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The iterative methods for solving nonlinear matrix equation $X + A^{\star}X^{-1}A + B^{\star}X^{-1}B = Q$

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Abstract

In this paper, we study the matrix equation $X + A^{\star}X^{-1}A + B^{\star}X^{-1}B = Q$, where A and B are square matrices, and Q is a positive definite matrix, and propose the iterative methods for finding positive definite solutions of the matrix equation. Also, general convergence results for the basic fixed point iteration for these equations are given. Some numerical examples are presented to show the usefulness of the iterations.

MSC: 65F10; 65F30; 65H10; 15A24

Keywords: nonlinear matrix equation; positive definite solution; inversion-free variant iterative method; convergence rate

1 Introduction

In this paper, we consider the matrix equation

$$X + A^{\star}X^{-1}A + B^{\star}X^{-1}B = Q, \tag{1}$$

where A and B are square matrices, Q is a positive definite matrix. It is easy to see that matrix equation (1) can be reduced to

$$X + A^{\star}X^{-1}A + B^{\star}X^{-1}B = I, \tag{2}$$

where I is the identity matrix. Trying to solve special linear systems [1] leads to solving nonlinear matrix equations of the above types as follows.

For a linear system $Mx = f$ with

$$M = \begin{pmatrix} Q & 0 & A \\ 0 & Q & B \\ A^{\star} & B^{\star} & Q \end{pmatrix}$$

positive definite, we rewrite $M = \tilde{M} + K$, where

$$\tilde{M} = \begin{pmatrix} X & 0 & A \\ 0 & X & B \\ A^{\star} & B^{\star} & Q \end{pmatrix}, \qquad K = \begin{pmatrix} Q-X & 0 & 0 \\ 0 & Q-X & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, we decompose $\tilde{M}$ to the LU decomposition

$$\tilde{M} = \begin{pmatrix} X & 0 & A \\ 0 & X & B \\ A^\star & B^\star & Q \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ A^\star X^{-1} & B^\star X^{-1} & I \end{pmatrix} \begin{pmatrix} X & 0 & A \\ 0 & X & B \\ 0 & 0 & X \end{pmatrix}.$$

A decomposition of $\tilde{M}$ exists if and only if X is a positive definite solution of matrix equation (1). Solving the linear system $\tilde{M}y = f$ is equivalent to solving two linear systems with a lower and upper block triangular system matrix. To compute the solution of $Mx = f$ from y, the Woodbury formula [2] can be applied.

The matrix equation $X + A^\star X^{-1} A = Q$ has been studied extensively by many authors [3–14]. Several conditions for the existence of positive definite solutions and some iterations to find maximal positive definite solutions for these equations were discussed. Apparently, matrix equation (1) generalizes the matrix equation $X + A^\star X^{-1} A = Q$.

Matrix equation (2) was studied in [15], and based on some conditions, the authors proved that matrix equation (2) has positive definite solutions. They also proposed two iterative methods to find the Hermitian positive definite solutions of matrix equation (2). They did not analyze the convergence rate of proposed algorithms.

In this paper, we propose two algorithms. We will show that Algorithm (7) is more accurate than Algorithm (3) pointed out in [15]. Also, Algorithm (10) needs less operation in comparison with Algorithm (3). The following notations are used throughout the rest of the paper. The notation $A \geq 0$ ($A > 0$) means that A is Hermitian positive semidefinite (positive definite). For Hermitian matrices A and B, we write $A \geq B$ ($A > B$) if $A - B \geq 0$ (> 0). Similarly, by $\lambda_1(A)$ and λ_n we denote, respectively, the maximal and the minimal eigenvalues of A. The norm used in this paper is the spectral norm of the matrix A, *i.e.*, $\|A\| = (\lambda_1(A^\star A))^{\frac{1}{2}}$.

2 Fixed point theorems

Lemma 1 [8] *If C and P are Hermitian matrices of the same order with $P > 0$, then $CPC + P^{-1} \geq 2C$.*

In [15] an algorithm that avoids matrix inversion for every iteration, called inversion-free variant of the basic fixed point iteration, and a theorem to find a Hermitian positive definite solution of matrix equation (2) were proposed as follows.

