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Research article

Solution of a 3-D cubic functional equation and its stability

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Abstract: In this paper, we define and find the general solution of the following 3-D cubic functional equation

$$\begin{aligned} f(2x_1+x_2+x_3) = {} & 3f(x_1+x_2+x_3)+f(-x_1+x_2+x_3)+2f(x_1+x_2)+2f(x_1+x_3) \\ & -6f(x_1-x_2)-6f(x_1-x_3)-3f(x_2+x_3)+2f(2x_1-x_2) \\ & +2f(2x_1-x_3)-18f(x_1)-6f(x_2)-6f(x_3). \end{aligned}$$

We also prove the Hyers-Ulam stability of this functional equation in fuzzy normed spaces by using the direct method and the fixed point method.

Keywords: cubic functional equation; fuzzy normed space; Hyers-Ulam stability; fixed point method
Mathematics Subject Classification: 39B52, 39B72, 39B82, 47H10, 47S40

1. Introduction

Stability problem of a functional equation was first posed in [31] which was answered in [7] and then generalized in [1, 28] for additive mappings and linear mappings respectively. Since then several stability problems for various functional equations have been investigated in [9, 10, 12, 23]. Fuzzy version was discussed in [13, 14]. Recently, the stability problem for Jensen functional equation and cubic functional equation were considered in [15,20] respectively in intuitionistic fuzzy normed spaces; while the idea of intuitionistic fuzzy normed space was introduced in [30] and further studied in [16–19, 21, 22, 24–27, 29] to deal with some summability problems. Several results for the Hyers-Ulam stability of many functional equations have been proved by several researchers [4–6, 8, 11, 23]

In modeling applied problems only partial information may be known (or) there may be a degree of uncertainty in the parameters used in the model or some measurments may be imprecise. Due to such features, many authors have considered the study of functional equations in the fuzzy setting. Jun *et al.* introduced the following functional equations

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

and

$$f(3x+y)+f(3x-y)=3f(x+y)+3f(x-y)+48f(x) \tag{1.2}$$

and investigated its general solution and the Hyers-Ulam stability respectively. The functional equations (1.1) and (1.2) are called cubic functional equations because the function $f(x)=cx^3$ is a solution of the above functional equations (1.1) and (1.2).

Rassias introduced the following new cubic equation $f : X \to Y$ satisfying the cubic functional equation

$$f(x+2y)+3f(x)=3f(x+y)+f(x-y)+6f(y) \tag{1.3}$$

for all $x, y \in X$, with X a linear space, Y a real complete linear space, and then solved the Hyers-Ulam stability problem for the above functional equation.

In this paper, we define and find the general solution of the 3-D cubic functional equation

$$\begin{aligned} f(2x_1+x_2+x_3) = {} & 3f(x_1+x_2+x_3)+f(-x_1+x_2+x_3)+2f(x_1+x_2)+2f(x_1+x_3) \\ & -6f(x_1-x_2)-6f(x_1-x_3)-3f(x_2+x_3)+2f(2x_1-x_2) \\ & +2f(2x_1-x_3)-18f(x_1)-6f(x_2)-6f(x_3). \end{aligned} \tag{1.4}$$

We also prove the Hyers-Ulam stability of this functional equation in fuzzy normed spaces by using the direct method and the fixed point method.

2. Preliminaries

Definition 2.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and $a, b \in \mathbb{R}$,

(N_1) $N(x,c)=0$ for $c \le 0$;
(N_2) $x=0$ if and only if $N(x,c)=1$ for all $c>0$;
(N_3) $N(cx,b)=N\left(x,\frac{b}{|c|}\right)$ if $c \neq 0$;
(N_4) $N(x+y,a+b) \ge \min\{N(x,a),N(y,b)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{b\to\infty} N(x,b)=1$;
(N_6) for $x \neq 0, N(x,\cdot)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(x,b)$ as the truth value of the statement the norm of x is less than or equal to the real number b.

Definition 2.2. Let (X,N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, b) = 1$ for all $b>0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.

Definition 2.3. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $b > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N\left(x_{n+p} - x_n, b\right) > 1 - \epsilon$.

Every convergent sequence in a fuzzy normed space is Cauchy.

Definition 2.4. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Definition 2.5. A mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If f is continuous at each point of $x_0 \in X$, then f is said to be continuous on X.

We bring the following theorems which some results in fixed point theory. These results play a fundamental role to arrive our purpose of this paper.

