Generalization of Quantum Error Correction via the Heisenberg Picture

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We show that the theory of operator quantum error correction can be naturally generalized by allowing constraints not only on states but also on observables. The resulting theory describes the correction of algebras of observables (and may therefore suitably be called ''operator algebra quantum error correction''). In particular, the approach provides a framework for the correction of hybrid quantum-classical information and it does not require the state to be entirely in one of the corresponding subspaces or subsystems. We discuss applications to quantum teleportation and to the study of information flows in quantum interactions.

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Error correction methods are of crucial importance for quantum computing and the so far most general framework, called operator quantum error correction (OQEC) [1,2], encompasses active error correction [3–7] (QEC), together with the concepts of decoherence-free and noiseless subspaces and subsystems [8–14]. The OQEC approach has enabled more efficient correction procedures in active error correction [15–18], has led to improved threshold results in fault tolerant quantum computing [19], and has motivated the development of a structure theory for passive error correction [20,21] which has recently been used in quantum gravity [22–25].

In this Letter, we introduce a natural generalization of this theory. To this end, we change the focus from that of states to that of observables: conservation of a state by a given noise model implies the conservation of all of its observables, and is therefore a rather strong requirement. This can be alleviated by specifically selecting only some observables to be conserved. In this context it is natural to consider *algebras* of observables [26]. Hence our codes take the form of operator algebras that are closed under Hermitian conjugation; that is, finite-dimensional C^*-algebras [27]. As a convenience, we shall simply refer to such operator algebras as ''algebras''. Correspondingly, we refer to the new theory as ''operator algebra quantum error correction'' (OAQEC). We present results that establish testable conditions for correctability in OAQEC. We also discuss illustrative examples and consider applications to quantum teleportation and to information flow in quantum interactions. We shall present the proofs and more examples in [28].

Noise models in quantum information are described by *channels*, which are (in the Schrödinger picture) trace-preserving (TP) and completely positive (CP) linear maps $\mathcal{E}$ on mixed states, ρ, which are operators acting on a Hilbert space $\mathcal{H}$. If ρ is a density matrix we can always write $\rho \mapsto \mathcal{E}(\rho) = \sum_a E_a \rho E_a^\dagger$, where $\{E_a\}$ is a nonunique family of channel *elements*. The QEC framework addresses the question of whether a given subspace of states $P\mathcal{H}$, called the code, can be corrected in the sense that there exists a correction channel $\mathcal{R}$ such that $\mathcal{R}(\mathcal{E}(\rho)) = \rho$ for all states ρ in the subspace; that is, all ρ which satisfy $\rho = P\rho P$. This amounts to asking for a subspace on which $\mathcal{E}$ has a left inverse that is a physical map. From QEC to OQEC the scope of error correction is generalized by only requiring the states of a subsystem to be conserved: $\mathcal{R}(\mathcal{E}(\rho \otimes \tau)) = \rho \otimes \tau'$ for all $\rho \otimes \tau$ in the subspace. As we will show, this amounts to the correction of special types of algebras. In general, every algebra $\mathcal{A}$ of observables induces a decomposition of the Hilbert space $\mathcal{H}$ into $\mathcal{H} = \oplus_{k=1}^d (A_k \otimes B_k) \oplus C$. Here all operators in the algebra have C in their kernel and act irreducibly on each A_k, while acting trivially on the subsystem B_k. This means that the algebra decomposes as

$$\mathcal{A} = \oplus_{k=1}^d [\mathcal{L}(A_k) \otimes \mathbf{1}^{B_k}] \oplus 0_C, \quad (1)$$

where $\mathcal{L}(A_k)$ denotes the set of all operators on A_k, $\mathbf{1}^{B_k}$ is the identity operator on B_k, and 0_C is the zero operator on C. From this perspective, we can view the QEC framework as focusing on codes $\mathcal{L}(A) \otimes \mathbf{1}^B$ (or subspaces A) with $\dim B = 1$. Moreover, OQEC considers ''subsystem codes'' encoded in algebras of the form $\mathcal{L}(A) \otimes \mathbf{1}^B$ for general subsystems A and B. Classical information is captured by the case in which $\dim A = 1$: commutative algebras. Thus, in addition to the classical, QEC and OQEC cases, our new OAQEC approach also provides a framework for the correction of hybrid quantum-classical information and memory [29] exposed to external noise. In particular, this includes cases in which separate (orthogonal) parcels of quantum information are labeled by classical ''addresses''.

