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Research article

On approximate solution of lattice functional equations in Banach f-algebras

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Abstract: The aim of the current manuscript is to prove the Hyers-Ulam stability of supremum, infimum and multiplication preserving functional equations in Banach f-algebras. In fact, by using the direct method and the fixed point method, the Hyers-Ulam stability of the functional equations is proved.

Keywords: Hyers-Ulam stability; functional equation; Banach lattice; f-algebra; fixed point method
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1. Introduction and preliminaries

In 1940, Ulam [20] suggested a problem of stability on group homomorphisms in metric groups. Hyers is the first mathematician who answered the question of Ulam in 1941. He demonstrated the following theorem in [8].

Theorem 1.1. *Let X and Y be two Banach spaces and $f : X \to Y$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for some $\delta > 0$ and all $x, y \in X$. Next there is an exclusive additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \delta$$

for all $x \in X$.

The Hyers' stability theorem was developed by other mathematicians. Recently, numerous consequences concerning the stability of various functional equations in different normed spaces and various control functions have been obtained. The problem of stability of some functional equations have been widely explored by direct methods and there are numerous exciting outcomes regarding this problem ([4, 7, 9–11, 13, 15, 17]). The fixed point approach has been used in other Hyers-Ulam stability investigations. The relationship between Hyers-Ulam stability and fixed point theory has been defined in ([5, 6, 14, 16]).

The first definition of Riesz spaces was done by Riesz in 1930. In [18], Riesz introduced the vector lattice spaces and their properties. A Riesz space (vector lattice) is a vector space which is also a lattice, so that the two structures are compatible in a certain natural way. If, in addition, the space is a Banach space (and, again, a certain natural compatibility axiom is satisfied), it is a Banach lattice. We present some of the terms and concepts of the Riesz spaces used in this article, concisely. However, we relegate the reader to [1, 12, 19, 24], for the fundamental notions and theorems of Riesz spaces and Banach lattices.

A real vector space X is supposed to be a partially orderly vector space or an ordered vector space, if it is equipped with a partial ordering " $\leq$ " that satisfies

1. $x \leq x$ for every $x \in X$.
2. $x \leq y$ and $y \leq x$ implies that $x = y$.
3. $x \leq y$ and $y \leq z$ implies that $x \leq z$.

A Riesz space (or vector lattice) is an ordered vector space in which for all $x, y \in X$ the infimum and supremum of $\{x, y\}$, denoted by $x \wedge y$ and $x \vee y$ respectively, exist in X . The negative part, the positive part, and the absolute value of $x \in X$, are defined by $x^- := -x \vee 0,\ x^+ := x \vee 0,$ and $|x| := x \vee -x$, respectively. Let X be a Riesz space. X is called Archimedean if $\inf\{\frac{x}{n} : n \in \mathbb{N}\} = 0$ for all $x \in X^+$. If $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in X$, then $\|.\|$ is called a lattice norm or Riesz norm on X.

1. $x + y = x \vee y + x \wedge y \quad , \quad -(x \vee y) = -x \wedge -y$.
2. $x + (y \vee z) = (x + y) \vee (x + z) \quad , \quad x + (y \wedge z) = (x + y) \wedge (x + z)$.
3. $|x| = x^+ + x^- \quad , \quad |x + y| \leq |x| + |y|$.
4. $x \leq y$ is equivalent to $x^+ \leq y^+$ and $y^- \leq x^-$.
5. $(x \vee y) \wedge z = (x \wedge y) \vee (y \wedge z) \quad , \quad (x \wedge y) \vee z = (x \vee y) \wedge (y \vee z)$.

Let X be a Riesz space. The sequence $\{x_n\}$ is called uniformly bounded if there exists an element $e \in X^+$ and sequence $\{a_n\} \in l^1$ such that $x_n \leq a_n \cdot e$. A Riesz space X is called uniformly complete if $\sup\{\sum_{i=1}^n x_i \ : \ n \in \mathbb{N}\}$ exists for every uniformly bounded sequence $x_n \subset X^+$. Let X and Y be Banach lattices. Then the function $F : X \to Y$ is called a cone-related function if $F(X^+) = \{F(|x|) : x \in X\} \subset Y^+$.