Theorem 2.6. (Banach Contraction Principle) *Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is,*
(A1) if $d\left(T_x, T_y\right) \leq Ld(x, y)$ for some (Lipschitz constant) $L < 1$, then

1) the mapping T has one and only fixed point $x^* = T(x^*)$;
2) the fixed point for each given element x^* is globally attractive, that is,

(A2) $\lim\limits_{n\to\infty} T^n x = x^*$ for any starting point $x \in X$;

1) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L} d\left(T^n x, T^{n+1} x\right)$ for all $n \geq 0, \quad x \in X$,
(A4) $d(x, x^*) \leq \frac{1}{1-L} d(x, x^*), \quad \forall\ x \in X$.

Theorem 2.7. (The Alternative of fixed point) *For a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L, and for each given element $x \in X$, either*
(B1) $d(T^n x, T^{n+1} x) = +\infty$, for all $n \geq 0$, or
(B2) There exists a natural number n_0 such that

i) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
ii) the sequence $(T^n x)$ is convergent to a fixed point y^* of T;
iii) y^* is the unique fixed point of T in the set $Y = \{y \in X; d(T^{n_0} x, y) < \infty\}$;
iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in Y$.

3. General solution of the functional equation (1.4)

In this section, we discuss the general solution of the functional equation (1.4).

Theorem 3.1. *If an odd mapping $f : X \to Y$ satisfies the functional equation*

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{3.1}$$

for all $x, y \in X$ if and only if $f : X \to Y$ satisfies the functional equation

$$\begin{aligned}f(2x_1 + x_2 + x_3) = 3f(x_1 + x_2 + x_3) + f(-x_1 + x_2 + x_3) + 2f(x_1 + x_2) + 2f(x_1 + x_3)\\ -6f(x_1 - x_2) - 6f(x_1 - x_3) - 3f(x_2 + x_3) + 2f(2x_1 - x_2)\\ +2f(2x_1 - x_3) - 18f(x_1) - 6f(x_2) - 6f(x_3)\end{aligned} \tag{3.2}$$

for all $x_1, x_2, x_3 \in X$.

Proof. Let $f : X \to Y$ satisfy the functional equation (3.1). Setting $(x, y) = (0, 0)$ in (3.1), we get $f(0) = 0$. Replacing (x, y) by $(x, 0)$, (x, x) and $(x, 2x)$ respectively in (3.1), we obtain

$$f(2x) = 2^3 f(x), \quad f(3x) = 3^3 f(x) \ \ \textit{and} \ \ f(4x) = 4^3 f(x) \tag{3.3}$$

for all $x \in X$. In general for any positive integer a, we have

$$f(ax) = a^3 f(x) \tag{3.4}$$

for all $x \in X$. It follows from (3.4) that

$$f(a^2 x) = a^6 f(x) \ \ \textit{and} \ \ f(a^3 x) = a^9 f(x) \tag{3.5}$$

for all $x \in X$. Replacing (x, y) by $(x_1, x_2 + x_3)$ in (3.1), we get

$$f(2x_1 + x_2 + x_3) - 2f(x_1 + x_2 + x_3) = f(-2x_1 + x_2 + x_3) + 2f(x_1 - x_2 - x_3) + 12f(x_1) \tag{3.6}$$

for all $x_1, x_2, x_3 \in X$. Again replacing (x, y) by $(x_2 + x_3, -2x_1)$ in (3.1), we get

$$4f(-x_1 + x_2 + x_3) + 4f(x_1 + x_2 + x_3) - f(2x_1 + x_2 + x_3) - 6f(x_2 + x_3) = f(-2x_1 + x_2 + x_3) \tag{3.7}$$

for all $x_1, x_2, x_3 \in X$. Substituting (3.7) in (3.6), we get

$$f(2x_1 + x_2 + x_3) - 3f(x_1 + x_2 + x_3) = f(-x_1 + x_2 + x_3) - 3f(x_2 + x_3) + 6f(x_1) \tag{3.8}$$

for all $x_1, x_2, x_3 \in X$. Setting (x, y) by $(x_2, 2x_1)$ in (3.1). we obtain

$$4f(x_1 + x_2) - 4f(x_1 - x_2) - 6f(x_2) = f(2x_1 + x_2) - f(2x_1 - x_2) \tag{3.9}$$

for all $x_1, x_2 \in X$. Switching (x, y) by $(x_3, 2x_1)$ in (3.1), we obtain

$$4f(x_1 + x_3) - 4f(x_1 - x_3) - 6f(x_3) = f(2x_1 + x_3) - f(2x_1 - x_3) \tag{3.10}$$

