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New Construction of Strongly Relatively Nonexpansive Sequences by Firmly Nonexpansive-Like Mappings

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Abstract: In recent works, many authors generated strongly relatively nonexpansive sequences of mappings by the sequences of firmly nonexpansive-like mappings. In this paper, we introduce a new method for construction of strongly relatively nonexpansive sequences from firmly nonexpansive-like mappings.

Keywords: Banach space; firmly nonexpansive-like mapping; fixed point; mapping of type (r); mapping of type (sr)

1. Introduction and Preliminaries

The class of firmly nonexpansive-like mappings has been introduced in [1]. Fixed point theory for such mappings can be applied to several nonlinear problems such as zero point problems for monotone operators, convex feasibility problems, convex minimization problems, equilibrium problems (see, [1–5] for more details).

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space X, J be a normalized duality mapping from X into dual X^*, and $S, T\colon C \to X$ are firmly nonexpansive-like mappings. The set of all fixed points of T is denoted by $F(T)$. It is known that if C is a bounded subset, then $F(T)$ is nonempty ([1], Theorem 7.4). We investigate asymptotic behavior of the following sequence $\{x_n\}$ in a uniformly smooth and 2-uniformly convex Banach space X.

$$x_{n+1} = Q_C J^{-1}(JTx_n - (\mu_X)^{-2}J(x_n - Sx_n)) \tag{1}$$

for all $n \in \mathbb{N}$, where $x_1 \in C$, μ_X denotes the uniform convexity constant of X, and Q_C denotes the generalized projection of X onto C. If X is a Hilbert space, then (1) is reduced to

$$x_{n+1} = Tx_n, \qquad \text{for all } n \in \mathbb{N}. \tag{2}$$

Throughout the present paper, we denote by $\mathbb{N}$ the set of all positive integers, $\mathbb{R}$ the set of all real numbers, X a real Banach space with dual X^*, $\|.\|$ the norms of X and X^*, $\langle x, x^* \rangle$ the value of $x^* \in X^*$

at $x \in X$, $x_n \longrightarrow x$ strong convergence of a sequence $\{x_n\}$ of X to $x \in X$, $x_n \rightharpoonup x$ weak convergence of a sequence $\{x_n\}$ of X to $x \in X$, S_X the unit sphere of X, and B_X the closed unit ball of X.

Now, we present some definitions which are needed in the sequel. The normalized duality mapping of X into X^* is defined by

$$Jx = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \tag{3}$$

for all $x \in X$. The space X is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{4}$$

exists for all $x, y \in S_X$. The space X is said to be uniformly smooth, if (4) converges uniformly in $x, y \in S_X$. It is said to be strictly convex, if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_X$ and $x \neq y$. It is said to be uniformly convex, if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$, where δ_X is the modulus of convexity of X defined by

$$\delta_X(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in B_X, \|x - y\| \geqslant \varepsilon\} \tag{5}$$

for all $\varepsilon \in [0, 2]$.

The space X is said to be 2-uniformly convex, if there exists $c > 0$ such that $\delta_X(\varepsilon) \geqslant c\varepsilon^2$ for all $\varepsilon \in [0, 2]$.

It is obvious that every 2-uniformly convex Banach space is uniformly convex. It is known that all Hilbert spaces are uniformly smooth and 2-uniformly convex. It is also known that all the Lebesgue spaces L_p are uniformly smooth and 2-uniformly convex whenever $1 < p \leqslant 2$.

For a smooth Banach space, J is said to be weakly sequentially continuous if $\{Jx_n\}$ converges weak to Jx, whenever $\{x_n\}$ is a sequence of X such that $x_n \rightharpoonup x \in X$.

Define $\varphi : X \times X \to \mathbb{R}$ by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \tag{6}$$

for all $x, y \in X$. It is known that

$$\varphi(x, y) = \varphi(x, z) + \varphi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{7}$$

for all $x, y, z \in X$.