Algorithm 1 [15] Let

$$\begin{cases} X_0 = Y_0 = I, \\ X_{n+1} = I - A^\star Y_n A - B^\star Y_n B, \\ Y_{n+1} = 2Y_n - Y_n X_n Y_n, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{3}$$

Theorem 1 [15] *Assume that matrix equation* (2) *has a positive definite solution. Then Algorithm* (3) *defines a monotonically decreasing matrix sequence $\{X_n\}$ converging to the positive definite matrix X which is a solution of matrix equation* (2). *Also, the sequence $\{Y_n\}$ defined in Algorithm* (3) *defines a monotonically increasing sequence converging to X^{-1}.*

Although it is not mentioned in the previous theorem that the sequence $\{X_n\}$ converges to the maximal Hermitian positive definite solution of equation (2), during the proof of the theorem in [15], it is obvious. So, $X = X_\infty$, where X_∞ is the maximal positive definite solution of matrix equation (2) in Theorem 1.

The problem of convergence rate for Algorithm (3) was not considered in [15]. We now establish the following result.

Theorem 2 *If matrix equation* (2) *has a positive definite solution, for Algorithm* (1) *and any* $\epsilon > 0$, *we have*

$$\|Y_{n+1} - X_\infty^{-1}\| \leq \left(\|AX_\infty^{-1}\| + \|BX_\infty^{-1}\| + \epsilon\right)^2 \|Y_{n-1} - X_\infty^{-1}\| \tag{4}$$

and

$$\|X_{n+1} - X_\infty\| \leq \left(\|A\|^2 + \|B\|^2\right)\|Y_n - X_\infty^{-1}\| \tag{5}$$

for all n large enough.

Proof From Algorithm (3), we have

$$\begin{aligned} Y_{n+1} &= 2Y_n - Y_n\left(I - A^\star Y_{n-1}A - B^\star Y_{n-1}B\right)Y_n \\ &= 2Y_n - Y_n^2 + Y_nA^\star\left(X_\infty^{-1} + Y_{n-1} - X_\infty^{-1}\right)AY_n + Y_nB^\star\left(X_\infty^{-1} + Y_{n-1} - X_\infty^{-1}\right)BY_n \\ &= 2Y_n - Y_nX_\infty Y_n + Y_nA^\star\left(Y_{n-1} - X_\infty^{-1}\right)AY_n + Y_nB^\star\left(Y_{n-1} - X_\infty^{-1}\right)BY_n. \end{aligned}$$

Thus

$$\begin{aligned} X_\infty^{-1} - Y_{n+1} &= X_\infty^{-1} - Y_n + Y_nX_\infty Y_n - Y_n + Y_nA^\star\left(X_\infty^{-1} - Y_{n-1}\right)AY_n + Y_nB^\star\left(X_\infty^{-1} - Y_{n-1}\right)BY_n \\ &= \left(X_\infty^{-1} - Y_n\right)X_+\left(X_\infty^{-1} - Y_n\right) + Y_nA^\star\left(X_\infty^{-1} - Y_{n-1}\right)AY_n + Y_nB^\star\left(X_\infty^{-1} - Y_{n-1}\right)BY_n. \end{aligned}$$

Now, since $\|Y_n - X_\infty^{-1}\| \leq \|Y_{n-1} - X_\infty^{-1}\|$ and $\lim Y_n = X_\infty^{-1}$, inequality (4) follows. Also, inequality (5) is true since

$$X_{n+1} - X_\infty = A^\star\left(X_\infty^{-1} - Y_n\right)A + B^\star\left(X_\infty^{-1} - Y_n\right)B. \tag{6}$$

This completes the proof. □

The above proof shows that Algorithm (3) should be modified as follows to improve the preceding convergence properties.

