1,2\}} \sum_{s\in S} f(2sx_{l_1 1}, 2sx_{l_2 2}, \ldots, 2sx_{l_n n}) \right\|\\ &= \left\| \frac{1}{2^{2n}} \sum_{s\in S} \Gamma(f)(2(sx_1, sx_2)) \right\|\\ &\le \frac{1}{|2|^{2n}} \max_{s\in S} \|\Gamma(f)(2(sx_1, sx_2))\|\\ &\le \frac{1}{|2|^{2n}} \max_{s\in S} \phi(2(sx_1, sx_2))\end{aligned}$$

for all $x_1, x_2 \in V^n$. Assume that (3.9) is true for an $l \in \mathbb{N}$. Then

$$\|\Gamma(\mathcal{T}^{l+1} f)(x_1, x_2)\|$$

$$= \left\| 2^n \sum_{q\in\{-1,1\}^n} (\mathcal{T}^{l+1} f)\left(\frac{x_1 + qx_2}{2}\right) - \sum_{l_1,\ldots,l_n\in\{1,2\}} (\mathcal{T}^{l+1} f)(x_{l_1 1}, x_{l_2 2}, \ldots, x_{l_n n}) \right\|$$
$$= \left\| \frac{1}{2^n} \sum_{q\in\{-1,1\}^n} \sum_{s\in S} \mathcal{T}^l f(sx_1 + sqx_2) - \frac{1}{2^{2n}} \sum_{l_1,\ldots,l_n\in\{1,2\}} \sum_{s\in S} \mathcal{T}^l f(2sx_{l_1 1}, 2sx_{l_2 2}, \ldots, 2sx_{l_n n}) \right\|$$
$$= \left\| \frac{1}{2^{2n}} \sum_{s\in S} \Gamma(\mathcal{T}^l f)(2(sx_1, sx_2)) \right\|$$
$$\leq \frac{1}{|2|^{2n}} \max_{s\in S} \left\| \Gamma(\mathcal{T}^l f)(2(sx_1, sx_2)) \right\|$$
$$\leq \left(\frac{1}{|2|^{2n}}\right)^{l+1} \max_{s\in S} \phi(2^{l+1}(sx_1, sx_2)) \tag{3.10}$$

for all $x_1, x_2 \in V^n$. Letting $l \to \infty$ in (3.9) and applying (3.3), we arrive at $\Gamma Q(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping Q satisfies (3.2) and the proof is now completed. □

The following example is an application of Theorem 3.2 concerning the stability of multi-quadratic mappings when the norm of $\Gamma f(x_1, x_2)$ is controlled by the powers sum of norms of components of vectors x_1 and x_2 in V^n.

Example 3.3. Let $p \in \mathbb{R}$ fulfills $p > 2n$. Let V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2 such that $|2| < 1$. Suppose that $f : V^n \longrightarrow W$ is a mapping satisfying the inequality

$$\|\Gamma f(x_1, x_2)\| \leq \sum_{k=1}^{2} \sum_{j=1}^{n} \|x_{kj}\|^p$$

for all $x_1, x_2 \in V^n$. Putting $\phi(x_1, x_2) = \sum_{k=1}^{2} \sum_{j=1}^{n} \|x_{kj}\|^p$, we have $\phi(2^l x_1, 2^l x_2) = |2|^{lp} \phi(x_1, x_2)$ and so

$$\lim_{l\to\infty} \left(\frac{1}{|2|^{2n}}\right)^l max_{s\in S} \sum_{k=1}^{2} \sum_{j=1}^{n} \|2^l sx_{kj}\|^p = \lim_{l\to\infty} \left(\frac{|2|^p}{|2|^{2n}}\right)^l \sum_{k=1}^{2} \sum_{j=1}^{n} \|x_{kj}\|^p = 0$$

for all $x_1, x_2 \in V^n$. On the other hand,

$$sup_{l\in\mathbb{N}} \left(\frac{1}{|2|^{2n}}\right)^{l+1} max_{s\in S} \phi(2^l sx, 0) = \frac{1}{|2|^{2n}} \sum_{j=1}^{n} \|x_{1j}\|^p.$$

By Theorem 3.2, there exists a unique multi-quadratic mapping $Q : V^n \longrightarrow W$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|2|^{2n}} \sum_{j=1}^{n} \|x_{1j}\|^p$$

for all $x \in V^n$.

Recall that a functional equation $\mathcal{F}$ is *hyperstable* if any mapping f satisfying the equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$. Under some conditions functional Eq. (3.2) can be hyperstable as follows.

Corollary 3.4. *Suppose that $p_{kj} > 0$ for $k \in \{1,2\}$ and $j \in \{1,\ldots,n\}$ fulfill $\sum_{k=1}^{2}\sum_{j=1}^{n} p_{kj} > 2n$. Let V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characterisitic different from 2 such that $|2| < 1$. If $f : V^n \longrightarrow W$ is a mapping satifying the inequality*

$$\|\Gamma f(x_1, x_2)\| \leq \prod_{k=1}^{2}\prod_{j=1}^{n} \|x_{kj}\|^{p_{kj}}$$

for all $x_1, x_2 \in V^n$, then f is multi-quadratic.

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Conflict of interest

The authors declare no conflicts of interest.

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