

## HYERS-ULAM STABILITY OF AN ADDITIVE $(\rho_1, \rho_2)$ -FUNCTIONAL INEQUALITY IN BANACH SPACES

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ABSTRACT. In this paper, we introduce and solve the following additive  $(\rho_1, \rho_2)$ -functional inequality

$$(0.1) \quad \|f(x+y+z) - f(x) - f(y) - f(z)\| \leq \|\rho_1(f(x+z) - f(x) - f(z))\| \\ + \|\rho_2(f(y+z) - f(y) - f(z))\|,$$

where  $\rho_1$  and  $\rho_2$  are fixed nonzero complex numbers with  $|\rho_1| + |\rho_2| < 2$ .

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the additive  $(\rho_1, \rho_2)$ -functional inequality (0.1) in complex Banach spaces.

### 1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [18] concerning the stability of group homomorphisms.

The functional equation  $f(x+y) = f(x) + f(y)$  is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [16] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability of quadratic functional equation was proved by Skof [17] for mappings  $f : E_1 \rightarrow E_2$ , where  $E_1$  is a normed space and  $E_2$  is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain  $E_1$  is replaced by an Abelian group.

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Park [13, 14] defined additive  $\rho$ -functional inequalities and proved the Hyers-Ulam stability of the additive  $\rho$ -functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors (see [2, 6, 19]).

We recall a fundamental result in fixed point theory.

**Theorem 1.1** ([3, 8]). *Let  $(X, d)$  be a complete generalized metric space and let  $J : X \rightarrow X$  be a strictly contractive mapping with Lipschitz constant  $\alpha < 1$ . Then for each given element  $x \in X$ , either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers  $n$  or there exists a positive integer  $n_0$  such that

- (1)  $d(J^n x, J^{n+1} x) < \infty$ ,  $\forall n \geq n_0$ ;
- (2) the sequence  $\{J^n x\}$  converges to a fixed point  $y^*$  of  $J$ ;
- (3)  $y^*$  is the unique fixed point of  $J$  in the set  $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$ ;
- (4)  $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$  for all  $y \in Y$ .

In 1996, G. Isac and Th.M. Rassias [11] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 15]).

In Section 2, we solve the additive  $(\rho_1, \rho_2)$ -functional inequality (0.1) and prove the Hyers-Ulam stability of the additive  $(\rho_1, \rho_2)$ -functional inequality (0.1) in Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive  $(\rho_1, \rho_2)$ -functional inequality (0.1) in Banach spaces by using the direct method.

Throughout this paper, let  $X$  be a real or complex normed space with norm  $\|\cdot\|$  and  $Y$  a complex Banach space with norm  $\|\cdot\|$ . Assume that  $\rho_1$  and  $\rho_2$  are fixed nonzero complex numbers with  $|\rho_1| + |\rho_2| < 2$ .

## 2. ADDITIVE $(\rho_1, \rho_2)$ -FUNCTIONAL INEQUALITY (0.1): A FIXED POINT METHOD

In this section, we solve and investigate the additive  $(\rho_1, \rho_2)$ -functional inequality (0.1) in complex Banach spaces.

**Lemma 2.1.** *If a mapping  $f : X \rightarrow Y$  satisfies  $f(0) = 0$  and*

$$(2.1) \quad \|f(x + y + z) - f(x) - f(y) - f(z)\| \leq \|\rho_1(f(x + z) - f(x) - f(z))\| \\ + \|\rho_2(f(y + z) - f(y) - f(z))\|$$

for all  $x, y, z \in X$ , then  $f : X \rightarrow Y$  is additive.

*Proof.* Assume that  $f : X \rightarrow Y$  satisfies (2.1).

Letting  $x = y = z = 0$  in (2.1), we get  $2\|f(0)\| \leq (|\rho_1| + |\rho_2|)\|f(0)\|$  and so  $f(0) = 0$ , since  $|\rho_1| + |\rho_2| < 2$ .