Let us discuss in more detail the motivation for considering algebras. We begin by recalling that a general observable is a positive operator-valued measure (POVM)

$X(\Delta)$, where $\Delta \subset \Omega$, the set in which the observable X takes values. For simplicity we consider observables with a finite number of outcomes which can be characterized by a family of positive operators $\{X_a\}$. In the Heisenberg picture an observable evolves according to the unital CP map $\mathcal{E}^\dagger$ with elements $E_a^\dagger$ instead of E_a. If for all values of the label a there exists an operator Y_a such that $X_a = \mathcal{E}^\dagger(Y_a)$ then all the statistical information about X has been conserved by $\mathcal{E}$ since for any initial state ρ we have, $\mathrm{Tr}(\rho X_a) = \mathrm{Tr}[\rho \mathcal{E}^\dagger(Y_a)] = \mathrm{Tr}[\mathcal{E}(\rho) Y_a]$, the latter equality following from the definition of $\mathcal{E}^\dagger$. In this case, to correct for the errors induced by $\mathcal{E}$ we need a channel $\mathcal{R}$ that maps each X_a to one of the operators Y_a through $\mathcal{R}^\dagger(X_a) = Y_a$, so that $(\mathcal{R} \circ \mathcal{E})^\dagger(X_a) = (\mathcal{E}^\dagger \circ \mathcal{R}^\dagger)(X_a) = X_a$. In such a scenario, we will say that X_a is *correctable* for $\mathcal{E}$ and *conserved* by $\mathcal{R} \circ \mathcal{E}$. In particular, if X is a standard projective measurement, and so $X_a^2 = X_a$ for all a, then the projectors X_a linearly span the algebra they generate. Hence, in this case $\mathcal{R} \circ \mathcal{E}$ conserves an entire commutative algebra. Therefore, focusing on the correctability of sets of observables which have the structure of an algebra, apart from allowing a complete characterization, is also sufficient for the study of all the correctable *projective* observables.

One result of this Letter will be to show that there always exists a single channel $\mathcal{R}$ correcting all projective observables correctable in the above sense. In fact, we study a more general question: If we have some control on the initial states, which is expected in a quantum computation, then we can ask for an observable to be conserved only if the state starts in a certain subspace $P\mathcal{H}$. That is, $P(\mathcal{E}^\dagger \circ \mathcal{R}^\dagger)(X)P = PXP$. We derive a necessary and sufficient condition for an entire algebra of operators on $P\mathcal{H}$ to be simultaneously correctable in that sense. The resulting theory contains OQEC and QEC in the special cases discussed above.

We remark that our approach differs from that of the stabilizer formalism [7] where observables in the Heisenberg picture are used as a way to characterize a subspace of states. Our approach is closer in spirit to that of [30]. The idea that observables naturally characterize subsystems has also been exploited in [31].

Noiseless subsystems.—First we recall the definition of a noiseless subsystem and we give an equivalent definition in terms of the dual channel $\mathcal{E}^\dagger$. Consider a decomposition of a finite-dimensional Hilbert space $\mathcal{H}$ as $\mathcal{H} = (A \otimes B) \oplus C$. We introduce the projector P onto the subspace $A \otimes B$. By definition, A is a noiseless subsystem for $\mathcal{E}$ if for all $\rho \in \mathcal{L}(A)$ and $\sigma \in \mathcal{L}(B)$ there exists $\tau \in \mathcal{L}(B)$ such that $\mathcal{E}(\rho \otimes \sigma) = \rho \otimes \tau$. In terms of the dual channel $\mathcal{E}^\dagger$, the subsystem A is noiseless for $\mathcal{E}$ if and only if

$$P\mathcal{E}^\dagger(X \otimes \mathbf{1})P = X \otimes \mathbf{1} \tag{2}$$

for all operators X acting on A. This is a consequence of the noiseless subsystem characterization from [1] as can be readily verified. In Eq. (2), the projectors P are needed since the definition of the noiseless subsystem is only concerned with what happens to states initially in the subspace $A \otimes B$. In general, an initial component outside this space may, after evolution, disturb the otherwise noiseless observables.