Theorem 1.2. *[1] For a mapping $F : X \to Y$ defined between two Riesz spaces, the following statements are equivalent:*

1. *F is a lattice homomorphism.*
2. *$F(x)^+ = F(x^+)$ for all $x \in X$.*
3. *$F(x) \wedge F(y) = F(x \wedge y)$, for all $x, y \in X$.*
4. *If $x \wedge y = 0$ in X, then $F(x) \wedge F(y) = 0$ in Y.*
5. *$F(|x|) = |F(x)|$, for every $x \in X$.*

Definition 1.3. [23] The (real) vector lattice (Riesz space) X is named a lattice ordered algebra (Riesz algebra, concisely, l-algebra) if it is a linear algebra (not essentially associative) so that if $a, b \in X^+$, then $ab \in X^+$. The latter property is equal to every of the following declarations:

(i) $|ab| \leq |a||b|$ for all $a, b \in X$;
(ii) $(ab)^+ \leq a^+b^+ + a^-b^-$ for all $a, b \in X$;
(ii) $(ab)^- \leq a^+b^- + a^-b^+$ for all $a, b \in X$.

Definition 1.4. [2] Assume that X be an l-algebra.

(i) If for all $a, b \in X$, $a \wedge b = 0$ implies $ab = 0$, then X is named an almost f-algebra.
(ii) If $c(a \vee b) = ca \vee cb$ and $(a \vee b)c = ac \vee ab$ for all $a, b \in X$ and $c \in X^+$, then X is named a d-algebra.
(iii) If $a \wedge b = 0$ implies $ca \wedge b = ac \wedge b = 0$ for all $a, b \in X$ and $c \in X^+$, then X is named an f-algebra.

Proposition 1.5. *[2] For an l-algebra X, the following statements are equivalent:*

(i) X is a d-algebra;
(ii) $c|a| = |ca|$ *and* $|a|c = |ac|$ *for all* $a \in X$ *and* $c \in X^+$*;*
(iii) for all $a, b \in X$ *and* $c \in X^+$, $c(a \vee b) = ca \vee cb$ *and* $(a \wedge b)c = ac \wedge bc$.

Definition 1.6. [21] Any lattice ordered algebra X that is meantime a Banach lattice is named a Banach lattice algebra when $\|ab\| \leq \|a\|\|b\|$ keeps for all $a, b \in X^+$. Moreover, if X is an f-algebra next it is named a Banach lattice f-algebra, explicitly, X is next a (real) Banach algebra.

It is explicit that every f-algebra is an almost f-algebra and a d-algebra. Every Archimedean f-algebra is commutative and associative. It turns out that every Archimedean almost f-algebra is commutative but not necessarily associative [2].

Theorem 1.7. *For an l-algebra A with unit element* $e > 0$*, the following are equivalent:*

(i) A is an f-algebra.
(ii) A is a d-algebra.
(iii) A is an almost f-algebra.
(iv) e is a weak order unit (i.e., $a \perp e$ *implies* $a = 0$*).*
(v) For all $a \in A$, $aa^+ \geq 0$.
(vi) For all $a \in A$, $a^2 \geq 0$.

We gather several simple f-algebra properties. Let A be an f-algebra. In this case, for every $a, b \in A$, we have the following:

(1) $|ab| = |a||b|$.
(2) $a \perp b$ implies that $ab = 0$.
(3) $a^2 = (a^+)^2 + (a^-)^2 \geq 0$.
(4) $0 \leq (a^+)^2 = aa^+$.
(5) $(a \vee b)(a \wedge b) = ab$.
(6) If $a^2 = 0$, then $ab = 0$.
(7) If A is semiprime (i.e., the only nilpotent element in A is 0 or, equivalently, $a^2 = 0$ in A implies $a = 0$), then $a^2 \leq b^2$ if and only if $|a| \leq |b|$.
(8) If A is semiprime, then $a \perp b$ if and only if $ab = 0$.