for all $x_1, x_3 \in X$. Adding (3.9) and (3.10), we get

$$\begin{aligned}4f(x_1 + x_2) - 4f(x_1 - x_2) + 4f(x_1 + x_3) - 4f(x_1 - x_3) - 6f(x_2) - 6f(x_3)\\ -f(2x_1 + x_2) + f(2x_1 - x_2) - f(2x_1 + x_3) + f(2x_1 - x_3) = 0\end{aligned} \tag{3.11}$$

for all $x_1, x_2, x_3 \in X$. Adding (3.8) and (3.11), we get

$$f(2x_1 + x_2 + x_3) = 3f(x_1 + x_2 + x_3) + f(-x_1 + x_2 + x_3) - 3f(x_2 + x_3)$$

$$+6f(x_1)+4f(x_1+x_2)-4f(x_1-x_2)+4f(x_1+x_3)-4f(x_1-x_3)$$

$$-6f(x_2)-6f(x_2)-f(2x_1+x_2)+f(2x_1-x_2)-f(2x_1+x_3)+f(2x_1-x_3) \tag{3.12}$$

for all $x_1, x_2, x_3 \in X$. Replacing (x, y) by $(-x_1, x_2)$ in (3.1), we obtain

$$-f(2x_1+x_2)=f(2x_1-x_2)-2f(x_1-x_2)-2f(x_1+x_2)-12f(x_1) \tag{3.13}$$

for all $x_1, x_2 \in X$. Switching (x, y) by $(-x_1, x_3)$ in (3.1), we have

$$-f(2x_1+x_3)=f(2x_1-x_3)-2f(x_1-x_3)-2f(x_1+x_3)-12f(x_1) \tag{3.14}$$

for all $x_1, x_3 \in X$. Adding (3.13) and (3.14), we obtain

$$\begin{aligned}-f(2x_1+x_2)-f(2x_1+x_3)=&f(2x_1-x_2)-2f(x_1-x_2)-2f(x_1+x_2)\\&+f(2x_1-x_3)-2f(x_1-x_3)-2f(x_1+x_3)-24f(x_1)\end{aligned} \tag{3.15}$$

for all $x_1, x_2, x_3 \in X$. Substituting (3.15) in (3.12), we have

$$\begin{aligned}f(2x_1+x_2+x_3)=&3f(x_1+x_2+x_3)+f(-x_1+x_2+x_3)+2f(x_1+x_2)\\&+2f(x_1+x_3)-6f(x_1-x_2)-6f(x_1-x_3)-3f(x_2+x_3)\\&+2f(2x_1-x_2)+2f(2x_1-x_3)-18f(x_1)-6f(x_2)-6f(x_3)\end{aligned} \tag{3.16}$$

for all $x_1, x_2, x_3 \in X$.

Conversely, let $f : X \to Y$ satisfy the functional equation (3.2). Replacing (x_1, x_2, x_3) by $(x, 0, 0)$, $(0, x, 0)$ and $(0, 0, x)$ respectively in (3.16), we get

$$f(2x)=2^3f(x),\quad f(x)=1^3f(x)\ \ and\ \ f(x)=1^3f(x). \tag{3.17}$$

One can easy to verify from (3.17) that, replacing (x_1, x_2, x_3) by $(x, y, 0)$ in (3.2), we have

$$f(2x+y)-2f(2x-y)=5f(x+y)-7f(x-y)-6f(x)-9f(y) \tag{3.18}$$

for all $x, y \in X$. Again replacing (x_1, x_2, x_3) by $(x, 0, -y)$ in (3.2), we obtain

$$f(2x-y)-2f(2x+y)=-7f(x+y)+5f(x-y)-6f(x)+9f(y) \tag{3.19}$$

for all $x, y \in X$. Adding the equations (3.18) and (3.19), we get our result. □

Throughout the upcoming sections, assume that X, (Z, N') and (Y, N) are linear space, fuzzy normed space and fuzzy Banach space, respectively. Let us denote

$$\begin{aligned}Df(x_1,x_2,x_3)=f(2x_1+x_2+x_3)=&3f(x_1+x_2+x_3)+f(-x_1+x_2+x_3)+2f(x_1+x_2)\\&+2f(x_1+x_3)-6f(x_1-x_2)-6f(x_1-x_3)-3f(x_2+x_3)+2f(2x_1-x_2)\\&+2f(2x_1-x_3)-18f(x_1)-6f(x_2)-6f(x_3)\end{aligned}$$

4. Stability results for the functional equation (1.4): Direct method

In this section, we investigate the Hyers-Ulam stability of the functional equation (1.4) in fuzzy normed space via direct method.