Conserved observables.—If $P\mathcal{E}^\dagger(X_a)P = PX_aP$ for all a we say that the observable X is *conserved* by $\mathcal{E}$ for states in $P\mathcal{H}$. More generally, let us say an algebra $\mathcal{A}$ is *conserved* by $\mathcal{E}$ for states in $P\mathcal{H}$ if every element of $\mathcal{A}$ is conserved; that is,

$$P\mathcal{E}^\dagger(X)P = PXP \quad \forall\ X \in \mathcal{A}. \tag{3}$$

Notice that Eq. (3) gives a generalization of noiseless subsystems. Indeed, any subalgebra $\mathcal{A}$ of $\mathcal{L}(P\mathcal{H})$ for which all elements $X \in \mathcal{A}$ satisfy Eq. (3) is a direct sum of noiseless subsystems. This can be seen by first noting that any algebra $\mathcal{A}$ has a decomposition of the form given in Eq. (1), and then applying Eq. (2). In particular, focusing on the so-called "simple" algebras $\mathcal{L}(A) \otimes \mathbf{1}^B$ captures standard noiseless subsystems as in Eq. (2).

The following theorem provides testable conditions that characterize when an algebra is conserved on states in a given subspace by a channel, strictly in terms of the operation elements for the channel. The result comes as an adaptation of results from [20]. It is a generalization because, here, the algebra need not contain the projector P.

Theorem 1.—A subalgebra $\mathcal{A}$ of $\mathcal{L}(P\mathcal{H})$ is conserved on states in $P\mathcal{H}$ by a channel $\mathcal{E}$ if and only if $[E_aP, X] = 0$ for all elements E_a and all $X \in \mathcal{A}$.

Heuristically, an algebra supported on a subspace is conserved by a channel precisely when elements of the algebra commute with the generators of the noise, restricted to the subspace.

Error correction of observables.—We say that an algebra $\mathcal{A}$ is *correctable* for $\mathcal{E}$ on states in the subspace $P\mathcal{H}$ if there exists a channel $\mathcal{R}$ such that

$$P(\mathcal{R} \circ \mathcal{E})^\dagger(X)P = PXP \quad \forall\ X \in \mathcal{A}. \tag{4}$$

This notion of correctability is more general than the one addressed by the framework of OQEC. Indeed, OQEC focuses on simple algebras, $\mathcal{L}(A) \otimes \mathbf{1}^B$. Here, correctability is defined for any finite-dimensional algebra. A further generalization is that we do not require P to belong to the algebra considered.

We now state the main result of the Letter, which generalizes the fundamental result for both QEC [4] and OQEC [1,17]. It provides conditions for testing whether an algebra is correctable for a given channel in terms of its operation elements.

Theorem 2.—A subalgebra $\mathcal{A}$ of $\mathcal{B}(P\mathcal{H})$ is correctable on $P\mathcal{H}$ for the channel $\mathcal{E}$ if and only if

$$[PE_c^\dagger E_bP, X] = 0 \quad \forall\ X \in \mathcal{A} \quad \forall\ c, b. \tag{5}$$

We present the complete proof in [28]. Note that the operators $\{E_c^\dagger E_b\}$ play a key role as in other error correction settings. The necessity of the condition follows directly from Theorem 1. The sufficiency is obtained by the explicit construction of the correction channel $\mathcal{R}$. An important property of the channel $\mathcal{R}$ constructed in the proof is that it corrects any channel whose elements are linear combinations of the elements E_a. Thus, as in the original theory of error correction, we can in practice neglect the channel $\mathcal{E}$ and focus instead directly on the discrete *error operators* E_a.

It is instructive to consider the special case of classical OAQEC codes for $P = \mathbf{1}$. A classical channel has elements $E_{ij} = \sqrt{p_{ij}}|i\rangle\langle j|$, where (p_{ij}) forms a stochastic matrix with transitional probabilities p_{ij} from j to i. Thus, $E_{ij}^\dagger E_{kl} = \delta_{ik}\sqrt{p_{ij}p_{il}}|j\rangle\langle l|$, and Theorem 2 shows that if $\alpha = (\alpha_j)$ are the diagonal components of a classical (diagonal) observable, then α can be corrected if and only if $\alpha_j = \alpha_k$ for all k, j such that there is an i with $p_{ij} \neq 0$ and $p_{ik} \neq 0$. Heuristically, two states cannot be distinguished from each other after the channel has acted precisely when there is a nonzero probability of a transition from both states to a common state.