Algorithm 2 Let

$$\begin{cases} X_0 = Y_0 = I, \\ Y_{n+1} = 2Y_n - Y_nX_nY_n, \\ X_{n+1} = I - A^\star Y_{n+1}A - B^\star Y_{n+1}B, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{7}$$

Theorem 3 *Assume that matrix equation* (2) *has a positive definite solution. Then Algorithm* (7) *defines a monotonically decreasing matrix sequence* $\{X_n\}$ *converging to* X_∞ *which* *is the maximal Hermitian positive definite solution of equation* (2). *Also, the sequence* $\{Y_n\}$ *defined in Algorithm* (7) *defines a monotonically increasing sequence converging to* X_∞^{-1}.

Proof Let X_+ be a positive definite solution of matrix equation (2). It is clear that

$$X_0 \geq X_1 \geq \cdots \geq X_n \geq X_+, \qquad Y_0 \leq Y_1 \leq \cdots \leq Y_n \leq X_+^{-1} \tag{8}$$

is true for $n = 1$. Assume (8) is true for $n = k$. By Lemma 1, we have that

$$Y_{k+1} = 2Y_k - Y_k X_k Y_k \leq X_K^{-1} \leq X_+^{-1}.$$

Therefore,

$$X_{k+1} = I - A^\star Y_{k+1} A \geq I - A^\star X_+^{-1} A = X_+.$$

Since $Y_k \leq X_{k-1}^{-1} \leq X_k^{-1}$, we have $Y_k^{-1} \geq X_k$. Thus

$$Y_{k+1} - Y_k = Y_k \big(Y_k^{-1} - X_k\big) Y_k \geq 0$$

and

$$X_{k+1} - X_k = -A^\star (Y_{k+1} - Y_k) A \leq 0.$$

We have now proved (8) for $n = k + 1$. Therefore, (8) is true for all n, and $\lim_{n\to\infty} X_n$ and $\lim_{n\to\infty} Y_n$ exist. So, we have $\lim X_n = X_\infty$ and $\lim Y_n = X_\infty^{-1}$. □

Similar to Theorem 2, we can state the following theorem.

Theorem 4 *If matrix equation* (2) *has a positive definite solution for Algorithm* (7) *and any* $\epsilon > 0$, *then we have*

$$\|Y_{n+1} - X_\infty^{-1}\| \leq \big(\|AX_\infty^{-1}\| + \|BX_\infty^{-1}\| + \epsilon\big)^2 \|Y_n - X_\infty^{-1}\|$$

and

$$\|X_n - X_\infty\| \leq \big(\|A\|^2 + \|B\|^2\big) \|Y_n - X_\infty^{-1}\| \tag{9}$$

for all n large enough.

Now, we can see that Algorithm (7) can be faster than Algorithm (3) from the estimates in Theorems 2 and 4.

Algorithm 3 Take

$$\begin{cases} X_0 = I, \qquad Y_0 = I, \\ Y_{n+1} = (I - X_n) Y_n + I, \\ X_{n+1} = I - A^\star Y_{n+1} A - B^\star Y_{n+1} B, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{10}$$

Algorithm (10) requires only five matrix multiplications per step, whereas Algorithm (3) requires six matrix multiplications per step.

Theorem 5 *If matrix equation* (2) *has a positive definite solution and the two sequences* $\{X_n\}$ *and* $\{Y_n\}$ *are determined by Algorithm* (10), *then* $\{X_n\}$ *is monotone decreasing and converges to the maximal Hermitian positive definite solution* X_∞. *Also, the sequence* $\{Y_n\}$ *defined in Algorithm* (10) *is a monotonically increasing sequence converging to* X_∞^{-1}.

Proof Let X_+ be a positive definite solution of equation (2). We prove that

$$X_0 \geq X_1 \geq \cdots \geq X_n \geq X_+ \tag{11}$$

and

$$Y_0 \leq Y_1 \leq \cdots \leq Y_n \leq X_+^{-1}. \tag{12}$$

Since X_+ is a solution of matrix equation (1), $X_0 = I \geq X_+$. Also, we have $Y_0 = Y_1 = I$, and so

$$X_1 = I - A^\star A - B^\star B \geq I - A^\star X_+^{-1} A - B^\star X_+^{-1} B = X_+,$$

i.e., $X_0 \geq X_1 \geq X_+$.