Theorem 4.1. *Let $\omega \in \{-1, 1\}$ be fixed and $\Gamma : X^3 \to Z$ be a mapping such that for some $\rho > 0$ with $\left(\frac{\rho}{2^3}\right)^{\omega} < 1$*

$$N'\left(\Gamma\left(2^{\omega}x, 0, 0\right), \varepsilon\right) \geq N'\left(\rho^{\omega}\Gamma(x, 0, 0), \varepsilon\right) \tag{4.1}$$

for all $x \in X$ and all $\varepsilon > 0$ and

$$\lim_{n\to\infty} N'\left(\Gamma(2^{\omega n}x_1, 2^{\omega n}x_2, 2^{\omega n}x_3), 2^{3\omega n}\varepsilon\right) = 1$$

for all $x_1, x_2, x_3 \in X$ and all $\varepsilon > 0$. Suppose an odd mapping $f : X \to Y$ satisfies the inequality

$$N\left(D_f(x_1, x_2, x_3), \varepsilon\right) \geq N'\left(\Gamma(x_1, x_2, x_3), \varepsilon\right) \tag{4.2}$$

for all $\varepsilon > 0$ and all $x_1, x_2, x_3 \in X$. Then the limit

$$C(x) = N - \lim_{n\to\infty} \frac{f(2^{\omega n}x)}{2^{3\omega n}}$$

exists for all $x \in X$ and the mapping $C : X \to Y$ is a unique cubic mapping such that

$$N\left(f(x) - C(x), \varepsilon\right) \geq N'\left(\Gamma(x, 0, 0), 3\varepsilon \mid 2^3 - \rho \mid\right) \tag{4.3}$$

for all $x \in X$ and all $\varepsilon > 0$.

Proof. First assume $\omega = 1$. Replacing (x_1, x_2, x_3) by $(x, 0, 0)$ in (4.2), we get

$$N\left(3f(2x) - 24f(x), \varepsilon\right) \geq N'\left(\Gamma(x, 0, 0), \varepsilon\right) \tag{4.4}$$

for all $x \in X$ and all $\varepsilon > 0$. From (4.4), we have

$$N\left(f(2x) - 8f(x), \frac{\varepsilon}{3}\right) \geq N'\left(\Gamma(x, 0, 0), \varepsilon\right) \tag{4.5}$$

for all $x \in X$ and all $\varepsilon > 0$. Replacing x by $2^n x$ in (4.5), we obtain

$$N\left(\frac{f(2^{n+1}x)}{2^3} - f(2^n x), \frac{\varepsilon}{3(2^3)}\right) \geq N'\left(\Gamma(2^n x, 0, 0), \varepsilon\right) \tag{4.6}$$

for all $x \in X$ and $\varepsilon > 0$. Using (4.1), (N_3) in (4.6) we get

$$N\left(\frac{f(2^{n+1}x)}{2^3} - f(2^n x), \frac{\varepsilon}{3(2^3)}\right) \geq N'\left(\Gamma(x, 0, 0), \frac{\varepsilon}{\rho^n}\right) \tag{4.7}$$

for all $x \in X$ and all $\varepsilon > 0$. It is easy to verify from (4.7), that

$$N\left(\frac{f(2^{n+1}x)}{2^{3(n+1)}} - \frac{f(2^n x)}{2^{3n}}, \frac{\varepsilon}{3(2^3)(2^{3n})}\right) \geq N'\left(\Gamma(x, 0, 0), \frac{\varepsilon}{\rho^n}\right) \tag{4.8}$$

holds for all $x \in X$ and all $\varepsilon > 0$. Replacing ε by $\rho^n\varepsilon$ in (4.8), we get

$$N\left(\frac{f(2^{n+1}x)}{2^{3(n+1)}} - \frac{f(2^n x)}{2^{3n}}, \frac{\rho^n \varepsilon}{2^{3(n+1)}3}\right) \geq N^{'}\left(\Gamma(x,0,0),\varepsilon\right) \tag{4.9}$$

for all $x \in X$ and all $\varepsilon > 0$. It is easy to see that

$$\frac{f(2^n x)}{2^{3n}} - f(x) = \sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{2^{3(i+1)}} - \frac{f(2^i x)}{2^{3i}} \tag{4.10}$$

