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Fuzzy Filters of Hoops Based on Fuzzy Points

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Abstract: In this paper, we define the concepts of $(\in, \in)$ and $(\in, \in \vee q)$-fuzzy filters of hoops, discuss some properties, and find some equivalent definitions of them. We define a congruence relation on hoops by an $(\in, \in)$-fuzzy filter and show that the quotient structure of this relation is a hoop.

Keywords: sub-hoop; $(\in, \in)$-fuzzy sub-hoop; $(\in, \in \vee q)$-fuzzy sub-hoop; $(\in, \in)$-fuzzy filter; $(\in, \in \vee q)$-fuzzy filter

1. Introduction

The hoop, which was introduced by Bosbach in [1,2], is naturally-ordered commutative residuated integral monoids. Several properties of hoops are displayed in [3–14]. The idea of the quasi-coincidence of a fuzzy point with a fuzzy set, which was introduced in [15], has played a very important role in generating fuzzy subalgebras of BCK/BCI-algebras, called (α, β)-fuzzy subalgebras of BCK/BCI-algebras, introduced by Jun [16]. Moreover, $(\in, \in \vee q)$-fuzzy subalgebra is a useful generalization of a fuzzy subalgebra in BCK/BCI-algebras. Many researcher applied the fuzzy structures on logical algebras [17–22]. Now, in this paper, we want to introduce these notions and investigate the existing fuzzy subsystems on hoops. Borzooei and Aaly Kologani in [8] defined the concepts of filters, (positive) implicative and fantastic filters of the hoop, and discussed their properties. Then, they defined a congruence relation on the hoop by a filter and proved that the quotient structure of this relation is a hoop. Finally, they investigated under what conditions that quotient structure will be the Brouwerian semilattice, Heyting algebra, and the Wajsberg hoop.

The aim of the paper is to define the concepts of $(\in, \in)$-fuzzy filters and $(\in, \in \vee q)$-fuzzy filters of hoops, discuss some properties and find equivalent definitions of them. By using an $(\in, \in)$-fuzzy filter of hoops, we define a congruence relation on hoops, and we show that the quotient structure of this relation forms a hoop.

2. Preliminaries

By a *hoop*, we mean an algebraic structure $(H, \odot, \to, 1)$ in which $(H, \odot, 1)$ is a commutative, monoid and for any $x, y, z \in H$, the following assertions are valid.

(H1) $x \to x = 1$.
(H2) $x \odot (x \to y) = y \odot (y \to x)$.
(H3) $x \to (y \to z) = (x \odot y) \to z$(See [1,2]).

For any $x, y \in H$, we can define a relation $\leq$ on hoop H by $x \leq y$ if and only if $x \to y = 1$. It is easy to see that $(H, \leq)$ is a poset. Therefore, in any hoop H, if for any $x \in H$, there exists an element $0 \in H$ such that $0 \leq x$, then H is called a *bounded hoop*. Let $x^0 = 1$, $x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$. If H

is a bounded hoop, then we define a negation " $'$ " on H by $x' = x \to 0$, for all $x \in H$. By a *sub-hoop* of a hoop H, we mean a subset S of H that satisfies the condition:

$$(\forall x, y \in H)(x, y \in S \;\Rightarrow\; x \odot y \in S,\; x \to y \in S). \tag{1}$$

Note that every non-empty sub-hoop contains the element 1.

Proposition 1. [23] *Let $(H, \odot, \to, 1)$ be a hoop. Then, the following conditions hold, for all $x, y, z \in H$:*
(i) $(H, \leq)$ is a meet-semilattice such that $x \wedge y = x \odot (x \to y)$.
(ii) $x \odot y \leq z$ if and only if $x \leq y \to z$.
(iii) $x \odot y \leq x, y$ and $x^n \leq x$, for any $n \in \mathbb{N}$.
(iv) $x \leq y \to x$.
(v) $1 \to x = x$ and $x \to 1 = 1$.
(vi) $x \leq (x \to y) \to y$.
(vii) $(x \to y) \odot (y \to z) \leq (x \to z)$.
$(viii)$ $x \leq y$ implies $x \odot z \leq y \odot z$, $z \to x \leq z \to y$ and $y \to z \leq x \to z$.

Let F be a non-empty subset of a hoop H. Then, F is called a *filter* of H if, for any $x, y \in F$, $x \odot y \in F$ and, for any $y \in H$ and $x \in F$, if $x \leq y$, then $y \in F$ (see [23]).

Let X be a nonempty set, $x \in X$ and $t \in (0, 1]$. The *fuzzy point* with support x and value t is defined as:

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set λ in a set X, Pu and Liu [15] defined the symbol $x_t \alpha \lambda$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$. This means that, $x_t \in \lambda$ (resp. $x_t q \lambda$) if $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$). Then, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set λ. Moreover, $x_t \in\vee q\, \lambda$ (resp. $x_t \in\wedge q\, \lambda$) means that $x_t \in \lambda$ or $x_t q \lambda$ (resp. $x_t \in \lambda$ and $x_t q \lambda$).

From now one, we let H denote a hoop, unless otherwise specified.

3. (α, β)-Fuzzy Filters for $(\alpha, \beta) \in \{(\in, \in), (\in, \in\vee q)\}$

In this section, we define (α, β)-fuzzy filters of hoops for $(\alpha, \beta) \in \{(\in, \in), (\in, \in\vee q)\}$, and we investigate some of their properties. Furthermore, we define a congruence relation on hoops by these filters and prove that the corresponding quotients are a bounded hoop.

Definition 1. *Let $(\alpha, \beta) \in \{(\in, \in), (\in, \in\vee q)\}$. Let λ be a fuzzy set of H. Then, λ is called an (α, β)-fuzzy filter of H if the following assertions are valid.*

$$(\forall x \in H)(\forall t \in (0,1])(x_t \alpha \lambda \;\Rightarrow\; 1_t \beta \lambda), \tag{2}$$

$$(\forall x, y \in H)(\forall t, k \in (0,1])(x_t \alpha \lambda,\; (x \to y)_k \alpha \lambda \;\Rightarrow y_{\min\{t,k\}} \beta \lambda). \tag{3}$$

Example 1. *On the set $H = \{0, a, b, 1\}$, we define two operations $\odot$ and $\to$ on H by:*

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then, $(H, \odot, \rightarrow, 1)$ is a hoop. Define the fuzzy set λ in H by $\lambda(0) = \lambda(a) = 0.4$, $\lambda(b) = 0.6$, and $\lambda(1) = 0.8$. Then, λ is an $(\in, \in)$-fuzzy filter of H, and it is clear that λ is an $(\in, \in \vee q)$-fuzzy filter of H.