Letting  $z = 0$  in (2.1), we get

$$\|f(x + y) - f(x) - f(y)\| \leq 0$$

for all  $x, y \in X$ . So  $f(x + y) = f(x) + f(y)$  for all  $x, y \in X$ . Thus  $f$  is additive.  $\square$

Using the fixed point method, we prove the Hyers-Ulam stability of the additive  $(\rho_1, \rho_2)$ -functional inequality (2.1) in complex Banach spaces.

**Theorem 2.2.** *Let  $\varphi : X^3 \rightarrow [0, \infty)$  be a function such that there exists an  $L < 1$  with*

$$(2.2) \quad \varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2}\varphi(x, y, z)$$

for all  $x, y, z \in X$ . Let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and

$$(2.3) \quad \|f(x + y + z) - f(x) - f(y) - f(z)\| \leq \|\rho_1(f(x + z) - f(x) - f(z))\| \\ + \|\rho_2(f(y + z) - f(y) - f(z))\| + \varphi(x, y, z)$$

for all  $x, y, z \in X$ . Then there exists a unique additive mapping  $A : X \rightarrow Y$  such that

$$\|f(x) - A(x)\| \leq \frac{L}{2(1-L)}\varphi(x, x, 0)$$

for all  $x \in X$ .

*Proof.* Letting  $y = x$  and  $z = 0$  in (2.3), we get

$$(2.4) \quad \|f(2x) - 2f(x)\| \leq \varphi(x, x, 0)$$

for all  $x \in X$ .

Consider the set

$$S := \{h : X \rightarrow Y, h(0) = 0\}$$

and introduce the generalized metric on  $S$ :

$$d(g, h) = \inf \{ \mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu \varphi(x, x, 0), \forall x \in X \},$$

where, as usual,  $\inf \emptyset = +\infty$ . It is easy to show that  $(S, d)$  is complete (see [12]).

Now we consider the linear mapping  $J : S \rightarrow S$  such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all  $x \in X$ .

Let  $g, h \in S$  be given such that  $d(g, h) = \varepsilon$ . Then

$$\|g(x) - h(x)\| \leq \varepsilon \varphi(x, x, 0)$$

for all  $x \in X$ . Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\| 2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right) \right\| \leq 2\varepsilon \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \\ &\leq 2\varepsilon \frac{L}{2} \varphi(x, x, 0) = L\varepsilon \varphi(x, x, 0) \end{aligned}$$

for all  $x \in X$ . So  $d(g, h) = \varepsilon$  implies that  $d(Jg, Jh) \leq L\varepsilon$ . This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all  $g, h \in S$ .

It follows from (2.4) that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \leq \frac{L}{2} \varphi(x, x, 0)$$

for all  $x \in X$ . So  $d(f, Jf) \leq \frac{L}{2}$ .

By Theorem 1.1, there exists a mapping  $A : X \rightarrow Y$  satisfying the following:

(1)  $A$  is a fixed point of  $J$ , i.e.,

$$(2.5) \quad A(x) = 2A\left(\frac{x}{2}\right)$$

for all  $x \in X$ . The mapping  $A$  is a unique fixed point of  $J$  in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that  $A$  is a unique mapping satisfying (2.5) such that there exists a  $\mu \in (0, \infty)$  satisfying

$$\|f(x) - A(x)\| \leq \mu \varphi(x, x, 0)$$

for all  $x \in X$ ;

(2)  $d(J^l f, A) \rightarrow 0$  as  $l \rightarrow \infty$ . This implies the equality

$$\lim_{l \rightarrow \infty} 2^l f\left(\frac{x}{2^l}\right) = A(x)$$

for all  $x \in X$ ;

$$(3) \quad d(f, A) \leq \frac{1}{1-L} d(f, Jf), \text{ which implies}$$

$$\|f(x) - A(x)\| \leq \frac{L}{2(1-L)} \varphi(x, x, 0)$$

for all  $x \in X$ .