for all $x \in X$. From (4.9) and (4.10), we have

$$N\left(\frac{f(2^n x)}{2^{3n}} - f(x), \sum_{i=0}^{n-1} \frac{\varepsilon\rho^i}{3(2^{3(i+1)})}\right) \tag{4.11}$$
$$\geq \min\{N\left(\frac{f(2^{i+1}x)}{2^{3(i+1)}} - \frac{f(2^i x)}{2^{3i}}, \frac{\varepsilon\rho^i}{3(2^{3(i+1)})}\right) : i = 0, 1, \cdots, n-1\} \geq N^{'}\left(\Gamma\left(x,0,0\right),\varepsilon\right)$$

for all $x \in X$ and all $\varepsilon > 0$. Replacing x by $2^m x$ in (4.11) and using (4.1), (N_3), we get

$$N\left(\frac{f(2^{n+m}x)}{2^{3(n+m)}} - \frac{f(2^m x)}{2^{3m}}, \sum_{i=0}^{n-1} \frac{\varepsilon\rho^i}{3(2^{3(i+1)})}\right) \geq N^{'}\left(\Gamma(2^m x,0,0),\varepsilon\right)$$

$$\geq N^{'}\left(\Gamma(x,0,0),\tfrac{\varepsilon}{\rho^m}\right)$$

and so

$$N\left(\frac{f(2^{n+m}x)}{2^{3(n+m)}} - \frac{f(2^m x)}{2^{3m}}, \sum_{i=m}^{n+m-1} \frac{\varepsilon\rho^i}{3(2^{3(i+1)})}\right) \geq N^{'}\left(\Gamma(x,0,0),\varepsilon\right) \tag{4.12}$$

for all $x \in X$, $\varepsilon > 0$ and all $m, n \geq 0$. Replacing ε by $\frac{\varepsilon}{\sum_{i=m}^{n+m-1}\frac{\rho^i}{3(2^{3(i+1)})}}$ in (4.12), we get

$$N\left(\frac{f(2^{n+m}x)}{2^{3(n+m)}} - \frac{f(2^m x)}{2^{3m}}, \varepsilon\right) \geq N^{'}\left(\Gamma(x,0,0), \frac{\varepsilon}{\sum_{i=m}^{n+m-1}\frac{\rho^i}{3(2^{3(i+1)})}}\right) \tag{4.13}$$

for all $x \in X$, $\varepsilon > 0$ and all $m, n \geq 0$. Since $0 < \rho < 2^3$ and $\sum_{i=0}^{n}\left(\frac{\rho}{2^3}\right)^i < \infty$, the Cauchy criterion for convergence and (N_5) imply that $\{\frac{f(2^n x)}{2^{3n}}\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is complete, this sequence converges to some point $C(x) \in Y$. So one can define the mapping $C : X \to Y$ by

$$C(x) := N - \lim_{n\to\infty} \frac{f(2^n x)}{2^{3n}}$$

for all $x \in X$. Since f is odd, C is odd. Letting $m = 0$ in (4.13), we obtain

$$N\left(\frac{f(2^n x)}{2^{3n}} - f(x), \varepsilon\right) \geq N^{'}\left(\Gamma(x,0,0), \frac{\varepsilon}{\sum_{i=0}^{n-1}\frac{\rho^i}{3(2^{3(i+1)})}}\right) \tag{4.14}$$

for all $x \in X$ and all $\varepsilon > 0$. Taking the limit as $n \to \infty$ in (4.14) and using (N_6), we get

$$N\left(f(x) - C(x), \varepsilon\right) \geq N^{'}\left(\Gamma(x, 0, 0), 3\varepsilon(2^3 - \rho)\right)$$

for all $x \in X$ and all $\varepsilon > 0$. Now we claim that C is cubic. Replacing (x_1, x_2, x_3) by $(2^n x_1, 2^n x_2, 2^n x_3)$ in (4.2) respectively, we have

$$N\left(\frac{1}{2^{3n}} D_f\left(2^n x_1, 2^n x_2, 2^n x_3\right), \varepsilon\right) \geq N^{'}\left(\Gamma(2^n x_1, 2^n x_2, 2^n x_3), 2^{3n}\varepsilon\right)$$

for all $x \in X$ and all $\varepsilon > 0$. Since

$$\lim_{n\to\infty} N^{'}\left(\Gamma(2^n x_1, 2^n x_2, 2^n x_3), 2^{3n}\varepsilon\right) = 1,$$