Theorem 1. *A fuzzy set λ in H is an $(\in, \in)$-fuzzy filter of H if and only if the following conditions hold:*

$$(\forall x \in H)(\lambda(1) \geq \lambda(x)), \tag{4}$$

$$(\forall x, y \in H)(\lambda(y) \geq \min\{\lambda(x), \lambda(x \rightarrow y)\}). \tag{5}$$

Proof. Let λ be an $(\in, \in)$-fuzzy filter of H and $x \in H$ such that $\lambda(x) = t$. Since λ is an $(\in, \in)$-fuzzy filter of H, by Definition 1, $\lambda(1) \geq t = \lambda(x)$. Therefore, $\lambda(1) \geq \lambda(x)$. Now, let $x, y \in H$. If $\lambda(x) = t$ and $\lambda(x \rightarrow y) = k$, then $x_t \in \lambda$ and $(x \rightarrow y)_k \in \lambda$. Since λ is an $(\in, \in)$-fuzzy filter of H, we have $y_{\min\{t,k\}} \in \lambda$, so:

$$\lambda(y) \geq \min\{t, k\} = \min\{\lambda(x), \lambda(x \rightarrow y)\}$$

Conversely, suppose $x \in H$ and $t \in (0, 1]$. If $x_t \in \lambda$, then $\lambda(x) \geq t$. Since $\lambda(1) \geq \lambda(x)$, we have $\lambda(1) \geq t$, and so, $1_t \in \lambda$. Furthermore, if $x_t \in \lambda$ and $(x \rightarrow y)_k \in \lambda$, then by assumption,

$$\lambda(y) \geq \min\{\lambda(x), \lambda(x \rightarrow y)\} \geq \min\{t, k\}$$

Hence, $y_{\min\{t,k\}} \in \lambda$. Therefore, λ is an $(\in, \in)$-fuzzy filter of H. □

Proposition 2. *If λ is an $(\in, \in)$-fuzzy filter of H, then the following statement holds.*

$$(\forall x, y \in H)(x \leq y \ \Rightarrow\ \lambda(x) \leq \lambda(y)). \tag{6}$$

Proof. Let $x, y \in H$ such that $x \leq y$. Therefore, it is clear that $x \rightarrow y = 1$. Since λ is an $(\in, \in)$-fuzzy filter of H, by Theorem 1, $\lambda(1) \geq \lambda(x)$ and $\lambda(y) \geq \min\{\lambda(x), \lambda(x \rightarrow y)\}$, for any $x, y \in H$. Then:

$$\lambda(y) \geq \min\{\lambda(x), \lambda(x \rightarrow y)\} = min\{\lambda(x), \lambda(1)\} = \lambda(x)$$

Hence, $\lambda(y) \geq \lambda(x)$. □

Theorem 2. *A fuzzy set λ in H is an $(\in, \in)$-fuzzy sub-hoop of H such that, for any $x \in H$, $\lambda(x) \leq \lambda(1)$. Then, for any $x, y, z \in H$, the following statements are equivalent:*
(i) λ is an $(\in, \in)$-fuzzy filter of H,
(ii) if $(x \rightarrow y)_t \in \lambda$ and $(y \rightarrow z)_k \in \lambda$, then $(x \rightarrow z)_{\min\{t,k\}} \in \lambda$,
(iii) if $(x \rightarrow y)_t \in \lambda$ and $(x \odot z)_k \in \lambda$, then $(y \odot z)_{\min\{t,k\}} \in \lambda$.

Proof. $(i) \Rightarrow (ii)$ Let λ be an $(\in, \in)$-fuzzy filter of H such that $(x \rightarrow y)_t \in \lambda$ and $(y \rightarrow z)_k \in \lambda$. Then, $\lambda(x \rightarrow y) \geq t$ and $\lambda(y \rightarrow z) \geq k$. By Proposition 1(vii), $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$. Then, by Proposition 2,

$$\lambda(x \rightarrow y) \leq \lambda((y \rightarrow z) \rightarrow (x \rightarrow z))$$

Since $(x \rightarrow y)_t \in \lambda$, we have $((y \rightarrow z) \rightarrow (x \rightarrow z))_t \in \lambda$. Thus, by (i),

$$\begin{aligned} \min\{t, k\} &\leq \min\{\lambda(x \rightarrow y), \lambda(y \rightarrow z)\} \\ &\leq \min\{\lambda((y \rightarrow z) \rightarrow (x \rightarrow z)), \lambda(y \rightarrow z)\} \\ &\leq \lambda(x \rightarrow z) \end{aligned}$$

Hence, $(x \rightarrow z)_{\min\{t,k\}} \in \lambda$.
$(ii) \Rightarrow (i)$ It is enough to let $x = 1$.
$(i) \Rightarrow (iii)$ Let λ be an $(\in, \in)$-fuzzy filter of H such that $(x \rightarrow y)_t \in \lambda$ and $(x \odot z)_k \in \lambda$. Then,

by Proposition 1(vi), $z \odot x \odot (x \to y) \leq z \odot y$. Thus, $x \to y \leq (z \odot x) \to (z \odot y)$. By Proposition 2, $\lambda(x \to y) \leq \lambda((z \odot x) \to (z \odot y))$, so:

$$\begin{aligned} \min\{t,k\} &\leq \min\{\lambda(x \to y), \lambda(z \odot x)\} \\ &\leq \min\{\lambda(z \odot x), \lambda((z \odot x) \to (z \odot y))\} \\ &\leq \lambda(z \odot y) \end{aligned}$$

Hence, $(y \odot z)_{\min\{t,k\}} \in \lambda$.
$(iii) \Rightarrow (i)$ It is enough to let $z = 1$. □

Theorem 3. *Let λ be an $(\in,\in)$-fuzzy filter of H, $x, y \in H$, and $t \in (0,1]$. Define:*

$$x \equiv_\lambda y \text{ if and only if } (x \to y)_t \in \lambda \text{ and } (y \to x)_t \in \lambda$$

Then, $\equiv_\lambda$ is a congruence relation on H.