For the sequence $\{Y_n\}$, since $I \leq X_+^{-1}$, $Y_0 = Y_1 = I \leq X_+^{-1}$.

Thus inequalities (11) and (12) are true for $n = 1$. Now, assume that inequalities (11) and (12) are true for $n = k$, *i.e.*,

$$X_0 \geq X_1 \geq \cdots \geq X_k \geq X_+$$

and

$$Y_0 \leq Y_1 \leq \cdots \leq Y_k \leq X_+^{-1}.$$

We show that inequalities (11) and (12) are true for $n = k + 1$. We have

$$Y_{k+1} = (I - X_k) Y_k + I \geq (I - X_{k-1}) Y_{k-1} + I = Y_k$$

and

$$Y_{k+1} = (I - X_k) Y_k + I \leq (I - X_+) X_+^{-1} + I = X_+^{-1},$$

i.e., $Y_k \leq Y_{k+1} \leq X_+^{-1}$. Then

$$X_k - X_{k+1} = A^\star (Y_{k+1} - Y_k) A + B^\star (Y_{k+1} - Y_k) B,$$

and $X_{k+1} \leq X_k$, since $Y_k \leq Y_{k+1}$. On the other hand,

$$\begin{aligned} X_{k+1} &= I - A^\star Y_{k+1} A - B^\star Y_{k+1} B \\ &\geq I - A^\star X_+^{-1} A - B^\star X_+^{-1} B \\ &= X_+, \end{aligned} \tag{13}$$

i.e., $X_k \geq X_{k+1} \geq X_+$.

Then the above inequalities are true for all n, also $\lim_{n\to\infty} X_n$ and $\lim_{n\to\infty} Y_n$ exist. By taking limit on Algorithm (10), we have $\lim_{n\to\infty} X_n = X_\infty$ and $\lim_{n\to\infty} Y_n = X_\infty^{-1}$, where X_∞ is the maximal positive definite solution of matrix equation (2). □

By Algorithm (10), we have $I - X_n Y_n = Y_{n+1} - Y_n$. Then, for small $\epsilon > 0$, $\|I - X_n Y_n\|$ can be one stopping condition.

Theorem 6 *If matrix equation* (2) *has a positive definite solution and after n iterative steps of Algorithm* (10), *the inequality* $\|I - X_n Y_n\| < \epsilon$ *implies*

$$\left\|X_n + A^\star X_n^{-1}A + B^\star X_n^{-1}B - I\right\| \le \epsilon\left(\|A\|^2 + \|B\|^2\right)\left\|X_\infty^{-1}\right\|.$$

Proof Since

$$\begin{aligned} X_n + A^\star X_n^{-1}A + B^\star X_n^{-1}B - I &= X_n - X_{n+1} + A^\star\left(X_n^{-1} - Y_{n+1}\right)A + B^\star\left(X_n^{-1} - Y_{n+1}\right)B \\ &= A^\star\left(Y_{n+1} - X_n^{-1} + X_n^{-1} - Y_n\right)A \\ &\quad + B^\star\left(Y_{n+1} - X_n^{-1} + X_n^{-1} - Y_n\right)B \\ &\quad + A^\star\left(X_n^{-1} - Y_{n+1}\right)A + B^\star\left(X_n^{-1} - Y_{n+1}\right)B \qquad (14) \\ &= A^\star X_n^{-1}(I - X_n Y_n)A + B^\star X_n^{-1}(I - X_n Y_n)B, \\ \left\|X_n + A^\star X_n^{-1}A + B^\star X_n^{-1}B - I\right\| &\le \left(\|A\|^2 + \|B\|^2\right)\left\|X_\infty^{-1}\right\|\|I - X_n Y_n\| \\ &\le \epsilon\left(\|A\|^2 + \|B\|^2\right)\left\|X_\infty^{-1}\right\|. \end{aligned}$$