Does OAQEC offer more powerful error correction procedures than OQEC? It is easy to see that if an algebra $\mathcal{A}$ is correctable according to this scheme then each simple sector $\mathcal{L}(A) \otimes \mathbf{1}^B$ is individually correctable through OQEC (or QEC when $\dim B = 1$). However, OAQEC codes have at least two attractive features: First, all simple sectors can be corrected simultaneously by the *same* correction channel. Second, each simple sector can be corrected even if the initial state is not entirely in the corresponding subspace. In particular, the initial state could be in a quantum superposition between various sectors even though combined sectors may not be correctable in the traditional sense.

As an illustrative example, consider a 2-qubit system exposed to noise inducing a phase flip error Z (a unitary Pauli operator) with probability p on the first qubit and probability $1 - p$ on the second qubit. The channel $\mathcal{E}$ has elements $\{\sqrt{p}Z_1, \sqrt{1-p}Z_2\}$. Let $\mathcal{C}_1$ be the (subspace) code with basis $|0_L\rangle = |00\rangle$, $|1_L\rangle = |11\rangle$ and $\mathcal{C}_2$ the code with basis $|0_L\rangle = |10\rangle$, $|1_L\rangle = |01\rangle$. Each of $\mathcal{C}_1$ and $\mathcal{C}_2$, or from the OAQEC perspective the algebras $\mathcal{L}(\mathcal{C}_1)$ and $\mathcal{L}(\mathcal{C}_2)$, is correctable individually for $\mathcal{E}$. Indeed, those two codes are stabilizer subspaces for Z_1Z_2 and $-Z_1Z_2$, respectively. However, the combined code $\mathcal{C}_1 \oplus \mathcal{C}_2$, or equivalently $\mathcal{L}(\mathcal{C}_1 \oplus \mathcal{C}_2) = \mathcal{L}(\mathbb{C}^4)$, is clearly not correctable for $\mathcal{E}$ because $\mathcal{E}$ is not a unitary operation. Nevertheless, the hybrid bit-qubit OAQEC code $\mathcal{A} = \mathcal{L}(\mathcal{C}_1) \oplus \mathcal{L}(\mathcal{C}_2)$ *is* correctable for $\mathcal{E}$. This follows from Theorem 2 and the observation that Z_1Z_2 and $\mathbf{1}$ generate the algebra $\mathcal{A}' = \mathbb{C}P_1 \oplus \mathbb{C}P_2$ where we have written P_i for the projector onto $\mathcal{C}_i$. Indeed $\mathcal{A}$ is the set of operators commuting with $\mathcal{A}'$ (its commutant). In this example one can check directly that $\mathcal{R} = \mathcal{E}^\dagger$ is a channel that corrects $\mathcal{A}$. As discussed above, separate (orthogonal) codes that determine the simple summands of an OAQEC code can all be corrected by the same correction operation. In addition, the algebra can be corrected without any restriction on the initial state of the two separate qubits.

Application: Noisy quantum teleportation.—In the standard picture for quantum teleportation [32], Alice can send Bob an entire qubit by sending two classical bits, provided they share a maximally entangled pair of qubits initially. Consider a pair of qubits in the maximally entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. We assume that Alice and Bob each possess one qubit of this pair. Consider also the unitary Pauli operators $\{U_0 = \mathbf{1}, U_1 = X, U_2 = Y, U_3 = Z\}$. In the language of channels, teleportation can be viewed as the channel $\mathcal{E}$ that has for input the qubit $|\psi\rangle$ to be teleported and for output three qubits: two which represent the bits Alice needs to send to Bob, and Bob's qubit: $\mathcal{E}(|\psi\rangle\langle\psi|) = \frac{1}{4}\sum_{i=0}^{3}|i\rangle\langle i| \otimes U_i(|\psi\rangle\langle\psi|)U_i^\dagger$, where $|i\rangle$ forms an orthonormal basis of the four-dimensional Hilbert space representing the 2 bits that Alice must send to Bob over a classical channel. We can readily verify that Bob can indeed fully correct the channel. The channel elements are of the form $E_i \propto |i\rangle \otimes U_i$, and hence the condition of Theorem 2 is met since $E_i^\dagger E_j \propto \langle i||j\rangle \otimes U_i^\dagger U_j = \delta_{ij}U_i^\dagger U_i = \delta_{ij}\mathbf{1}$. Thus all operators on the initial qubit can be corrected, and all the information can be recovered. Note that even though the operators E_i map between two different spaces, the operators $E_i^\dagger E_j$ map the initial qubit space to itself. One can teleport as many qubits as desired in parallel provided that one starts with one shared entangled pair per qubit to be teleported.