It follows from (2.2) and (2.3) that

$$\begin{aligned} & \|A(x+y+z) - A(x) - A(y) - A(z)\| \\ &= \lim_{n \rightarrow \infty} 2^n \left\| f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} 2^n |\rho_1| \left\| f\left(\frac{x+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{z}{2^n}\right) \right\| \\ &\quad + \lim_{n \rightarrow \infty} 2^n |\rho_2| \left\| f\left(\frac{y+z}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right) \right\| + \lim_{n \rightarrow \infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\ &= \|\rho_1(A(x+z) - A(x) - A(z))\| + \|\rho_2(A(y+z) - A(y) - A(z))\| \end{aligned}$$

for all  $x, y, z \in X$ . So

$$\begin{aligned} \|A(x+y+z) - A(x) - A(y) - A(z)\| &\leq \|\rho_1(A(x+z) - A(x) - A(z))\| \\ &\quad + \|\rho_2(A(y+z) - A(y) - A(z))\| \end{aligned}$$

for all  $x, y, z \in X$ . By Lemma 2.1, the mapping  $A : X \rightarrow Y$  is additive.  $\square$

**Corollary 2.3.** *Let  $r > 1$  and  $\theta$  be nonnegative real numbers, and let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and*

$$(2.6) \quad \begin{aligned} \|f(x+y+z) - f(x) - f(y) - f(z)\| &\leq \|\rho_1(f(x+z) - f(x) - f(z))\| \\ &\quad + \|\rho_2(f(y+z) - f(y) - f(z))\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned}$$

for all  $x, y, z \in X$ . Then there exists a unique additive mapping  $A : X \rightarrow Y$  such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^r - 2} \|x\|^r$$

for all  $x \in X$ .

*Proof.* The proof follows from Theorem 2.2 by taking  $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$  for all  $x, y, z \in X$ . Choosing  $L = 2^{1-r}$ , we obtain the desired result.  $\square$

**Theorem 2.4.** *Let  $\varphi : X^3 \rightarrow [0, \infty)$  be a function such that there exists an  $L < 1$*

with

$$\varphi(x, y, z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all  $x, y, z \in X$ . Let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and (2.3). Then there exists a unique additive mapping  $A : X \rightarrow Y$  such that

$$\|f(x) - A(x)\| \leq \frac{1}{2(1-L)}\varphi(x, x, 0)$$

for all  $x \in X$ .

*Proof.* Let  $(S, d)$  be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping  $J : S \rightarrow S$  such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all  $x \in X$ .

It follows from (2.4) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{1}{2}\varphi(x, x, 0)$$

for all  $x \in X$ .

The rest of the proof is similar to the proof of Theorem 2.2.  $\square$

**Corollary 2.5.** Let  $r < 1$  and  $\theta$  be positive real numbers, and let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and (2.6). Then there exists a unique additive mapping  $A : X \rightarrow Y$  such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2-2^r}\|x\|^r$$

for all  $x \in X$ .

*Proof.* The proof follows from Theorem 2.4 by taking  $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$  for all  $x, y, z \in X$ . Choosing  $L = 2^{r-1}$ , we obtain the desired result.  $\square$

**Remark 2.6.** If  $\rho$  is a real number such that  $|\rho_1| + |\rho_2| < 2$  and  $Y$  is a real Banach space, then all the assertions in this section remain valid.

### 3. ADDITIVE $(\rho_1, \rho_2)$ -FUNCTIONAL INEQUALITY (0.1): A DIRECT METHOD

In this section, we prove the Hyers-Ulam stability of the additive  $(\rho_1, \rho_2)$ -functional inequality (2.1) in complex Banach spaces by using the direct method.

**Theorem 3.1.** *Let  $\varphi : X^3 \rightarrow [0, \infty)$  be a function such that*

$$(3.1) \quad \Psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty$$

for all  $x, y, z \in X$ . Let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and (2.3). Then there exists a unique additive mapping  $A : X \rightarrow Y$  such that

$$(3.2) \quad \|f(x) - A(x)\| \leq \frac{1}{2} \Psi(x, x, 0)$$

for all  $x \in X$ .