Definition 1 ([3]). *The metric projection P_C from X onto C and the generalized projection Q_C from X onto C are defined by*

$$P_C x = \operatorname*{argmin}_{y \in C} \|y - x\|, \quad Q_C x = \operatorname*{argmin}_{y \in C} \varphi(y, x) \tag{8}$$

for all $x \in X$, respectively.

Obviously, for $x \in X$ and $z \in C$,

$$z = P_C x \Longleftrightarrow \langle y - z, J(x - z) \rangle, \ (\forall y \in C). \tag{9}$$

Also, for $x \in X$ and $z \in C$,

$$z = Q_C x \Longleftrightarrow \langle y - z, Jx - Jz \rangle, \ (\forall y \in C). \tag{10}$$

Definition 2 ([1]). *A mapping $T : C \longrightarrow X$ is said to be a firmly nonexpansive-like mapping, if*

$$\langle Tx - Ty, J(x - Tx) - J(y - Ty) \rangle \geqslant 0 \tag{11}$$

for all $x, y \in C$.

Definition 3 ([1]). *Let* $T : C \longrightarrow X$ *be a mapping. A point* $p \in C$ *is said to be an asymptotic fixed point of* T*, if there exists a sequence* $\{x_n\}$ *of* C *such that* $x_n \rightharpoonup p$ *and* $x_n - Tx_n \longrightarrow 0$*. The set of all asymptotic fixed points of* T *is denoted by* $\hat{F}(T)$*.*

Definition 4 ([1]). *The mapping* T *is said to be of type* (r)*, if* $F(T)$ *is nonempty and* $\varphi(u, Tx) \leqslant \varphi(u, x)$ *for all* $u \in F(T)$ *and* $x \in C$*.*

It is known that if T is a mapping of type (r), then $F(T)$ is closed and convex.

Definition 5 ([4]). *The mapping* T *is said to be of type* (sr)*, if* T *is of type* (r) *and* $\varphi(Tz_n, z_n) \longrightarrow 0$*, whenever* $\{z_n\}$ *is a bounded sequence of* C *such that* $\varphi(u, z_n) - \varphi(u, Tz_n) \longrightarrow 0$ *for some* $u \in F(T)$*.*

Definition 6 ([4]). *The sequence* $\{T_n\}$ *is said to satisfy the condition* (Z)*, if every weak subsequential limit of* $\{x_n\}$ *belongs to* $F(\{T_n\})$*, whenever* $\{x_n\}$ *is a bounded sequence of* C *such that* $x_n - T_nx_n \longrightarrow 0$*.*

Now, we give some results which will be used in our main results.

Theorem 1 ([5]). *The space* X *is* 2*-uniformly convex if and only if there exists* $\mu \geqslant 0$ *such that*

$$\frac{\|x+y\|^2 + \|x-y\|^2}{2} \geqslant \|x\|^2 + \|\mu^{-1}y\|^2, \quad \text{for all } x, y \in X. \tag{12}$$

Lemma 1 ([4], Lemma 2.2). *Suppose that* X *is* 2*-uniformly convex. Then*

$$(\frac{1}{\mu_X}\|x-y\|)^2 \leqslant \varphi(x, y), \quad \text{for all } x, y \in X. \tag{13}$$

Lemma 2 ([1]). *If* $T : C \longrightarrow X$ *is a firmly nonexpansive-like mapping, then* $F(T)$ *is a closed convex subset of* X *and* $\hat{F}(T) = F(T)$*.*

Lemma 3 ([4]). *Suppose that* X *is uniformly convex. If* $S : X \longrightarrow X$ *and* $T : C \longrightarrow X$ *are mappings of type* (r) *such that* $F(S) \cap F(T)$ *is nonempty and* S *or* T *is of type* (sr)*, then* $ST : C \longrightarrow X$ *is of type* (r) *and* $F(ST) = F(S) \cap F(T)$*. Further, if both* S *and* T *are of type* (sr)*, then so is* ST*.*