Proof. It is clear that $\equiv_\lambda$ is reflexive and symmetric. Now, we prove that $\equiv_\lambda$ is transitive. For this, suppose $x \equiv_\lambda y$ and $y \equiv_\lambda z$. Then, there exists $t, m \in (0,1]$ such that $(x \to y)_t \in \lambda$ and $(y \to x)_t \in \lambda$, and also, $(y \to z)_m \in \lambda$ and $(z \to y)_m \in \lambda$. By Proposition 1(vii), $x \to y \leq (y \to z) \to (x \to z)$, and by Proposition 2, we have:

$$\begin{aligned} \min\{t,m\} &\leq \min\{\lambda(x \to y), \lambda(y \to z)\} \\ &\leq \min\{\lambda(y \to z), \lambda((y \to z) \to (x \to z))\} \\ &\leq \lambda(x \to z) \end{aligned}$$

Hence, $(x \to z)_{\min\{t,m\}} \in \lambda$. In a similar way, we get that:

$$\begin{aligned} \min\{t,m\} &\leq \min\{\lambda(z \to y), \lambda(y \to x)\} \\ &\leq \min\{\lambda(y \to x), \lambda((y \to x) \to (z \to x))\} \\ &\leq \lambda(z \to x) \end{aligned}$$

Hence, $(z \to x)_{\min\{t,m\}} \in \lambda$. Therefore, $x \equiv_\lambda z$. Suppose that $x \equiv_\lambda y$. We show that $x \odot z \equiv_\lambda y \odot z$, for any $x, y, z \in H$. Since $x \equiv_\lambda y$, for any $t \in (0,1]$, we have $(x \to y)_t \in \lambda$ and $(y \to x)_t \in \lambda$. Since $y \odot z \leq y \odot z$, $y \leq z \to (y \odot z)$. By Proposition 1(viii), $x \to y \leq x \to (z \to (y \odot z))$, and so, $x \to y \leq (x \odot z) \to (y \odot z)$. Since λ is an $(\in,\in)$-fuzzy filter of H, by Proposition 2, we have:

$$t \leq \lambda(x \to y) \leq \lambda((x \odot z) \to (y \odot z))$$

Hence, $((x \odot z) \to (y \odot z))_t \in \lambda$. In a similar way, since $x \odot z \leq x \odot z$, we get $x \leq z \to (x \odot z)$. By Proposition 1(viii), $y \to x \leq y \to (z \to (x \odot z))$, and so, $y \to x \leq (y \odot z) \to (y \odot z)$. Since λ is an $(\in,\in)$-fuzzy filter of H, by Proposition 2, we have:

$$t \leq \lambda(y \to x) \leq \lambda((y \odot z) \to (x \odot z))$$

Hence, $((y \odot z) \to (x \odot z))_t \in \lambda$. Therefore, $x \odot z \equiv_\lambda y \odot z$. Finally, suppose that $x \equiv_\lambda y$; we show that $x \to z \equiv_\lambda y \to z$, for any $x, y, z \in H$. Since $x \equiv_\lambda y$, for any $t \in (0,1]$, we have $(x \to y)_t \in \lambda$ and $(y \to x)_t \in \lambda$. By Proposition 1(vii) and Proposition 2,

$$t \leq \lambda(x \to y) \leq \lambda((y \to z) \to (x \to z))$$

and:

$$t \leq \lambda(y \to x) \leq \lambda((x \to z) \to (y \to z))$$

Hence, $x \to z \equiv_\lambda y \to z$. It is easy to see that $z \to x \equiv_\lambda z \to y$. Therefore, $\equiv_\lambda$ is a congruence relation on H. □

Theorem 4. *Let* $\frac{H}{\equiv_\lambda} = \{[a]_\lambda \mid a \in H\}$, *and operations* $\otimes$ *and* $\rightsquigarrow$ *on* $\frac{H}{\equiv_\lambda}$ *are defined as follows:*

$$[a]_\lambda \otimes [b]_\lambda = [a \odot b]_\lambda \text{ and } [a]_\lambda \rightsquigarrow [b]_\lambda = [a \to b]_\lambda$$

Then, $(\frac{H}{\equiv_\lambda}, \otimes, \rightsquigarrow, [1]_\lambda)$ *is a hoop.*

Proof. We have $[a]_\lambda = [b]_\lambda$ and $[c]_\lambda = [d]_\lambda$ if and only if $a \equiv_\lambda b$ and $c \equiv_\lambda d$. Since $\equiv_\lambda$ is the congruence relation on H, then all above operations are well-defined. Thus, by routine calculation, we can see that $\frac{H}{\equiv_\lambda}$ is a hoop. □

Now, we define a relation on $\frac{H}{\equiv_\lambda}$ by:

$$[a]_\lambda \leq [b]_\lambda \text{ if and only if } (a \to b)_t \in \lambda, \text{ for any } a, b \in H \text{ and } t \in (0,1]$$

It is easy to see that $(\frac{H}{\equiv_\lambda}, \leq)$ is a poset.

Note: According to the definition of the congruence relation, it is clear that:

$$[1]_\lambda = \{a \in H \mid (a \to 1)_t \in \lambda \text{ and } (1 \to a)_t \in \lambda\} = \{a \in H \mid a_t \in \lambda\}.$$

Therefore, as we define a relation on quotient, $[a]_\lambda \leq [b]_\lambda$ if and only if $(a \to b)_t \in \lambda$, it is similar to writing $[a]_\lambda \leq [b]_\lambda$ if and only if $[a]_\lambda \to [b]_\lambda \in [1]_\lambda$.

Theorem 5. *If* λ *is a non-zero* $(\in, \in)$*-fuzzy filter of* H, *then the set:*

$$H_0 := \{x \in H \mid \lambda(x) \neq 0\} \tag{7}$$

is a filter of H.