(9) Every unital f-algebra is semiprime.

See [3,22,23] for consider more properties of f-algebras.

Definition 1.8. Let X and Y be Banach f-algebras and $F : X \to Y$ be a cone-related function. We describe the following:

(P_1) Supremum preserving functional equation:

$$F(|x|) \vee F(|y|) = F(|x| \vee |y|), \quad \text{for all } \ x, y \in X.$$

(P_2) Multiplication preserving functional equation:

$$F(|x|)F(|y|) = F(|x||y|), \quad \text{for all } \ x, y \in X.$$

(P_3) Semi-homogeneity:

$$F(\tau|x|) = \tau F(|x|), \quad \text{for all } \ x \in X \text{and } \ \tau \in [0, \infty).$$

(P_4) Positive Cauchy additive functional equation:

$$F(|x|) + F(|y|) = F(|x| + |y|), \quad \text{for all } \ x, y \in X.$$

Definition 1.9. [16] A function $d : X \times X \to [0, \infty]$ is named a generalized metric on set X if d satisfies the following conditions:

(a) for each $x, y \in X$, $d(x, y) = 0$ iff $x = y$;
(b) for all $x, y \in X$, $d(x, y) = d(y, x)$;
(c) for all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

Notice that the just generalized metric significant difference from the metric is that the generalized metric range contains the infinite.

Theorem 1.10. *[16] Let (X, d) be a complete generalized metric space and $J : X \to X$ be a contractive mapping with Lipschitz constant $L < 1$. Then for every $x \in X$, either*

$$d(J^{n+1}x, J^n x) = \infty$$

for all nonnegative integers n or there exists an integer $n_0 > 0$ such that

(a) *For all $n \geq n_0$, $d(J^n x, J^{n+1}x) < \infty$;*
(b) *$J^n(x) \to y^*$, where y^* is a fixed point of J;*
(c) *y^* is the unique fixed point of J in the set $Y = \{y \in X \ : \ d(J^{n_0}(x), y) < \infty\}$;*
(d) *For each $y \in Y$, $d(y^*, y) \leq \frac{1}{1-L} d(Jy, y)$.*

2. Main results

In this part, we will investigate the stability of lattice multiplication functional equation in Banach f-algebra by using the fixed point method.

Lemma 2.1. *Let X and Y be Riesz spaces and $F : X \to Y$ be a cone-related mapping such that*

$$F\left(\frac{|x|+|z|}{2} \vee |y|\right) \vee 2F\left(|z| - |x| \wedge |y|\right) = F\left(|z|\right) \vee F\left(|y|\right) \tag{2.1}$$

or

$$F\left(\frac{|x|+|z|}{2} \vee |y|\right) \vee 2F\left(|z| - |x| \wedge |y|\right) = F(|y| + |z| \vee |x|) \vee F(|z| + |y| \wedge |x|) \tag{2.2}$$

for all $x, y, z \in X$. Then F satisfies $(P1)$.

Proof. It is easy to indicate that $F(0) = 0$. Replacing z by x in (2.1), we get

$$F(|x| \vee |y|) \vee 2F(0 \wedge |y|) = F(|x|) \vee F(|y|),$$

which means that F satisfies $(P1)$. By the same reasoning, we can show that if (2.2) holds, then F satisfies $(P1)$. □

Lemma 2.2. *Let X and Y be Riesz spaces. If $F : X \to Y$ is a cone-related mapping, which satisfies*

$$F(|x| - |y| \vee |x| + |z|) + F(|y| - |z| \vee |x|) = 2F(|y|) + F(|z|) \tag{2.3}$$

or

$$F(|x| + |y| \vee |z|) + 2F(|x| \wedge |y| - |z|) = F(|x|) + F(|z|) \tag{2.4}$$

or

$$F(|x| \vee |y| + |z|) + F(|x| - |y| \vee |z|) = F(|x|) + 2F(|z|) \tag{2.5}$$

for all $x, y, z \in X$, then $F : X \to Y$ satisfies (P4).