Lemma 4 ([4]). *Suppose that* X *is uniformly convex. Let* $\{S_n\}$ *be a sequence of mappings of* X *into itself and* $\{T_n\}$ *a sequence of mappings of* C *into* X *such that* $F(\{S_n\}) \cap F(\{T_n\})$ *is nonempty, both* $\{S_n\}$ *and* $\{T_n\}$ *are of type* (sr)*, and* S_n *or* T_n *is of type* (sr) *for all* $n \in \mathbb{N}$*. Then the following holds:*

- (i) $\{S_nT_n\}$ *is of type* (sr)*;*
- (ii) *if* X *is uniformly smooth and both* $\{S_n\}$ *and* $\{T_n\}$ *satisfy the condition* (Z)*, then so does* $\{S_nT_n\}$*.*

Theorem 2 ([4]). *Let* X *be a smooth and uniformly convex Banach space,* C *a nonempty closed convex subset of* X*, and* $\{T_n\}$ *a sequence of mappings of* C *into* X *such that* $\{T_n\}$ *is of type* (sr) *and* $\{T_n\}$ *satisfies the condition* (Z)*. If* $T_n(C) \subset C$ *for all* $n \in \mathbb{N}$ *and* J *is weakly sequentially continuous, then the sequence* $\{x_n\}$ *defined by* $x_1 \in C$ *and* $x_{n+1} = T_nx_n$ *for all* $n \in \mathbb{N}$ *converges weakly to the strong limit of* $\{Q_Fx_n\}$*.*

Now, we construct a new strongly relatively nonexpansive sequence from a given sequence of firmly nonexpansive-like mappings with a common fixed point in Banach spaces.

2. Main Results

The following results will be used in the sequel of the paper.

Lemma 5. *Let C be a nonempty closed convex subset of a smooth, strictly convex, 2-uniformly convex and reflexive Banach space X. Suppose that (S, T) is a pair of firmly nonexpansive-like mappings of C into X and let $F = F(S) \cap F(T) \neq \emptyset$. Let U be a mapping of C into X defined by $U = J^{-1}(JT - \beta J(I - S))$, where $\beta > 0$ and I denotes the identity mapping on C. Then*

$$\varphi(u, Ux) + \frac{1}{2}(\frac{2}{\mu_X^2} - \beta)\|Ux - Tx\|^2 \leqslant \varphi(u, Tx)$$

for all $u \in F(U)$ and $x \in C$.

Proof. Let $u \in F(U)$ and $x \in C$ be given. Then, from (7) and the definition of U, it follows that

$$\begin{aligned} \varphi(u, Ux) + \varphi(Ux, Tx) - \varphi(u, Tx) &= 2\langle u - Ux, JTx - JUx\rangle \\ &= 2\beta\langle u - Ux, J(x - Sx)\rangle. \end{aligned} \tag{14}$$

Since S is firmly nonexpansive-like and $u \in F(S)$, we know that

$$\begin{aligned} \langle u - Ux, J(x - Sx)\rangle &= \langle u - Sx, J(x - Sx)\rangle + \langle Sx - Ux, J(x - Sx)\rangle \\ &= \langle Sx - Ux, J(x - Sx)\rangle. \end{aligned} \tag{15}$$

On the other hand, we have

$$\begin{aligned} \langle Sx - Ux, J(x - Sx)\rangle &= -\|Sx - Tx\|^2 + \langle Tx - Ux, J(x - Sx)\rangle \\ &\leqslant -(\|Sx - Tx\|^2 - \|Tx - Ux\|\|x - Sx\|) \\ &\leqslant -(\|Sx - x\|^2 - \frac{1}{2}\|Ux - Tx\|)^2 + \frac{1}{4}\|Ux - Tx\|^2 \\ &\leqslant \|Ux - Tx\|^2. \end{aligned} \tag{16}$$

Since $\beta > 0$, from (14)–(16), we deduce that

$$\varphi(u, Ux) + \varphi(Ux, Tx) - \varphi(u, Tx) \leqslant 2\beta\|Ux - Tx\|^2. \tag{17}$$

Since X is 2-uniformly convex, Lemma 1 implies that

$$(\mu_X)^{-2}\|Ux - Tx\|^2 \leqslant \varphi(Ux, Tx). \tag{18}$$

By (17) and (18), we obtain the desired inequality. □

Now, we present the construction of strongly relatively nonexpansive sequences in the following.