Consider now the case in which the classical step of the teleportation process, when Alice transmits bits to Bob, is implemented over a noisy classical channel. Let $\{U_g\}_{g\in S}$ be a family of unitary operators in $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. In the standard teleportation protocol, this is the family of single qubit Pauli operators. Now suppose that a noisy stochastic channel is applied on the classical bits $|g\rangle$, $g \in S$, that Alice must send to Bob with transition probabilities p_{gh}. Then one could ask, what information can be recovered by Bob once he is in possession of all the data? The overall channel is given by composing $\mathcal{E}$ with the classical channel. Thus, the error operators F_{ghl} satisfy $F_{ghl} \propto \sqrt{p_{gh}}|g\rangle\langle h||l\rangle \otimes U_l = \delta_{hl}\sqrt{p_{gh}}|g\rangle \otimes U_h$. It follows that the commutant of the correctable algebra, which is contained in $\mathcal{L}(\mathcal{H})$, has generators $F_{ghl}^\dagger F_{g'h'l'} = \delta_{gg'}\sqrt{p_{gh}p_{gh'}}U_h^\dagger U_{h'}$. We are only interested in the nonzero generators; that is, those for which $p_{gh} \neq 0$ and $p_{gh'} \neq 0$. This condition means that classically, because of the noise, we can no longer distinguish between the state $|h\rangle$ and the state $|h'\rangle$. Therefore, the commutant of the conserved algebra has a generator $U_g^\dagger U_h$ for each pair of classical states g, h that became indistinguishable under the noisy channel. In short, if Bob is not sure whether Alice's classical message was g or h, then he can only completely recover those properties of the quantum states that are invariant under the transformations $U_g^\dagger U_h$. In particular, this implies more general code algebras will be obtained.

Application: Information flow in interactions.—Consider the interaction U between a system S and an apparatus A where the initial state of the apparatus is known to be ρ_A. Tracing out either over A or S after the evolution yields, respectively, the channel $\mathcal{E}_{SS}(\rho_S) = \mathrm{Tr}_A[U(\rho_S \otimes \rho_A)U^\dagger]$ from S to S or $\mathcal{E}_{SA}(\rho_S) = \mathrm{Tr}_S[U(\rho_S \otimes \rho_A)U^\dagger]$ from S to A.

Using OAQEC we can determine what observables of the system can be corrected, i.e., recovered, for either of the two channels. The algebra $\mathcal{A}_{SA}$ preserved by $\mathcal{E}_{SA}$, that represents the information about S which is transferred to A, can be computed to be the largest algebra of operators commuting with the range of $\mathcal{E}_{SS}^\dagger$. Hence a direct consequence of Theorem 2 is that in an open dynamics defined by a channel $\mathcal{E}$, full information about a projective observable can escape the system if and only if it commutes with all the operators in the range of the channel; that is, those observables whose first moment is correctable for $\mathcal{E}$. This is a generalization of work in [33]. Furthermore, this method characterizes those observables which are effectively duplicated; in other words, those whose information stayed in S and also flowed to A. They form the algebra $\mathcal{A}_{SS} \cap \mathcal{A}_{SA}$, which is commutative. Those observables have been nondestructively measured by the system A. This analysis has implications for the theory of decoherence [34,35]: a unique commutative algebra of observables emerges naturally as characterizing the information which is shared between the system and the environment after an interaction. This suggests that the *pointer observables* should be defined not just by their property of being stably encoded in the system but also by the requirement that the information they represent is transmitted to the environment. In this sense there is no *basis ambiguity* [36] for the interpretation of a unitary interaction as a measurement of the system by the apparatus.

Outlook.—We have presented a generalization of the theory of operator quantum error correction that allows for the correction of an arbitrary algebra of operators. Our main result gives a characterization of correctable codes in this scheme. Proofs and more applications will be provided in [28]. The recent experience with operator quantum error correction suggests a reinvestigation of codes that have appeared in the literature for possibly improved efficiency or other applications enabled by this approach. We also suggest that the applications to quantum teleportation and information flow presented here warrant further investigation. Furthermore, we have here focused on algebras of operators rather than general operator subspaces. It should be most interesting to consider the possible conservation of the statistics of general POVMs.

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