*Proof.* Letting  $y = x$  and  $z = 0$  in (2.3), we get

$$(3.3) \quad \|f(2x) - 2f(x)\| \leq \varphi(x, x, 0)$$

and so

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)$$

for all  $x \in X$ . Thus

$$(3.4) \quad \begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\leq \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right) \end{aligned}$$

for all nonnegative integers  $m$  and  $l$  with  $m > l$  and all  $x \in X$ . It follows from (3.4) that the sequence  $\{2^k f(\frac{x}{2^k})\}$  is Cauchy for all  $x \in X$ . Since  $Y$  is a Banach space, the sequence  $\{2^k f(\frac{x}{2^k})\}$  converges. So one can define the mapping  $A : X \rightarrow Y$  by

$$A(x) := \lim_{k \rightarrow \infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all  $x \in X$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (3.4), we get (3.2).

It follows from (2.3) and (3.1) that

$$\begin{aligned} &\|A(x + y + z) - A(x) - A(y) - A(z)\| \\ &= \lim_{n \rightarrow \infty} 2^n \left\| f\left(\frac{x + y + z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right) \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \lim_{n \rightarrow \infty} 2^n |\rho_1| \left\| f\left(\frac{x+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{z}{2^n}\right) \right\| \\
&\quad + \lim_{n \rightarrow \infty} 2^n |\rho_2| \left\| f\left(\frac{y+z}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right) \right\| + \lim_{n \rightarrow \infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&= \|\rho_1(A(x+z) - A(x) - A(z))\| + \|\rho_2(A(y+z) - A(y) - A(z))\|
\end{aligned}$$

for all  $x, y, z \in X$ . So

$$\begin{aligned}
\|A(x+y+z) - A(x) - A(y) - A(z)\| &\leq \|\rho_1(A(x+z) - A(x) - A(z))\| \\
&\quad + \|\rho_2(A(y+z) - A(y) - A(z))\|
\end{aligned}$$

for all  $x, y, z \in X$ . By Lemma 2.1, the mapping  $A : X \rightarrow Y$  is additive.

Now, let  $T : X \rightarrow Y$  be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\| 2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right) \right\| \\
&\leq \left\| 2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right) \right\| + \left\| 2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right) \right\| \\
&\leq 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}, 0\right),
\end{aligned}$$

which tends to zero as  $q \rightarrow \infty$  for all  $x \in X$ . So we can conclude that  $A(x) = T(x)$  for all  $x \in X$ . This proves the uniqueness of  $A$ .  $\square$

**Corollary 3.2.** *Let  $r > 1$  and  $\theta$  be nonnegative real numbers, and let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and (2.6). Then there exists a unique additive mapping  $A : X \rightarrow Y$  such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^r - 2} \|x\|^r$$

for all  $x \in X$ .

**Theorem 3.3.** *Let  $\varphi : X^3 \rightarrow [0, \infty)$  be a function and let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$ , (2.3) and*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty$$

for all  $x, y, z \in X$ . Then there exists a unique additive mapping  $A : X \rightarrow Y$  such that

$$(3.5) \quad \|f(x) - A(x)\| \leq \frac{1}{2} \Psi(x, x, 0)$$

for all  $x \in X$ .



*Proof.* It follows from (3.3) that

$$\left\| f(x) - \frac{1}{2}f(2x) \right\| \leq \frac{1}{2}\varphi(x, x, 0)$$

for all  $x \in X$ . Hence

$$\begin{aligned} \left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1} x) \right\| \\ (3.6) \qquad \qquad \qquad &\leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}}\varphi(2^j x, 2^j x, 0) \end{aligned}$$

for all nonnegative integers  $m$  and  $l$  with  $m > l$  and all  $x \in X$ . It follows from (3.6) that the sequence  $\{\frac{1}{2^n}f(2^n x)\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, the sequence  $\{\frac{1}{2^n}f(2^n x)\}$  converges. So one can define the mapping  $A : X \rightarrow Y$  by

$$A(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n}f(2^n x)$$

for all  $x \in X$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (3.6), we get (3.5).

The rest of the proof is similar to the proof of Theorem 3.1. □

**Corollary 3.4.** *Let  $r < 1$  and  $\theta$  be positive real numbers, and let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and (2.6). Then there exists a unique additive mapping  $A : X \rightarrow Y$  such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^r} \|x\|^r$$

for all  $x \in X$ .

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