Theorem 3. *Let C be a nonempty closed convex subset of a smooth and 2-uniformly convex Banach space X;*

(i) *$\{T_n\}$, $\{S_n\}$ are sequences of firmly nonexpansive-like mappings from C into X such that $F = F(\{T_n\}) \cap F(\{S_n\})$ is nonempty;*

(ii) *$\{U_n\}$ is a sequence of mappings from C into X defined by*

$$U_n = J^{-1}(JT_n - \beta_n J(I - S_n))$$

for all $n \in \mathbb{N}$, where β_n is a sequence of real numbers such that $0 < \inf_n \beta_n$ and $\sup_n \beta_n < 2(\mu_X)^{-2}$ and I denotes the identity mapping on C.

Then $F(\{U_n\}) \subset F(\{S_n\}) \cap F(\{T_n\})$ and $\{U_n\}$ is of type (sr). Also, if X is uniformly smooth and $\{S_n\}$ satisfies the condition (Z), then $\{U_n\}$ satisfies the condition (Z).

Proof. We can easily see that $F(\{U_n\}) \subset F(\{S_n\}) \cap F(\{T_n\})$. At first, we show that $\{U_n\}$ is of type (sr).
Note that $F(\{U_n\})$ is nonempty. By Lemma 5, we also know that each U_n is a mapping of type (r) from C into X.
Suppose that $\{T_n z_n\}$ is a bounded sequence of C such that

$$\varphi(u, T_n z_n) - \varphi(u, U_n T_n z_n) \longrightarrow 0$$

for some $u \in F(\{U_n\})$. Then, it follows from Lemma 5 that

$$0 \leqslant \frac{1}{2}(\frac{2}{\mu_X^2} - \beta_n)\|U_n z_n - T_n z_n\|^2 \leqslant \varphi(u, T_n z_n) - \varphi(u, U_n z_n). \tag{19}$$

Thus, it follows from $\sup_n \beta_n < 2(\mu_X)^{-2}$ that $\|U_n z_n - T_n z_n\| \longrightarrow 0$. Consequently, we have $\varphi(U_n z_n, T_n z_n) \longrightarrow 0$ and hence$\{U_n\}$ is of type (sr). Now, we present the proof of part (ii). Suppose that X is uniformly smooth and $\{S_n\}$ satisfies the condition (Z). Let p be a weak subsequential limit of a bounded sequence $\{x_n\}$ of C such that $T_n x_n - U_n x_n \longrightarrow 0$. By the definition of U_n, we have

$$J(x_n - S_n x_n) = \frac{1}{\beta_n}(JT_n x_n - JU_n x_n) \tag{20}$$

for all $n \in \mathbb{N}$. Since J is uniformly norm-to-norm continuous on each nonempty bounded subset of X and $\sup_n \dfrac{1}{\beta_n} < \infty$, it follows from (20) that

$$\|x_n - S_n x_n\| = \frac{1}{\beta_n}\|JT_n x_n - JU_n x_n\| \longrightarrow 0.$$

From our assumptions, we know that $p \in F \supset F(\{U_n\})$. Therefore, $\{U_n\}$ satisfies the condition (Z). □

Remark 1. *It is notable that every nonexpansive mapping T is a mapping of type (r), but the converse is not necessarily satisfied in a Hilbert space. For instance, let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $Tx = x^2$, then T is of type (r) and is neither nonexpansive nor of type (sr). Also, let $T : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be defined by $Tx = \sqrt{x}$. Then T is a mapping of type (sr).*

Remark 2. *For a mapping T from C into X, the following assertions hold:*
(a) T is of type (sr) if and only if $\{T, T, \cdots\}$ is of type (sr);
(b) $\hat{F}(T) = F(T)$ if and only if $\{T, T, \cdots\}$ satisfies the condition (Z).