This completes the proof. □

Theorem 7 *If* $X_n > 0$ *for every n, then matrix equation* (2) *has a Hermitian positive definite solution.*

Proof Since $X_n > 0$ for every n, the proof of the monotonicity of $\{Y_n\}$ and $\{X_n\}$ noted in Theorem 5 remains valid. Therefore, the sequence $\{X_n\}$ is monotone decreasing and bounded from below by the zero matrix. Then $\lim_{n\to\infty} X_n = X$ exists. We claim that the sequence $\{Y_n\}$ is bounded above. Suppose that it does not hold. Then, for every $m > 0$, there exists n_m such that $mI < Y_{n_m}$. Since each X_n is positive definite for every n, we have

$$A^\star Y_n A + B^\star Y_n B < I \quad \text{for every } n.$$

Furthermore, since A or B are nonsingular for every $m > 0$, we have

$$m\left(A^\star A + B^\star B\right) < A^\star Y_{n_m} A + B^\star Y_{n_m} B < I.$$

By [16, Lemma 1.2],

$$m\left(A^\star A + B^\star B\right) < I \quad \text{for every } m,$$

which is a contradiction. Then the sequence $\{Y_n\}$ is bounded above and convergent. Suppose that $\lim_{n\to\infty} Y_n = Y$. As $Y_0 = I$ and $\{Y_n\}$ is monotone increasing, $Y \le I$. Taking limit in Algorithm (10) implies that

$$Y = (I - X)Y + I,$$
$$X = I - A^{\star}YA - B^{\star}YB.$$

Since $Y \leq I$, $X = Y^{-1} > 0$, and hence $X = I - A^{\star}X^{-1}A - B^{\star}X^{-1}B$. Then matrix equation (2) has a positive definite solution. □

Theorem 8 *If matrix equation* (2) *has a positive definite solution and* $\|A\| < \frac{1}{2}$ *and* $\|B\| < \frac{1}{2}$, *then the sequence* $\{X_n\}$ *defined in Algorithm* (10) *satisfies*

$$\left\| Y_{n+1} - X_{\infty}^{-1} \right\| \leq \left(\left\| AX_{\infty}^{-1} \right\| + \left\| BX_{\infty}^{-1} \right\| \right) \left\| Y_n - X_{\infty}^{-1} \right\| \tag{15}$$

and

$$\|X_{n+1} - X_{\infty}\| \leq \left(\|A\|^2 + \|B\|^2 \right) \left\| Y_n - X_{\infty}^{-1} \right\| \tag{16}$$

for all n large enough.

Proof From Algorithm (10), we have

$$\begin{aligned} Y_{n+1} &= (I - X_n)Y_n + I \\ &= A^{\star}Y_nAY_n + B^{\star}Y_nBY_n + I \\ &= A^{\star}\left(Y_n + X_{\infty}^{-1} - X_{\infty}^{-1}\right)AY_n + B^{\star}\left(Y_n + X_{\infty}^{-1} - X_{\infty}^{-1}\right)BY_n + I \\ &= A^{\star}\left(Y_n - X_{\infty}^{-1}\right)AY_n + B^{\star}\left(Y_n - X_{\infty}^{-1}\right)BY_n + A^{\star}X_{\infty}^{-1}AY_n + B^{\star}X_{\infty}^{-1}BY_n + Y_n - Y_n + I \\ &= A^{\star}\left(Y_n - X_{\infty}^{-1}\right)AY_n + B^{\star}\left(Y_n - X_{\infty}^{-1}\right)BY_n - \left(I - A^{\star}X_{\infty}^{-1}A - B^{\star}X_{\infty}^{-1}B\right)Y_n + Y_n + I \\ &= A^{\star}\left(Y_n - X_{\infty}^{-1}\right)AY_n + B^{\star}\left(Y_n - X_{\infty}^{-1}\right)BY_n - X_{\infty}Y_n + Y_n + I. \end{aligned}$$