Proof. Letting $y = x$ in (2.3), we have

$$F(|x| + |z|) + F(|x|) = 2F(|x|) + F(|z|),$$

for all $x, y, z \in X$ and so F is a positive Cauchy additive mapping. Letting $x = y = z = 0$ in (2.4), we get $F(0) = 0$. Now putting $z = y$ in (2.4), we show that F satisfies $(P2)$. Finally, letting $y = x$ in (2.5), we obtain that F is a positive Cauchy additive mapping. □

Theorem 2.3. *Let X and Y be two Banach lattices. Consider a cone-related function $F : X \to Y$ with $F(0) = 0$, that is,*

$$\left\| F\left(\frac{\tau|x| + \nu|z|}{2} \vee \eta|y|\right) \vee 2F(\nu|z| - \tau|x| \wedge \eta|y|) - \nu F(|z|) \vee \eta F(|y|) \right\| \leq \varphi(\tau|x|, \eta|y|, \nu|z|) \tag{2.6}$$

for all $x, y, z \in X$ and $\tau, \eta, \nu \in [1, \infty)$. Suppose that a function $\varphi : X^3 \to [0, \infty)$ satisfies

$$\varphi(|x|, |y|, |z|) \leq (\tau\eta\nu)^{\frac{\alpha}{3}} \varphi\left(\frac{|x|}{\tau}, \frac{|y|}{\eta}, \frac{|z|}{\nu}\right) \tag{2.7}$$

for all $x, y, z \in X$, $\tau, \eta, \nu \in [1, \infty)$ and there is a number $\alpha \in [0, \frac{1}{3})$. Then there is an individual cone-related mapping $T : X \to Y$ which satisfies the properties $(P1)$, $(P3)$ and

$$\|T(|x|) - F(|x|)\| \leq \frac{\tau^\alpha}{\tau - \tau^\alpha} \varphi(|x|, |x|, |x|) \tag{2.8}$$

for all $x \in X$ and $\tau \in [1, \infty)$.

Proof. Letting $x = y = z$ and $\tau = \eta = \nu$ in (2.6), we get

$$\|F(\tau|x|) - \tau F(|x|)\| \leq \varphi(\tau|x|, \tau|x|, \tau|x|).$$

By (2.7), we have

$$\|F(\tau|x|) - \tau F(|x|)\| \leq \tau^{\alpha}\varphi(|x|, |x|, |x|).$$

Dividing by τ in the above inequality, we get

$$\left\|\frac{1}{\tau}F(\tau|x|) - F(|x|)\right\| \leq \tau^{\alpha-1}\varphi(|x|, |x|, |x|)$$

for all $x \in X$, $\tau \geq 1$ and $\alpha \in [0, \frac{1}{3})$.

Consider the set

$$\Delta := \{G \mid \ G : X \to Y, G(0) = 0\}$$

and the generalized metric on Δ defined by

$$d(G, H) = \inf\{c \in \mathbb{R}^+ : \ \|G(|x|) - H(|x|)\| \leq c\varphi(|x|, |x|, |x|) \text{ for all } x \in X\},$$

where, as common, $\inf \emptyset = \infty$. It is easy to demonstrate that (Δ, d) is a complete generalized metric space (see [6, Theorem 3.1]).

Now, we define the operator $J : \Delta \to \Delta$ by

$$JG(|x|) = \frac{1}{\tau}G(\tau|x|) \quad \text{for all} \quad x \in X.$$

Given $G, H \in \Delta$, let $c \in [0, \infty)$ be a desired constant with $d(G, H) \leq c$, that means

$$\|G(|x|) - H(|x|)\| \leq c\varphi(|x|, |x|, |x|).$$

Then we have

$$\begin{aligned} \|JG(|x|) - JH(|x|)\| &= \frac{1}{\tau}\|G(\tau|x|) - H(\tau|x|)\| \\ &\leq \frac{1}{\tau}c\varphi(\tau|x|, \tau|x|, \tau|x|) \\ &= \tau^{\alpha-1}c\varphi(|x|, |x|, |x|). \end{aligned}$$

By Theorem 1.10, there is a mapping $T : X \to Y$ such that the following hold.