Proof. Let $x \in H_0$. Since $\lambda(x) \neq 0$, we conclude that there exists $t \in (0,1]$ such that $\lambda(x) \geq t$. Moreover, from λ being an $(\in, \in)$-fuzzy filter of H, by Definition 1, $x_t \in \lambda$, then $1_t \in \lambda$. Hence, $\lambda(1) \geq \lambda(x) = t \neq 0$, and so, $1 \in H_0$. Now, suppose that $x, x \to y \in H_0$. Then, there exist $t, k \in (0,1]$, such that $\lambda(x) \geq t$ and $\lambda(x \to y) \geq k$, and so, $x_t \in \lambda$ and $(x \to y)_k \in \lambda$. Thus, by Definition 1, $y_{\min\{t,k\}} \in \lambda$, and so, $\lambda(y) \geq \min\{t,k\} \neq 0$. Hence, $y \in H_0$. Therefore, H_0 is a filter of H. □

Proposition 3. *If* λ *is a non-zero* $(\in, \in \vee q)$*-fuzzy filter of* H, *then* $\lambda(1) > 0$.

Proof. Let $\lambda(1) = 0$. Since λ is an $(\in, \in \vee q)$-fuzzy filter of H, by Theorem 1, for any $x \in H$, $\lambda(x) \leq \lambda(1) = 0$. Hence, for any $x \in H$, $\lambda(x) = 0$, and so, λ is a zero $(\in, \in \vee q)$-fuzzy filter of H, which is a contradiction. Therefore, $\lambda(1) > 0$. □

Theorem 6. *For any filter* F *of* H *and* $t \in (0, 0.5]$, *there exists an* $(\in, \in \vee q)$*-fuzzy filter* λ *of* H *such that its* $\in$*-level set is equal to* F.

Proof. Let $t \in (0, 0.5]$ and $\lambda : H \to [0,1]$ be defined by $\lambda(x) = t$, for any $x \in F$, and $\lambda(x) = 0$, otherwise. By this definition, it is clear that $U(\lambda; t) = F$. Therefore, it is enough to prove that λ is an $(\in, \in \vee q)$-fuzzy filter of H. Let $x \in H$. Then, $\lambda(x) = 0$ or $\lambda(x) = t$. Since F is a filter of H and $1 \in F$, we have $t = \lambda(1) \geq \lambda(x)$, for any $x \in H$. Now, suppose that $x_t \in \lambda$ and $(x \to y)_k \in \lambda$. We consider

the following cases:
Case 1: If $\lambda(x) = t$ and $\lambda(x \to y) = t$, then $x, x \to y \in F$. Since F is a filter of H, we have $y \in F$, and so, $\lambda(y) \geq \min\{\lambda(x), \lambda(x \to y)\} = t$. Hence, $y_t \in \lambda$. Therefore, λ is an $(\in, \in \vee q)$-fuzzy filter of H.
Case 2: If $\lambda(x) = t$ and $\lambda(x \to y) = 0$, then it is clear that:

$$\lambda(y) \geq \min\{\lambda(x), \lambda(x \to y)\} = 0$$

Hence, $y_0 \in \lambda$. Therefore, λ is an $(\in, \in \vee q)$-fuzzy filter of H.
Case 3: If $\lambda(x) = 0$ and $\lambda(x \to y) = 0$, then it is clear that:

$$\lambda(y) \geq \min\{\lambda(x), \lambda(x \to y)\} = 0$$

Hence, $y_0 \in \lambda$. Therefore, λ is an $(\in, \in \vee q)$-fuzzy filter of H.
Therefore, in all cases, λ is an $(\in, \in \vee q)$-fuzzy filter of H and $U(\lambda; t) = F$. □

For any fuzzy set λ in H and $t \in (0, 1]$, we define three sets that are called the $\in$-*level set*, q-*set*, and $\in \vee q$-*set*, respectively, as follows.

$$U(\lambda; t) := \{x \in H \mid \lambda(x) \geq t\}.$$
$$\lambda_q^t := \{x \in H \mid x_t \, q \, \lambda\}.$$
$$\lambda_{\in \vee q}^t := \{x \in H \mid x_t \in \vee q \lambda\}.$$

Theorem 7. *Given a fuzzy set λ in H, the following statements are equivalent.*

(i) *The nonempty $\in$-level set $U(\lambda; t)$ of λ is a filter of H, for all $t \in (0.5, 1]$.*
(ii) *λ satisfies the following assertions.*

$$(\forall x \in H)(\lambda(x) \leq \max\{\lambda(1), 0.5\}). \tag{8}$$
$$(\forall x, y \in H)(\max\{\lambda(y), 0.5\} \geq \min\{\lambda(x \to y), \lambda(x)\}). \tag{9}$$

Proof. Let $x \in H$ and $t \in (0.5, 1]$ such that $\lambda(x) = t$. Then, $x \in U(\lambda; t)$. Since $U(\lambda; t)$ is a filter of H, $1 \in U(\lambda; t)$. Thus, $\lambda(1) \geq t$. Moreover, since $t \in (0.5, 1]$, we have $\max\{\lambda(1), 0.5\} \geq \lambda(1) \geq t = \lambda(x)$. Hence, $\max\{\lambda(1), 0.5\} \geq \lambda(x)$. Now, suppose $x, y \in H$ and $t, k \in (0.5, 1]$ such that $\lambda(x) = t$ and $\lambda(x \to y) = k$. Then, $x, x \to y \in U(\lambda; \min\{t, k\})$. Since $U(\lambda; \min\{t, k\})$ is a filter of H, we have $y \in U(\lambda; \min\{t, k\})$. Thus, $\lambda(y) \geq \min\{t, k\}$. From $t, k \in (0.5, 1]$, we conclude that,

$$\max\{\lambda(y), 0.5\} \geq \min\{t, k\} = \min\{\lambda(x), \lambda(x \to y)\}$$

Hence,

$$\max\{\lambda(y), 0.5\} \geq \min\{\lambda(x), \lambda(x \to y)\}$$

Conversely, let $x \in U(\lambda; t)$. Then, $\lambda(x) \geq t$. Since $t \in (0.5, 1]$, by assumption:

$$t \leq \lambda(x) \leq \max\{\lambda(1), 0.5\} = \lambda(1)$$

Thus, $\lambda(1) \geq t$, so $1 \in U(\lambda; t)$. Now, suppose that $x, x \to y \in U(\lambda; t)$, for any $x, y \in H$ and $t \in (0.5, 1]$. Then, $\lambda(x) \geq t$ and $\lambda(x \to y) \geq t$. By assumption,

$$\max\{\lambda(y), 0.5\} \geq \min\{\lambda(x \to y), \lambda(x)\} \geq t$$

Since $t \in (0.5, 1]$, we have $\lambda(y) \geq t$, so $y \in U(\lambda; t)$. Hence, $U(\lambda; t)$ is a filter of H. □

It is clear that every $(\in, \in)$-fuzzy filter of H is an $(\in, \in \vee q)$-fuzzy filter of H. However, the converse may not be true, in general.