Thus

$$\begin{aligned} X_{\infty}^{-1} - Y_{n+1} &= X_{\infty}^{-1} + A^{\star}\left(X_{\infty}^{-1} - Y_n\right)AY_n + B^{\star}\left(X_{\infty}^{-1} - Y_n\right)BY_n + X_{\infty}Y_n - Y_n - I \\ &= A^{\star}\left(X_{\infty}^{-1} - Y_n\right)AY_n + B^{\star}\left(X_{\infty}^{-1} - Y_n\right)BY_n + (I - X_{\infty})\left(X_{\infty}^{-1} - Y_n\right) \\ &= A^{\star}\left(X_{\infty}^{-1} - Y_n\right)AY_n + B^{\star}\left(X_{\infty}^{-1} - Y_n\right)BY_n \\ &\quad + A^{\star}X_{\infty}^{-1}A\left(X_{\infty}^{-1} - Y_n\right) + B^{\star}X_{\infty}^{-1}B\left(X_{\infty}^{-1} - Y_n\right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \left\| X_{\infty}^{-1} - Y_{n+1} \right\| &\leq \left(\left\| A^{\star} \right\| \|AY_n\| + \left\| B^{\star} \right\| \|BY_n\| + \left\| A^{\star}X_{\infty}^{-1}A \right\| + \left\| B^{\star}X_{\infty}^{-1}B \right\| \right) \left\| X_{\infty}^{-1} - Y_n \right\| \\ &\leq \left(\left(\|AY_n\| + \left\| X_{\infty}^{-1}A \right\| \right) \left\| A^{\star} \right\| + \left(\|BY_n\| + \left\| X_{\infty}^{-1}B \right\| \right) \left\| B^{\star} \right\| \right) \left\| X_{\infty}^{-1} - Y_n \right\|. \end{aligned}$$

Now, since $\lim_{n\to\infty} Y_n = X_{\infty}^{-1}$, inequality (15) follows. Also, inequality (16) is true since

$$X_{n+1} - X_{\infty} = A^{\star}\left(X_{\infty}^{-1} - Y_n\right)A + B^{\star}\left(X_{\infty}^{-1} - Y_n\right)B. \tag{17}$$

This completes the proof. □

3 Numerical examples

In this section, we present some numerical examples to show the effectiveness of the new inversion-free variant of the basic fixed point iteration methods. Hermitian positive definite solutions of matrix equation (2) for different matrices A and B are computed. We will compare the suggested algorithms, Algorithm (7) and Algorithm (10), by Algorithm (3). All programs were written in MATLAB.

Example 1 Consider equation (2) with

$$A = \frac{1}{200}\begin{pmatrix} 0.2 & -0.1 & 0.3 \\ 0.56 & 0.3 & -0.7 \\ 0.2 & 0.5 & 0.6 \end{pmatrix}, \qquad B = \frac{1}{20}\begin{pmatrix} 0.46 & -0.01 & 0.020 \\ -0.15 & -0.488 & -0.060 \\ 0.04 & -0.01 & -0.120 \end{pmatrix}.$$

Algorithm (3) needs six iterations to get the solution

$$X = \begin{pmatrix} 0.999400612248657 & -0.000176704506272 & -0.000028208026789 \\ -0.000176704506272 & 0.999395021004650 & -0.000077249011425 \\ -0.000028208026789 & -0.000077249011425 & 0.999930483901903 \end{pmatrix},$$

$$\|X + A^{\star}X^{-1}A + B^{\star}X^{-1}B - I\|_{\infty} = 1.3885\text{e}{-013}.$$

Algorithm (7) needs six iterations to get the solution

$$X = \begin{pmatrix} 0.999400612248567 & -0.000176704506276 & -0.000028208026792 \\ -0.000176704506276 & 0.999395021004514 & -0.000077249011443 \\ -0.000028208026792 & -0.000077249011443 & 0.999930483901898 \end{pmatrix},$$

$$\|X + A^{\star}X^{-1}A + B^{\star}X^{-1}B - I\|_{\infty} = 6.7763\text{e}{-021}.$$

We can easily see that Algorithm (7) is more accurate than Algorithm (3).