Corollary 1. *Let (S, T) be a pair of firmly nonexpansive-like mappings from C into X such that $F(T) \cap F(S)$ are nonempty and U be a mapping from C into X which is defined by*

$$U = J^{-1}(JT - \beta J(I - S))$$

where $0 < \beta < 2(\mu_X)^{-2}$. Then the following assertions hold:

(i) $F(U) \subset F(T) \cap F(S)$ and U is of type (sr);
(ii) if X is uniformly smooth, then $\hat{F}(U) = F(U)$.

Theorem 4. *Let $\{V_n\}$ be a sequence of mappings from C into itself which are defined by*

$$V_n = Q_C U_n$$

for all $n \in \mathbb{N}$. Then the following consequences hold:

(i) $F(\{V_n\}) \subset F$ *and* $\{V_n\}$ *is of type* (sr);
(ii) *if* X *is uniformly smooth and* $\{S_n\}$ *satisfies the condition* (Z), *then so does* $\{V_n\}$.

Proof. We know that $F(V_n) \subset F(T_n) \cap F(S_n)$ for all $n \in N$ and hence $F(\{V_n\}) \subset F \neq \emptyset$. We first show that $\{V_n\}$ is of type (sr). From (i) of Corollary 1, we know that each U_n is of type (sr). Since Q_C is of type (sr) from X into itself and

$$F(Q_C) \cap F(U_n) \subset F(T_n) \cap F(S_n) \supset F \neq \emptyset,$$

Lemma 3 implies that each $V_n = Q_C U_n$ is also of type (sr).
Since $\{Q_C, Q_C, ...\}$ is of type (sr) by Remark 2, $\{U_n\}$ is of type (sr) by Theorem 3, and

$$F(Q_C) \cap F(\{U_n\}) \subset F \neq \emptyset,$$

the part (i) of Lemma 4 implies that $\{V_n\}$ is of type (sr).
We finally show the part (ii). Suppose that X is uniformly smooth and $\{S_n\}$ satisfies the condition (Z). Since C is weakly closed, we can easily see that $\hat{F}(Q_C) = F(Q_C) = C$. This implies that $\{Q_C, Q_C, ...\}$ satisfies the condition (Z). From Theorem 3, we know that $\{U_n\}$ satisfies the condition (Z). Thus, the part (ii) of Lemma 4 implies the conclusion. □

As a direct consequence of Theorems 2 and 4, we obtain the following result.

Theorem 5. *Let* X *be a uniformly smooth and* 2*-uniformly convex Banach space,* C *be a nonempty closed convex subset of* X, $\{T_n\}$ *and* $\{S_n\}$ *be two sequences of firmly nonexpansive-like mappings from* C *into* X *such that* $F = F(\{T_n\}) \cap F(\{S_n\})$ *is nonempty and* $\{S_n\}$ *satisfies the condition* (Z), β_n *be a sequence of real numbers such that*

$$0 < \inf_n \beta_n, \ \ \sup_n \beta_n < 2(\mu_X)^{-2},$$

and $\{x_n\}$ *be a sequence defined by* $x_1 \in C$ *and*

$$x_{n+1} = Q_C J^{-1}\big(JT_n x_n - \beta_n J(x_n - S_n x_n)\big)$$

for all $n \in \mathbb{N}$. *If* J *is weakly sequentially continuous, then* $\{x_n\}$ *converges weakly to the strong limit of* $\{Q_F x_n\}$.

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