(1) T is a fixed point of J, that means

$$T(\tau|x|) = \tau T(|x|) \tag{2.9}$$

for all $x \in X$. Also the mapping T is an individual fixed point of J in the set

$$Z = \{G \in \Delta; \ d(G, T) < \infty\},$$

this implies that T is an individual mapping satisfying (2.9) such that there exists $c \in (0, \infty)$ satisfying

$$\|T(|x|) - F(|x|)\| \leq c\varphi(|x|, |x|, |x|)$$

for all $x \in X$.

(2) $d\left(J^n F, T\right) \to 0$, as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{F(\tau^n|x|)}{\tau^n} = T(|x|)$$

for all $x \in X$.

(3) $d(F, T) \leq \frac{1}{1-L} d(JF, F)$, which implies the inequality

$$d(F, T) \leq \frac{\tau^{\alpha-1}}{1-\tau^{\alpha-1}} = \frac{\tau^{\alpha}}{\tau-\tau^{\alpha}}.$$

So we obtain

$$\|F(|x|) - T(|x|)\| \leq \frac{\tau^{\alpha}}{\tau-\tau^{\alpha}} \varphi(|x|, |x|, |x|)$$

for all $x \in X$, $\tau \geq 1$ and $\alpha \in [0, \frac{1}{3})$. Accordingly the inequality (2.8) holds.

Next, we show that $(P1)$ is satisfied. Putting $\tau = \eta = \nu := \tau^n$ in (2.6), we get

$$\left\|F\left(\frac{\tau^n|x| + \tau^n|z|}{2} \vee \tau^n|y|\right) \vee 2F(\tau^n|z| - \tau^n|x| \wedge \tau^n|y|) - \tau^n F(|z|) \vee \tau^n F(|y|)\right\| \leq \varphi(\tau^n|x|, \tau^n|y|, \tau^n|z|)$$

and so

$$\left\|F\left(\tau^n\left(\frac{|x| + |z|}{2} \vee |y|\right)\right) \vee 2F(\tau^n(|z| - |x| \wedge |y|)) - \tau^n F(|z|) \vee \tau^n F(|y|)\right\| \leq \tau^{3n\alpha}\varphi(|x|, |y|, |z|).$$

Substituting x, y, z by $\tau^n y, \tau^n y, \tau^n z$, respectively, in the above inequality, we get

$$\begin{aligned}&\left\|F\left(\tau^n\left(\frac{\tau^n|x| + \tau^n|z|}{2} \vee \tau^n|y|\right)\right) \vee 2F(\tau^n(\tau^n|z| - \tau^n|x| \wedge \tau^n|y|)) - \tau^n F(\tau^n|z|) \vee \tau^n F(\tau^n|y|)\right\| \\ &\quad \leq \tau^{3n\alpha}\varphi(\tau^n|x|, \tau^n|y|, \tau^n|z|)\end{aligned}$$

and thus

$$\begin{aligned}&\left\|F\left(\tau^{2n}\left(\frac{|x| + |z|}{2} \vee |y|\right)\right) \vee 2F(\tau^{2n}(|z| - |x| \wedge |y|)) - \tau^n F(\tau^n|z|) \vee \tau^n F(\tau^n|y|)\right\| \\ &\quad \leq \tau^{6n\alpha}\varphi(|x|, |y|, |z|)\end{aligned} \tag{2.10}$$

for all $x, y, z \in X$, $\tau \geq 1$ and $\alpha \in [0, \frac{1}{3})$. Dividing both sides of (2.10) by τ^{2n}, we obtain

$$\begin{aligned}&\left\|\frac{1}{\tau^{2n}}F\left(\tau^{2n}\left(\frac{|x| + |z|}{2} \vee |y|\right)\right) \vee \frac{2}{\tau^{2n}}F(\tau^{2n}(|z| - |x| \wedge |y|)) - \frac{1}{\tau^n}F(\tau^n|z|) \vee \frac{1}{\tau^n}F(\tau^n|y|)\right\| \\ &\quad \leq \tau^{6n\alpha-2n}\varphi(|x|, |y|, |z|).\end{aligned}$$