Algorithm (10) needs six iterations to get the solution

$$X = \begin{pmatrix} 0.999400612248567 & -0.000176704506276 & -0.000028208026792 \\ -0.000176704506276 & 0.999395021004514 & -0.000077249011443 \\ -0.000028208026792 & -0.000077249011443 & 0.999930483901898 \end{pmatrix},$$

$$\|X + A^{\star}X^{-1}A + B^{\star}X^{-1}B - I\|_{\infty} = 6.7763\text{e}{-021}.$$

Example 2 Consider equation (2) with

$$A = \frac{1}{820}\begin{pmatrix} 41 & 15 & 23 & 35 & 66 \\ 25 & 12 & 27 & 45 & 21 \\ 23 & 27 & 28 & 16 & 24 \\ 15 & 45 & 16 & 52 & 65 \\ 66 & 21 & 24 & 65 & 35 \end{pmatrix}, \qquad B = \frac{1}{830}\begin{pmatrix} 23 & 21 & 23 & 25 & 32 \\ 21 & 45 & 60 & 42 & 33 \\ 23 & 24 & 34 & 18 & 17 \\ 13 & 42 & 18 & 44 & 30 \\ 32 & 33 & 26 & 30 & 26 \end{pmatrix}.$$

Algorithm (3) after 21 iterations gives the solution

$$X = \begin{pmatrix} 0.98393799066 & -0.01161748103 & -0.01233926321 & -0.01833845539 & -0.01633619168 \\ -0.01161748103 & 0.98497686219 & -0.01315828865 & -0.01745583944 & -0.01639741581 \\ -0.01233926321 & -0.01315828865 & 0.98561286596 & -0.01623773649 & -0.01467582916 \\ -0.01833845539 & -0.01745583944 & -0.01623773649 & 0.97439947749 & -0.02237728728 \\ -0.01633619168 & -0.01639741581 & -0.01467582916 & -0.02237728728 & 0.97634558763 \end{pmatrix},$$

$$\|X + A^\star X^{-1}A + B^\star X^{-1}B - I\|_\infty = 3.7975\text{e}{-013}.$$

Algorithm (7) after 21 iterations gives the solution

$$X = \begin{pmatrix} 0.98393799066 & -0.01161748103 & -0.01233926321 & -0.01833845539 & -0.01633619168 \\ -0.01161748103 & 0.98497686219 & -0.01315828865 & -0.01745583944 & -0.01639741581 \\ -0.01233926321 & -0.01315828865 & 0.98561286596 & -0.01623773649 & -0.01467582916 \\ -0.01833845539 & -0.01745583944 & -0.01623773649 & 0.97439947749 & -0.02237728728 \\ -0.01633619168 & -0.01639741581 & -0.01467582916 & -0.02237728728 & 0.97634558763 \end{pmatrix},$$

$$\|X + A^\star X^{-1}A + B^\star X^{-1}B - I\|_\infty = 6.0963\text{e}{-018}.$$

Algorithm (10) after 21 iterations gives the solution

$$X = \begin{pmatrix} 0.98393799066 & -0.01161748103 & -0.01233926321 & -0.01833845539 & -0.01633619168 \\ -0.01161748103 & 0.98497686219 & -0.01315828865 & -0.01745583944 & -0.01639741581 \\ -0.01233926321 & -0.01315828865 & 0.98561286596 & -0.01623773649 & -0.01467582916 \\ -0.01833845539 & -0.01745583944 & -0.01623773649 & 0.97439947749 & -0.02237728728 \\ -0.01633619168 & -0.01639741581 & -0.01467582916 & -0.02237728728 & 0.97634558763 \end{pmatrix},$$

$$\|X + A^\star X^{-1}A + B^\star X^{-1}B - I\|_\infty = 6.1220\text{e}{-018}.$$

We can see that Algorithm (10) needs to find a Hermitian positive definite solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the proof. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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