Example 2. *On the set $H = \{0, a, b, c, d, 1\}$, we define two operations $\odot$ and $\to$ as follows:*

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

By routine calculations, it is clear that $(H, \odot, \to, 0, 1)$ is a bounded hoop. Define a fuzzy set λ in H as follows:

$$\lambda : H \to [0,1],\ x \mapsto \begin{cases} 0.1 \text{ if } x = 0, \\ 0.1 \text{ if } x = a, \\ 0.2 \text{ if } x = b, \\ 0.3 \text{ if } x = c, \\ 0.1 \text{ if } x = d, \\ 0.5 \text{ if } x = 1 \end{cases}$$

It is easy to see that λ is an $(\in, \in \vee q)$-fuzzy filter of hoop H, but it is not an $(\in, \in \vee q)$-fuzzy filter of H; because:

$$0.2 = \lambda(b) \not\geq \min\{\lambda(c), \lambda(c \to b)\} = \min\{\lambda(c), \lambda(1)\} = \min\{0.3, 0.5\} = 0.3$$

However, it is not an $(\in, \in)$-fuzzy filter of H.

Now, we investigate under which conditions any $(\in, \in \vee q)$-fuzzy filter is an $(\in, \in)$-fuzzy filter.

Theorem 8. *If an $(\in, \in \vee q)$-fuzzy filter λ of H satisfies the condition:*

$$(\forall x \in H)(\lambda(x) < 0.5), \tag{10}$$

then λ is an $(\in, \in)$-fuzzy filter of H.

Proof. Let $x_t \in \lambda$, for any $x \in H$ and $t \in (0, 0.5)$. Since λ is an $(\in, \in \vee q)$-fuzzy filter of H, by Definition 1, $1_t \in \lambda$ or $1_t q\lambda$. If $1_t \in \lambda$, then the proof is clear. If $1_t q\lambda$, then $\lambda(1) + t > 1$. Since $t \in (0, 0.5)$, $1 - t \in [0.5, 1]$, then $\lambda(1) > t$. Hence, $1_t \in \lambda$. Now, suppose that $x_t \in \lambda$ and $(x \to y)_k \in \lambda$. From λ is an $(\in, \in \vee q)$-fuzzy filter of H, by Definition 1, $y_{\min\{t,k\}} \in \lambda$ or $y_{\min\{t,k\}} q\lambda$. If $y_{\min\{t,k\}} \in \lambda$, then the proof is complete. However, if $y_{\min\{t,k\}} q\lambda$, then $\lambda(y) + \min\{t, k\} > 1$, and so, $\lambda(y) > 1 - \min\{t, k\}$. Since $t \in (0, 0.5)$, we have $1 - \min\{t, k\} \in [0.5, 1]$, and so, $\lambda(y) > \min\{t, k\}$. Then, $y_{\min\{t,k\}} \in \lambda$. Therefore, λ is an $(\in, \in)$-fuzzy filter of H. □

Theorem 9. *If λ is an $(\in, \in \vee q)$-fuzzy filter of H, then the q-set λ_q^t is a filter of H, for all $t \in (0.5, 1]$.*

Proof. Let $x \in {\lambda_q}^t$, for any $x \in H$ and $t \in (0.5, 1]$. Then, $\lambda(x) + t > 1$, and so, $\lambda(x) > 1 - t$. Since λ is an $(\in, \in \vee q)$-fuzzy filter of H, by Definition 1, we have $1_{1-t} \in \lambda$ or $1_{1-t} q\lambda$. If $1_{1-t} q\lambda$, then it is clear that $\lambda(1) > t$. Since $t \in (0.5, 1]$, we have $\lambda(1) + t > 2t > 1$, and so, $1 \in {\lambda_q}^t$. If $1_{1-t} \in \lambda$, then $\lambda(1) \geq 1 - t$, and so, $\lambda(1) + t > 1$. Thus, in both cases, $1 \in {\lambda_q}^t$. Now, suppose that $x, x \to y \in {\lambda_q}^t$, for any $x, y \in H$ and $t \in (0.5, 1]$. Then, $\lambda(x) + t > 1$ and $\lambda(x \to y) + t > 1$, and so, $\lambda(x) > 1 - t$ and $\lambda(x \to y) > 1 - t$. Since λ is an $(\in, \in \vee q)$-fuzzy filter of H, by Definition 1, we have $y_{\min\{1-t, 1-t\}} \in \lambda$ or $y_{\min\{1-t, 1-t\}} q\lambda$. If $y_{1-t} \in \lambda$, then $\lambda(y) > 1 - t$, and so, $\lambda(y) + t > 1$. If $y_{1-t} q\lambda$, then $\lambda(y) + 1 - t > 1$, and so, $\lambda(y) > t$. Since $t \in (0.5, 1]$, we have $\lambda(y) + t > 2t > 1$. Hence, in both cases, $y \in {\lambda_q}^t$. Therefore, ${\lambda_q}^t$ is a filter of H, for any $t \in (0.5, 1]$. □

Theorem 10. *A fuzzy set λ in H is an $(\in, \in\vee q)$-fuzzy filter of H if and only if the following assertion is valid.*

$$(\forall x, y \in H)\left(\begin{array}{l}\lambda(1) \geq \min\{\lambda(x), 0.5\} \\ \lambda(y) \geq \min\{\lambda(x), \lambda(x \to y), 0.5\}\end{array}\right). \tag{11}$$