Since lattice operations are continuous, by (2), as $n \to \infty$, we get

$$\left\|T\left(\frac{|x| + |z|}{2} \vee |y|\right) \vee 2T\left(|z| - |x| \wedge |y|\right) - T\left(|z|\right) \vee T\left(|y|\right)\right\| \leq 0.$$

Therefore,

$$T\left(\frac{|x| + |z|}{2} \vee |y|\right) \vee 2T\left(|z| - |x| \wedge |y|\right) = T\left(|z|\right) \vee T\left(|y|\right)$$

for all $x, y, z \in X$. Accordingly, by Lemma 2.1, T satisfies $(P1)$. □

Theorem 2.4. *Let $F : X \to Y$ be a cone-related mapping with $F(0) = 0$ in two uniformly complete Banach f-algebras X and Y. Assume that*

$$\left\| F\left(\frac{\tau|x| + \nu|z|}{2} \vee \eta|y|\right) \vee 2F(\nu|z| - \tau|x| \wedge \eta|y|) - \nu F(|z|) \vee \eta F(|y|) \right\| \leq \varphi(\tau|x|, \eta|y|, \nu|z|)$$

and

$$\|F(|x||y|) - F(|x|)F(|y|)\| \leq \psi(|x|, |y|),$$

where $\varphi : X^3 \to [0, \infty)$ and $\psi : X^2 \to [0, \infty)$ satisfy

$$\varphi(\tau|x|, \eta|y|, \nu|z|) \leq (\tau\eta\nu)^{\frac{\alpha}{3}} \varphi(|x|, |y|, |z|)$$

and

$$\psi(\tau^n|x|, \tau^n|y|) = O(\tau^{2n}) \tag{2.11}$$

for all $x, y, z \in X$, $\tau, \eta, \nu \in [1, \infty)$, $\alpha \in [0, \frac{1}{3})$. Then there exists an individual cone-related mapping $T : X \to Y$ satisfying $(P1), (P2), (P3)$ and

$$\|F(|x|) - T(|x|)\| \leq \frac{\tau^\alpha}{\tau - \tau^\alpha} \varphi(|x|, |x|, |x|). \tag{2.12}$$

Proof. In the previous theorem, we proved that there exists an individual cone-related mapping $T : X \to Y$ such that

$$T(|x|) = \lim_{n\to\infty} \frac{F(\tau^n|x|)}{\tau^n}, \quad \|F(|x|) - T(|x|)\| \leq \frac{\tau^\alpha}{\tau - \tau^\alpha} \varphi(|x|, |x|, |x|) \tag{2.13}$$

for all $x \in X$ and $\tau \in [1, \infty)$. Moreover, we showed that T satisfies $(P1)$ and $(P3)$. It follows from (2.11),(2.12) and (2.13) that

$$\begin{aligned}
\|T(|x||y|) - T(|x|)T(|y|)\| &= \lim_{n\to\infty} \left\| \frac{F(\tau^{2n}|x||y|)}{\tau^{2n}} - \frac{F(\tau^n|x|)}{\tau^n} \frac{F(\tau^n|y|)}{\tau^n} \right\| \\
&= \lim_{n\to\infty} \frac{1}{\tau^{2n}} \|F(\tau^{2n}|x||y|) - F(\tau^n|x|)F(\tau^n|y|)\| \\
&\leq \lim_{n\to\infty} \frac{1}{\tau^{2n}} \psi(\tau^n|x|, \tau^n|y|) = 0.
\end{aligned}$$