A satisfies the functional equation (1.4). Hence $C : X \to Y$ is cubic. To prove the uniqueness of C, let $D : X \to Y$ be another cubic mapping satisfying (4.3). Fix $x \in X$. Clearly, $C(2^n x) = 2^{3n} C(x)$ and $D(2^n x) = 2^{3n} D(x)$ for all $x \in X$ and all $n \in \mathbb{N}$. It follows from (4.3) that

$$\begin{aligned}
N(C(x) - D(x), \varepsilon) &= N\left(\tfrac{C(2^n x)}{2^{3n}} - \tfrac{D(2^n x)}{2^{3n}}, \varepsilon\right) \\
&\geq \min\{N\left(\tfrac{C(2^n x)}{2^{3n}} - \tfrac{f(2^n x)}{2^{3n}}, \tfrac{\varepsilon}{2}\right), N\left(\tfrac{f(2^n x)}{2^{3n}} - \tfrac{D(2^n x)}{2^{3n}}, \tfrac{\varepsilon}{2}\right)\} \\
&\geq N^{'}\left(\Gamma(2^n x, 0, 0), \tfrac{3(2^n)\varepsilon(2^3-\rho)}{2}\right) \\
&\geq N^{'}\left(\Gamma(x, 0, 0), \tfrac{3(2^n)\varepsilon(2^3-\rho)}{2\rho^n}\right)
\end{aligned}$$

for all $x \in X$ and all $\varepsilon > 0$. Since $\lim_{n\to\infty} \frac{3(2^n)\varepsilon(2^3-\rho)}{2\rho^n} = \infty$, we have

$$\lim_{n\to\infty} N^{'}\left(\Gamma(x, 0, 0), \frac{3(2^n)\varepsilon(2^3 - \rho)}{2\rho^n}\right) = 1.$$

Thus $N(C(x) - D(x), \varepsilon) = 1$ for all $x \in X$ and all $\varepsilon > 0$, and so $C(x) = D(x)$.

For $\omega = -1$, we can prove the result by a similar method. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.1, concerning the stability for the functional equation (1.4).

Corollary 4.2. *Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$N(D_f(x_1, x_2, x_3), \varepsilon) \geq \begin{cases} N'(\theta, \varepsilon) \\ N'(\theta \sum_{i=1}^{3} \|x_i\|^s, \varepsilon) \\ N'(\theta(\sum_{i=1}^{3} \|x_i\|^{3s} + \Pi_{i=1}^{3} \|x_i\|^s), \varepsilon) \end{cases}$$

for all $x_1, x_2, x_3 \in X$ and all $\varepsilon > 0$, where θ, s are constants with $\theta > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$N(f(x) - C(x), \varepsilon) \geq \begin{cases} N'(\theta, \mid 21 \mid \varepsilon) & \\ N'\left(\theta\|x\|^s, 3 \mid 2^3 - 2^s \mid \varepsilon\right) & ; s \neq 3 \\ N'\left(\theta\|x\|^{3s}, 3 \mid 2^3 - 2^{ns} \mid \varepsilon\right) & ; s \neq \frac{3}{n} \end{cases}$$

for all $x \in X$ and all $r > 0$.

5. Stability results for the functional equation (1.4): Fixed point method

In this section, we establish the Hyers-Ulam stability of the functional equation (1.4) in fuzzy normed space via fixed point method.

To prove the stability result, we define the following: η_i is a constant such that

$$\eta_i = \begin{cases} 2 & if \quad i = 0 \\ \frac{1}{2} & if \quad i = 1 \end{cases}$$

and Ω is the set such that $\Omega = \{t : X \to Y, t(0) = 0\}$.

Theorem 5.1. *Let $f : X \to Y$ be a mapping for which there exists a mapping $\Gamma : X^3 \to Z$ with condition*

$$\lim_{k\to\infty} N'\left(\Gamma(\eta_i^k x_1, \eta_i^k x_2, \eta_i^k x_3), \eta_i^{3k}\varepsilon\right) = 1 \tag{5.1}$$

for all $x_1, x_2, x_3 \in X$ and all $\varepsilon > 0$ and satisfying the inequality

$$N(D_f(x_1, x_2, x_3), \varepsilon) \geq N'(\Gamma(x_1, x_2, x_3), \varepsilon) \tag{5.2}$$

for all $x_1, x_2, x_3 \in X$ and $\varepsilon > 0$. If there exists $L = L[i]$ such that the function $x \to \beta(x) = \frac{1}{3}\Gamma\left(\frac{x}{2}, 0, 0\right)$ has the property

$$N'\left(L\frac{1}{\eta_i^3}\beta(\eta_i x), \varepsilon\right) = N'\left(\beta(x), \varepsilon\right) \tag{5.3}$$

for all $x \in X$ and $\varepsilon > 0$, then there exists a unique cubic function $C : X \to Y$ satisfying the functional equation (1.4) and

$$N(f(x) - C(x), \varepsilon) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), \varepsilon\right)$$

for all $x \in X$ and $\varepsilon > 0$.