Proof. Let $x_t \in \lambda$, for any $x \in H$ and $t \in (0,1]$. Since λ is an $(\in, \in\vee q)$-fuzzy filter of H, we have $1_t \in \lambda$ or $1_t q\lambda$. It means that $\lambda(1) \geq t$ or $\lambda(1) > 1-t$. Therefore, $\lambda(1) \geq \min\{\lambda(x), 0.5\}$. In a similar way, if $x_t \in \lambda$ and $(x \to y)_k \in \lambda$, for any $x, y \in H$ and $t, k \in (0,1]$, since λ is an $(\in, \in\vee q)$-fuzzy filter of H, then we have $y_{\min\{t,k\}} \in \lambda$ or $y_{\min\{t,k\}} q\lambda$. This means that $\lambda(y) \geq \min\{t,k\}$ or $\lambda(1) > \min\{1-t, 1-k\}$. Therefore, $\lambda(y) \geq \min\{\lambda(x), \lambda(x \to y), 0.5\}$. Conversely, let $x_t \in \lambda$, for any $t \in (0,1]$ and $x \in H$. Then, by assumption, we have $\lambda(1) \geq \min\{\lambda(x), 0.5\}$. If $t \in (0, 0.5]$, then $\lambda(1) \geq \lambda(x) = t$, and so, $1_t \in \lambda$. If $t \in (0.5, 1]$, then $\lambda(1) \geq 0.5$, and so, $\lambda(1) + t > t + 0.5 > 1$; thus, $1_t q\lambda$. Hence, $1_t \in\vee q\lambda$. Now, suppose that $x_t \in \lambda$ and $(x \to y)_k \in \lambda$, for any $x, y \in H$ and $t, k \in (0,1]$. Then, by assumption, we have $\lambda(y) \geq \min\{\lambda(x), \lambda(x \to y), 0.5\}$. If $t, k \in (0, 0.5]$, then:

$$\lambda(y) \geq \min\{\lambda(x), \lambda(x \to y), 0.5\} = \min\{t, k\}$$

Hence, $y_{\min\{t,k\}} \in \lambda$. If $t, k \in (0.5, 1]$, then:

$$\lambda(y) \geq \min\{\lambda(x), \lambda(x \to y), 0.5\} = 0.5$$

Therefore, $\lambda(y) + \min\{t,k\} > \min\{t,k\} + 0.5 > 1$. Thus, $y_{\min\{t,k\}} q\lambda$. Hence, $y_{\min\{t,k\}} \in\vee q\lambda$. Therefore, λ is an $(\in, \in\vee q)$-fuzzy filter of H. □

Theorem 11. *A fuzzy set λ in H is an $(\in, \in\vee q)$-fuzzy filter of H if and only if the non-empty $\in$-level set $U(\lambda; t)$ of λ is a filter of H, for all $t \in (0, 0.5]$.*

Proof. Let λ be an $(\in, \in\vee q)$-fuzzy filter of H and $x \in U(\lambda; t)$, for any $t \in (0, 0.5]$. Then, $\lambda(x) \geq t$, and so, $x_t \in \lambda$. Since λ is an $(\in, \in\vee q)$-fuzzy filter of H, $1_t \in \lambda$ or $1_t q\lambda$. If $1_t \in \lambda$, then it is clear that $1 \in U(\lambda; t)$, and if $1_t q\lambda$, then $\lambda(1) > 1-t$. Since $t \in (0, 0.5]$, $1-t \in (0.5, 1]$, so $\lambda(1) > 1-t > t$. Thus, $1 \in U(\lambda; t)$. Now, suppose that $x, x \to y \in U(\lambda; t)$, then $\lambda(x) \geq t$ and $\lambda(x \to y) \geq t$, and so, $x_t \in \lambda$ and $(x \to y)_t \in \lambda$. Since λ is an $(\in, \in\vee q)$-fuzzy filter of H, $y_t \in \lambda$ or $y_t q\lambda$. If $y_t \in \lambda$, then it is clear that $y \in U(\lambda; t)$, and if $y_t q\lambda$, then $\lambda(y) > 1-t$. Since $t \in (0, 0.5]$, $1-t \in (0.5, 1]$, so $\lambda(y) > 1-t > t$. Thus, $y \in U(\lambda; t)$. Therefore, $U(\lambda; t)$ is a filter of H, for any $t \in (0, 0.5]$.

Conversely, suppose $U(\lambda; t)$ is a filter of H and $x_t \in \lambda$, for any $x \in H$ and $t \in (0, 0.5]$. Then, $\lambda(x) \geq t$, so $x \in U(\lambda; t)$. Since $U(\lambda; t)$ is a filter of H, $1 \in U(\lambda; t)$, so $\lambda(1) \geq t$. Hence, $1_t \in \lambda$, and so, $1_t \in\vee q\lambda$. Now, let $x_t \in \lambda$ and $(x \to y)_k \in \lambda$, for any $x, y \in H$ and $t, k \in (0, 0.5]$. Then, $\lambda(x) \geq t$ and $\lambda(x \to y) \geq k$, and so, $x, x \to y \in U(\lambda; \min\{t,k\})$. Since $U(\lambda; t)$ is a filter of H, $y \in U(\lambda; \min\{t,k\})$, and so, $\lambda(y) \geq \min\{t,k\}$. Hence, $y_{\min\{t,k\}} \in \lambda$. Therefore, $y_{\min\{t,k\}} \in\vee q\lambda$. Therefore, λ is an $(\in, \in\vee q)$-fuzzy filter of H, for any $t \in (0, 0.5]$. □

Theorem 12. *A fuzzy set λ in H is an $(\in, \in\vee q)$-fuzzy filter of H if and only if the following assertion is valid.*

$$(\forall x, y \in H)\left(\begin{array}{l}\lambda(x \odot y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \\ x \leq y \;\Rightarrow\; \lambda(y) \geq \min\{\lambda(x), 0.5\}\end{array}\right). \tag{12}$$

Proof. Assume that λ is an $(\in, \in\vee q)$-fuzzy filter of H and $x, y \in H$. Then, by Theorem 11, $U(\lambda; t)$ is a filter of H, for any $t \in (0, 0.5]$. If $x, y \in U(\lambda; t)$ and $t \in (0, 0.5]$, then $x \odot y \in U(\lambda; t)$. Thus, $\lambda(x \odot y) \geq t = \min\{\lambda(x), \lambda(y)\}$. If $t \in (0.5, 1]$, it is clear that $\lambda(x \odot y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$. Now, suppose $x \leq y$. If $\lambda(x) \geq t$ and $t \in (0, 0.5]$, then $x \in U(\lambda; t)$. Since $U(\lambda; t)$ is a filter of H, $y \in U(\lambda; t)$. Thus, $\lambda(y) \geq \lambda(x)$, for $t \in (0, 0.5]$. If $t \in (0.5, 1]$, then $\lambda(y) \geq \min\{\lambda(x), 0.5\}$.