Thus T satisfies $(P2)$. $\square$

Corollary 2.5. *Suppose that $\tau > 1$, $\alpha < \frac{1}{3}$ and $\theta < 1$ are nonnegative real numbers. Let X and Y be two uniformly complete Banach f-algebras and $F : X \to Y$ be a cone-related mapping with $F(0) = 0$. If F satisfies*

$$\left\| F\left(\frac{\tau|x| + \nu|z|}{2} \vee \eta|y|\right) \vee 2F(\nu|z| - \tau|x| \wedge \eta|y|) - \nu F(|z|) \vee \eta F(|y|) \right\| \leq \theta\left(\|x\|^\alpha + \|y\|^\alpha + \|z\|^\alpha\right)$$

and

$$\|F(|x||y|) - F(|x|)F(|y|)\| \leq \theta\left(\|x\|^\alpha + \|y\|^\alpha\right)$$

for all $x, y, z \in X$ and $\tau, \eta, \nu \in [0, \infty)$, then there exists an individual cone-related mapping $T : X \to Y$ satisfying $(P1)$, $(P2)$, $(P3)$ and

$$\|T(|x|) - F(|x|)\| \leq \frac{2\theta\tau^\alpha}{\tau - \tau^\alpha} \|x\|$$

for all $x \in X$.

Proof. Letting

$$\varphi(x,y,z) = \theta\left(\|x\|^{\alpha} + \|y\|^{\alpha}\right), \quad \psi(x,y) = \theta\left(\|x\|^{\alpha} + \|y\|^{\alpha}\right)$$

in the above theorem, we get the desired result. □

Theorem 2.6. *Suppose that r, θ, τ, η and ν are nonnegative real numbers with $\theta > 0$, $r \in [0, \frac{1}{3})$ and $\tau, \eta, \nu \in [1, \infty)$. If X is a Banach lattice and Y is Banach f-algebra and $F : X \to Y$ is a mapping such that*

$$\left\| F(\tau|x| - \eta|y| \vee \tau|x| + \nu|z|) + F(\eta|y| - \nu|z| \vee \tau|x|) - 2\eta F(|y|) - \nu F(|z|) \right\| \leq \theta \|\tau x\|^r \|\eta y\|^r \|\nu z\|^r \quad (2.14)$$

for all $x, y, z \in X$, then there exists an individual mapping $T : X \to Y$ such that the next properties hold:

(1) For all $x \in X$,

$$\|F(|x|) - T(|x|)\| \leq \frac{\tau^{3r}\theta}{\tau^3 - \tau^{r+2}} \|x\|^r.$$

(2) $T : X \to Y$ satisfies (P4).

Proof. By induction on n, we can easily show that the following inequality holds.

$$\left\| \frac{1}{\tau^n} F(\tau^n|x|) - F(|x|) \right\| \leq \tau^2 \theta \sum_{i=3}^{n} \tau^{i(r-1)} \|x\|^r. \quad (2.15)$$

Moreover, for all $x \in X$, we have

$$\begin{aligned} \left\| \frac{1}{\tau^n} F(\tau^n|x|) - \frac{1}{\tau^m} F(\tau^m|x|) \right\| &= \tau^{-m} \left\| \frac{1}{\tau^{n-m}} F(\tau^{n-m} F(\tau^m|x|)) - F(\tau^m|x|) \right\| \\ &\leq \theta \tau^{2-m} \sum_{i=3}^{n-m} \tau^{i(r-1)} \|\tau^m x\|^r \\ &= \theta \tau^2 \sum_{i=m+3}^{n} \tau^{i(r-1)} \|x\|^r \end{aligned}$$

for all $m, n \in \mathbb{N}$ with $n \geq m$ and all $x \in X$. As $m \to \infty$, the right side term in the above inequality tends to 0 and so the sequence $\left\{ \frac{1}{\tau^n} F(\tau^n|x|) \right\}$ is a Cauchy sequence. Hence it converges since Y is complete. Therefore we can define an operator $T : X \to Y$ by

$$T(|x|) := \lim_{n \to \infty} \frac{1}{\tau^n} F(\tau^n|x|) \quad (2.16)$$

for all $x \in X$. Then we obtain

$$\|T(|x|) - F(|x|)\| \leq \frac{\tau^{3r}\theta}{\tau^3 - \tau^{r+2}} \|x\|^r$$

by (2.15).