Proof. Set

$$\Gamma(x_1, x_2, x_3) = \begin{cases} \theta \\ \theta(\sum_{i=1}^3 \|x_i\|^s) \\ \theta(\prod_{i=1}^3 \|x_i\|^s + \sum_{i=1}^3 \|x_i\|^{ns}) \end{cases}$$

for all $x_1, x_2, x_3 \in X$. Then

$$N'\left(\Gamma\left(\eta_i^k x_1, \eta_i^k x_2, \eta_i^k x_3\right), \eta_i^{3k}\varepsilon\right) = \begin{cases} N'(\theta, \eta_i^{3k}\varepsilon) \\ N'\left(\theta \sum_{i=1}^3 \|x_i\|^s, \eta_i^{(3-s)k}\varepsilon\right) \\ N'\left(\theta(\sum_{i=1}^3 \|x_i\|^{ns} + \Pi_{i=1}^3\|x_i\|^s), \eta_i^{(3-ns)k}\varepsilon\right) \end{cases}$$

$$= \begin{cases} \longrightarrow \quad 1 \quad as \quad k \longrightarrow \infty, \\ \longrightarrow \quad 1 \quad as \quad k \longrightarrow \infty, \\ \longrightarrow \quad 1 \quad as \quad k \longrightarrow \infty. \end{cases}$$

Thus (5.1) holds. But we have

$$\beta(x) = \frac{1}{3}\Gamma\left(\frac{x}{2}, 0, 0\right)$$

has the property

$$N'\left(L\frac{1}{\eta_i^3}\beta(\eta_i x), \varepsilon\right) \geq N'(\beta(x), \varepsilon)$$

for all $x \in X$ and $\varepsilon > 0$. Hence

$$N'(\beta(x), \varepsilon) = N'\left(\Gamma\left(\frac{x}{2}, 0, 0\right), 3\varepsilon\right) = \begin{cases} N'(\theta, 3\varepsilon) \\ N'\left(\theta\|\frac{x}{2}\|^s, 3\varepsilon\right) \\ N'\left(\theta\|\frac{x}{2}\|^{ns}, 3\varepsilon\right). \end{cases}$$

Thus

$$N'\left(\frac{1}{\eta_i^3}\beta(\eta_i x), \varepsilon\right) = \begin{cases} N'\left(\frac{\theta}{\eta_i^3}, 3\varepsilon\right) \\ N'\left(\frac{\theta}{\eta_i^3}\left(\frac{2}{2^s}\right)\|\eta_i x\|^s, 3\varepsilon\right) \\ N'\left(\frac{\theta}{\eta_i^3}\left(\frac{2}{2^{ns}}\right)\|\eta_i x\|^{ns}, 3\varepsilon\right) \end{cases} = \begin{cases} N'(\eta_i^{-3}\beta(x), \varepsilon) \\ N'(\eta_i^{s-3}\beta(x), \varepsilon) \\ N'(\eta_i^{ns-3}\beta(x), \varepsilon). \end{cases}$$

Now we can decide the Lipschitz constant $0 < L < 1$ by η_i, given in the previous statement of Theorem 5.1. So we divide into the following 6 cases for the conditions of η_i as follows:

Case (i): $L = 2^{-3} \quad for \quad s = 0 \quad if \quad i = 0$:

$N(f(x) - C(x), \varepsilon) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), \varepsilon\right) \geq N'\left(\frac{\theta(2^{-3})}{1-2^{-3}}, 3\varepsilon\right) \geq N'(\theta, 21\varepsilon)$.

Case (ii): $L = 2^3 \quad for \quad s = 0 \quad if \quad i = 1$:
$N(f(x) - C(x), \varepsilon) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), \varepsilon\right) \geq N'\left(\frac{\theta}{1-2}, 3\varepsilon\right) \geq N'(\theta, -21\varepsilon)$.