Conversely, let λ be a fuzzy set in H that satisfies the condition (12). Since $x \leq 1$ for all $x \in H$, we have $\lambda(1) \geq \min\{\lambda(x), 0.5\}$ for all $x \in H$. Since $x \odot (x \to y) \leq y$ for all $x, y \in H$, we get:

$$\begin{aligned}\lambda(y) &\geq \min\{\lambda(x \odot (x \to y)), 0.5\}\\ &\geq \min\{\min\{\lambda(x), \lambda(x \to y), 0.5\}, 0.5\}\\ &= \min\{\lambda(x), \lambda(x \to y), 0.5\}\end{aligned}$$

for all $x, y \in H$. It follows from Theorem 10 that λ is an $(\in, \in \vee\, q)$-fuzzy filter of H. □

Theorem 13. *A fuzzy set λ in H is an $(\in, \in \vee\, q)$-fuzzy filter of H if and only if $\lambda^t_{\in\vee\, q}$ is a filter of H, for all $t \in (0, 1]$ (we call $\lambda^t_{\in\vee\, q}$ an $\in \vee\, q$-level filter of λ).*

Proof. Let λ be an $(\in, \in \vee\, q)$-fuzzy filter of H and $x \in \lambda^t_{\in\vee\, q}$, for any $x \in H$ and $t \in (0, 1]$. Then, $x \in U(\lambda; t)$ or $x \in \lambda^t_q$. This means that $x_t \in \lambda$ or $x_{1-t} \in \lambda$. Since λ is an $(\in, \in \vee\, q)$-fuzzy filter of H, we have, if $x_t \in \lambda$, then $1_t \in \lambda$ or $1_t q\lambda$. Furthermore, if $x_{1-t} \in \lambda$, then $1_{1-t} \in \lambda$ or $x_{1-t} q\lambda$; this means that $x_t q\lambda$ or $x_t \in \lambda$. Hence, in both cases, $1_t \in \vee\, q\lambda$, and so, $1 \in \lambda^t_{\in\vee\, q}$. In a similar way, let $x, x \to y \in \lambda^t_{\in\vee\, q}$, for $x, y \in H$ and $t \in (0, 1]$. Then, $x, x \to y \in U(\lambda; t)$ or $x, x \to y \in \lambda^t_q$ or $x \in U(\lambda; t)$ and $x \to y \in \lambda^t_q$. Therefore, we have the following cases:
<u>Case 1:</u> if $x, x \to y \in U(\lambda; t)$, then $x_t \in \lambda$ and $(x \to y)_t \in \lambda$. Since λ is an $(\in, \in \vee\, q)$-fuzzy filter of H, $y_t \in \lambda$ or $y_t q\lambda$. Therefore, $y \in \lambda^t_{\in\vee\, q}$.
<u>Case 2:</u> if $x, x \to y \in \lambda^t_q$, then $x_{1-t} \in \lambda$ and $(x \to y)_{1-t} \in \lambda$. Since λ is an $(\in, \in \vee\, q)$-fuzzy filter of H, $y_{1-t} \in \lambda$ or $y_{1-t} q\lambda$. It is equivalent to $y_t q\lambda$ or $y_t \in \lambda$, respectively. Therefore, $y \in \lambda^t_{\in\vee\, q}$.
<u>Case 3:</u> if $x \in U(\lambda; t)$ and $x \to y \in \lambda^t_q$, then $x_t \in \lambda$ and $(x \to y)_{1-t} \in \lambda$. Since λ is an $(\in, \in \vee\, q)$-fuzzy filter of H, $\lambda(y) \geq \min\{1 - t, t\}$, and so, it is equal to $y_t \in \lambda$ or $y_t q\lambda$. Thus, in both cases, $y \in \lambda^t_{\in\vee\, q}$.
Therefore, $\lambda^t_{\in\vee\, q}$ is a filter of H.

Conversely, let $x_t \in \lambda$, for any $x \in H$ and $t \in (0, 1]$. Since $\lambda^t_{\in\vee\, q}$ is a filter of H, $1 \in \lambda^t_{\in\vee\, q}$. Then, $1_t \in \vee\, q\lambda$. Now, suppose that $x_t \in \lambda$ and $(x \to y)_k \in \lambda$, for any $x, y \in H$ and $t, k \in (0, 1]$. Then, it is clear that $x, x \to y \in \lambda^{\min\{t,k\}}_{\in\vee\, q}$. Since $\lambda^t_{\in\vee\, q}$ is a filter of H, $y \in \lambda^{\min\{t,k\}}_{\in\vee\, q}$. Therefore, $y_{\min\{t,k\}} \in \lambda$ or $y_{\min\{t,k\}} q\lambda$. Hence, $y_{\min\{t,k\}} \in \vee\, q\lambda$. Therefore, λ is an $(\in, \in \vee\, q)$-fuzzy filter of H. □

Theorem 14. *Let $f : H \to K$ be a homomorphism of hoops. If λ and μ are $(\in, \in \vee\, q)$-fuzzy filters of H and K, respectively, then:*

(i) *$f^{-1}(\mu)$ is an $(\in, \in \vee\, q)$-fuzzy filter of H.*
(ii) *If f is onto and λ satisfies the condition:*

$$(\forall T \subseteq H)(\exists x_0 \in T)\left(\lambda(x_0) = \sup_{x \in T} \lambda(x)\right), \tag{13}$$

then $f(\lambda)$ is an $(\in, \in \vee\, q)$-fuzzy filter of K.