Next, we show that (2) is satisfied. Putting $\tau = \nu = \eta = \tau^n$ in (2.14), we have

$$\left\| F(\tau^n(|x| - |y| \vee |x| + |z|)) + F(\tau^n(|y| - |z| \vee |x|)) - 2\tau^n F(|y|) - \tau^n F(|z|) \right\| \leq \theta \tau^{3nr} \|x\|^r \|y\|^r \|z\|^r.$$

Replacing x by $\tau^n x$, y by $\tau^n y$ and z by $\tau^n z$ in the above inequality, we obtain

$$\left\|F(\tau^{2n}(|x| - |y| \vee |x| + |z|)) + F(\tau^{2n}(|y| - |z| \vee |x|)) - 2\tau^n F(\tau^n|y|) - \tau^n F(\tau^n|z|)\right\| \leq \theta\tau^{6nr}\|x\|^r\|y\|^r\|z\|^r.$$

Dividing the last inequality by τ^{2n}, we have

$$\left\|\frac{1}{\tau^{2n}}F(\tau^{2n}(|x| - |y| \vee |x| + |z|)) + \frac{1}{\tau^{2n}}F(\tau^{2n}(|y| - |z| \vee |x|)) - 2\frac{1}{\tau^n}F(\tau^n|y|) - \frac{1}{\tau^n}F(\tau^n|z|)\right\| \\ \leq \theta\tau^{2n(3r-1)}\|x\|^r\|y\|^r\|z\|^r.$$

Since lattice operations are continuous, as $n \to \infty$ we obtain

$$T(|x| - |y| \vee |x| + |z|) + T(|y| - |z| \vee |x|) = 2T(|y|) - T(|z|)$$

by (2.16). Therefore, by Lemma 2.2, T is a positive Cauchy additive mapping. □

Corollary 2.7. *Suppose that $r < \frac{1}{3}$ and θ are nonnegative real numbers and that X, Y are Banach f-algebras. If $F : X \to Y$ is a cone related mapping satisfying (2.14) such that*

$$\|F(|x||y|) - F(|x|)F(|y|)\| \leq \theta\|x\|^r\|y\|^r \tag{2.17}$$

for all $x, y \in X$, then there is an individual cone-related mapping $T : X \to Y$ such that satisfies $(P2)$ and $(P4)$.

Proof. In the above proof, we demonstrate that $T : X \to Y$ exists and satisfies $(P4)$. It follows from (2.16) and (2.17) that

$$\begin{aligned} \left\|T(|x||y|) - T(|x|)T(|y|)\right\| &= \lim_{n\to\infty}\left\|\frac{F(\tau^{2n}|x||y|)}{\tau^{2n}} - \frac{F(\tau^n|x|)}{\tau^n}.\frac{F(\tau^n|y|)}{\tau^n}\right\| \\ &= \lim_{n\to\infty}\frac{1}{\tau^{2n}}\left\|F(\tau^{2n}|x||y|) - F(\tau^n|x|)F(\tau^n|y|)\right\| \\ &\leq \lim_{n\to\infty}\theta\tau^{2n(r-1)}\|x\|^r\|y\|^r \\ &= 0. \end{aligned}$$

Thus $T(|x||y|) = T(|x|)T(|y|)$ and so T satisfies $(P2)$. □

3. Conclusion

In this paper, we have proved the Hyers-Ulam stability of supremum, infimum and multiplication preserving functional equations in Banach f-algebras.

Conflict of interest

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

The authors declare that they have no competing interests.

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