Case (iii): $L = 2^{s-3} \quad for \quad s < 3 \quad if \quad i = 0$:

$$\begin{aligned} N(f(x) - C(x), \varepsilon) &\geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), \varepsilon\right) \geq N'\left(\frac{2^{s-3}}{1-2^{s-3}}\frac{\theta\|x\|^s}{2^s}, 3\varepsilon\right) \\ &\geq N'\left(\theta\|x\|^s, 3\varepsilon(2^3 - 2^s)\right). \end{aligned}$$

Case (iv): $L = 2^{3-s} \quad for \quad s > 3 \quad if \quad i = 1$:

$$\begin{aligned} N(f(x) - C(x), \varepsilon) &\geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), \varepsilon\right) \geq N'\left(\frac{1}{1-2^{3-s}}\frac{\theta\|x\|^s}{2^s}, 3\varepsilon\right) \\ &\geq N'\left(\theta\|x\|^s, 3\varepsilon(2^s - 2^3)\right). \end{aligned}$$

Case (v): $L = 2^{ns-3} \quad for \quad s < \frac{3}{n} \quad if \quad i = 0$:

$$\begin{aligned} N(f(x) - C(x), \varepsilon) &\geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), \varepsilon\right) \geq N'\left(\frac{2^{ns-3}}{1-2^{ns-3}}\frac{\theta\|x\|^{ns}}{2^{ns}}, 3\varepsilon\right) \\ &\geq N'\left(\theta\|x\|^{ns}, 3\varepsilon(2^3 - 2^{ns})\right). \end{aligned}$$

Case (vi): $L = 2^{3-ns} \quad for \quad s < \frac{3}{n} \quad if \quad i = 1$:

$$N(f(x) - C(x), \varepsilon) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), \varepsilon\right) \geq N'\left(\frac{1}{1-2^{3-ns}}\frac{\theta\|x\|^{ns}}{2^{ns}}, 3\varepsilon\right)$$
$$\geq N'\left(\theta\|x\|^{ns}, -3\varepsilon(2^{ns} - 2^3)\right).$$

Hence the proof is completed. □

The following corollary is an immediate consequence of Theorem 5.1, concerning the stability of the functional equation (1.4).

Corollary 5.2. *Suppose a function $f : X \to Y$ satisfies the inequality*

$$N(D_f(x_1, x_2, x_3), \varepsilon) \geq \begin{cases} N'(\theta, \varepsilon) \\ N'(\theta \sum_{i=1}^{3} \|x_i\|^s, \varepsilon) \\ N'(\theta(\sum_{i=1}^{3} \|x_i\|^{3s} + \Pi_{i=1}^{3}\|x_i\|^s), \varepsilon) \end{cases}$$

for all $x_1, x_2, x_3 \in X$ and $\varepsilon > 0$, where θ, s are constants with $\theta > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$N(f(x) - C(x), \varepsilon) \geq \begin{cases} N'(\theta, \mid 21 \mid \varepsilon) & \\ N'\left(\theta\|x\|^s, 3 \mid 2^3 - 2^s \mid \varepsilon\right) & ; s \neq 3 \\ N'\left(\theta\|x\|^{3s}, 3 \mid 2^3 - 2^{ns} \mid \varepsilon\right) & ; s \neq \frac{3}{n} \end{cases}$$

for all $x \in X$ and $\varepsilon > 0$.

Remark 5.3. To prove the stability of functional equations and functional inequalities, we have two methods: direct method and fixed point method. For the direct method, we use the Hyers-Ulam method, which is a traditional method, and for the fixed point method, we use the Isac-Rassias method, which is a more recent method. The proofs for the stability of functional equations and functional inequalities are similar to the orginal direct method and/or the fixed point method.

In general, to prove the stability, we divide two cases, for an example, $p > 3$ and $0 < p < 3$ in $\|x\|^p$, appeared in a control function of cubic functional equations. In this paper, we just use one control function to prove the stability of a new 3-D cubic functional equation by using the direct method and by using the fixed point method.

6. Conclusion

We have introduced the following 3-D cubic functional equation

$$\begin{aligned} f(2x_1 + x_2 + x_3) = {} & 3f(x_1 + x_2 + x_3) + f(-x_1 + x_2 + x_3) + 2f(x_1 + x_2) + 2f(x_1 + x_3) \\ & -6f(x_1 - x_2) - 6f(x_1 - x_3) - 3f(x_2 + x_3) + 2f(2x_1 - x_2) \\ & +2f(2x_1 - x_3) - 18f(x_1) - 6f(x_2) - 6f(x_3). \end{aligned}$$

We have solved the 3-D cubic functional equation and we have proved the Hyers-Ulam stability of the 3-D functional equation in fuzzy normed spaces by using the direct method and the fixed point method.

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Conflict of interest

The authors declare that they have no competing interests.

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