Proof. (i) Let $x_t \in f^{-1}(\mu)$, for any $x \in H$ and $t \in (0, 1]$. Then, $f(x)_t \in \mu$. Since μ is an $(\in, \in \vee\, q)$-fuzzy filter of H, we have $f(1)_t \in \vee\, q\mu$. Thus, $1_t \in \vee\, qf^{-1}(\mu)$. Now, suppose $x_t \in f^{-1}(\mu)$ and $(x \to y)_k \in f^{-1}(\mu)$, for any $x, y \in H$ and $t, k \in (0, 1]$. Then, $f(x)_t \in \mu$ and $f(x \to y)_k \in \mu$. Since μ is an $(\in, \in \vee\, q)$-fuzzy filter of H, we have $f(y)_{\min\{t,k\}} \in \vee\, q\mu$. Hence, $y_{\min\{t,k\}} \in \vee\, qf^{-1}(\mu)$. Therefore, $f^{-1}(\mu)$ is an $(\in, \in \vee\, q)$-fuzzy filter of H.
(ii) Let $a \in K$ and $t \in (0, 1]$ be such that $a_t \in f(\lambda)$. Then, $(f(\lambda))(a) \geq t$. By assumption, there exists $x \in f^{-1}(a)$ such that $\lambda(x) = sup_{z \in f^{-1}(a)} \lambda(z)$. Then, $x_t \in \lambda$. Since λ is an $(\in, \in \vee\, q)$-fuzzy filter of H, we have $1_t \in \lambda$. Now, $1 \in f^{-1}(1)$, so $(f(\lambda))(1) \geq \lambda(1)$, then $(f(\lambda))(1) \geq t$ or $(f(\lambda))(1) + t > 1$. Thus, $1_t \in \vee\, qf(\lambda)$. In a similar way, let $a, a \to b \in K$ and $t, k \in (0, 1]$ be such that $a_t \in f(\lambda)$ and $(a \to b)_k \in f(\lambda)$. Then, $(f(\lambda))(a) \geq t$ and $(f(\lambda))(a \to b) \geq k$. By assumption, there exist $x \in f^{-1}(a)$

and $x \to y \in f^{-1}(a \to b)$ such that $\lambda(x) = sup_{z \in f^{-1}(a)} \lambda(z)$ and $\lambda(x \to y) = sup_{w \in f^{-1}(a \to b)} \lambda(w)$. Then, $x_t \in \lambda$ and $(x \to y)_k \in \lambda$. Since λ is an $(\in, \in \vee q)$-fuzzy filter of H, we have $y_{\min\{t,k\}} \in \lambda$. Now, $y \in f^{-1}(y)$, so $(f(\lambda))(y) \geq \lambda(y)$, then $(f(\lambda))(y) \geq \min\{t,k\}$ or $(f(\lambda))(y) + \min\{t,k\} > 1$. Thus, $y_{\min\{t,k\}} \in \vee q f(\lambda)$. □

Theorem 15. *Let λ be an $(\in, \in \vee q)$-fuzzy filter of H such that:*

$$|\{\lambda(x) \mid \lambda(x) < 0.5\}| \geq 2.$$

Then, there exist two $(\in, \in \vee q)$-fuzzy filters μ and ν of H such that:

(i) $\lambda = \mu \cup \nu$.
(ii) $\mathrm{Im}(\mu)$ *and* $\mathrm{Im}(\nu)$ *have at least two elements.*
(iii) *μ and ν do not have the same family of $\in \vee q$-level filters.*

Proof. Let $\{\lambda(x) \mid \lambda(x) < 0.5\} = \{t_1, t_2, \ldots, t_r\}$ where $t_1 > t_2 > \cdots > t_r$ and $r \geq 2$. Then, the chain of $\in \vee q$-level filters of λ is:

$$\lambda^{0.5}_{\in\vee q} \subseteq \lambda^{t_1}_{\in\vee q} \subseteq \lambda^{t_2}_{\in\vee q} \subseteq \cdots \subseteq \lambda^{t_r}_{\in\vee q} = H.$$

Define two fuzzy sets μ and ν in H by:

$$\mu(x) = \begin{cases} t_1 \text{ if } x \in \lambda^{t_1}_{\in\vee q}, \\ t_n \text{ if } x \in \lambda^{t_n}_{\in\vee q} \setminus \lambda^{t_{n-1}}_{\in\vee q} \text{ for } n = 2, 3, \cdots, r, \end{cases}$$

and:

$$\nu(x) = \begin{cases} \lambda(x) \text{ if } x \in \lambda^{0.5}_{\in\vee q}, \\ k \text{ if } x \in \lambda^{t_2}_{\in\vee q} \setminus \lambda^{0.5}_{\in\vee q}, \\ t_n \text{ if } x \in \lambda^{t_n}_{\in\vee q} \setminus \lambda^{t_{n-1}}_{\in\vee q} \text{ for } n = 3, 4, \cdots, r, \end{cases}$$

respectively, where $k \in (t_3, t_2)$. Then, μ and ν are $(\in, \in \vee q)$-fuzzy filters of H, and $\mu \subseteq \lambda$ and $\nu \subseteq \lambda$. The chains of $\in \vee q$-level filters of μ and ν are given by:

$$\mu^{t_1}_{\in\vee q} \subseteq \mu^{t_2}_{\in\vee q} \subseteq \cdots \subseteq \mu^{t_r}_{\in\vee q} \text{ and } \nu^{0.5}_{\in\vee q} \subseteq \nu^{t_2}_{\in\vee q} \subseteq \cdots \subseteq \nu^{t_r}_{\in\vee q},$$

respectively. It is clear that $\mu \cup \nu = \lambda$. This completes the proof.
□

4. Conclusions

Our aim was to define the concepts of $(\in, \in)$-fuzzy filters and $(\in, \in \vee q)$-fuzzy filters of hoops, and we discussed some properties and found some equivalent definitions of them. Then, we defined a congruence relation on the hoop by an $(\in, \in)$-fuzzy filter of the hoop and proved that the quotient structure of this relation is a hoop. For future works, we will introduce (α, β)-fuzzy (positive) implicative filters for $(\alpha, \beta) \in \{(\in, \in), (\in, \in \vee q)\}$ of hoops, investigate some of their properties, and try to find some equivalent definitions of them. Furthermore, we study the relation between them. Moreover, we can investigate the corresponding quotients.

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