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An note on S_r-covering approximation spaces *

Bin Qin† Xun Ge‡

February 21, 2013

Abstract: In this paper, we prove that a covering approximation space $(U, \mathcal{C})$ is an S_r-covering approximation space if and only if $\{N(x) : x \in U\}$ forms a partition of the universe of discourse U. Furthermore, we give some simple characterizations for S_r-space $(U, \mathcal{C})$ by using only a single covering approximation operator and by using only elements of covering $\mathcal{C}$. Results of this paper answer affirmatively an open problem posed by Z.Yun et al. in [16].

Keywords: Universe of discourse; Covering approximation space; S_r-covering approximation space; Covering lower (upper) approximation operation; Neighborhood; Partition.

1 Introduction

Rough set theory, which was first proposed by Z.Pawlak in [4], is a useful tool in researches and applications of process control, economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis and other fields [2, 3, 5, 6, 10, 14, 15, 18, 19]. In the classical rough set theory, Pawlak approximation spaces are based on partitions of the universe of discourse U, but this requirement is not satisfied in some situations [20]. In the past years, Pawlak approximation spaces have been extended to covering approximation spaces [1, 8, 12, 13, 16, 21].

Definition 1.1 ([21]). *Let U, the universe of discourse, be a finite set and $\mathcal{C}$ be a family of nonempty subsets of U.*

(1) $\mathcal{C}$ is called a covering of U if $\bigcup\{K : K \in \mathcal{C}\} = U$. Furthermore, $\mathcal{C}$ is called a partition of U if also $K \bigcap K' = \emptyset$ for all $K, K' \in \mathcal{C}$, where $K \neq K'$.

(2) The pair $(U, \mathcal{C})$ is called a covering approximation space (resp. a Pawlak approximation space) if $\mathcal{C}$ is a covering (resp. a partition) of U.

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(3) $\bigcap\{K : x \in K \in \mathcal{C}\}$ *is called the neighborhood of* x *and denoted as* $Neighbor_C(x)$. *When there is no confusion, we omit* C *at the lowercase and abbreviate* $Neighbor_C(x)$ *to* $N(x)$.

For a covering approximation spaces $(U, \mathcal{C})$, it is interesting to study the condition for $\{N(x) : x \in U\}$ to form a partition of universe U. In particular, it is an important issue in covering approximation spaces theory to characterize this condition by covering lower (upper) approximation operations [8, 16]. In order to give a more detailed description for this issue, we present some covering lower (upper) approximation operations as follows.

Definition 1.2 ([16]). *Let* (U, C) *be a covering approximation space and* $X \subseteq U$. *Put*

(1) $\underline{\mathcal{C}_2}(X) = \bigcup\{K : K \in \mathcal{C} \wedge K \subseteq X\}$, $\overline{\mathcal{C}_2}(X) = U - \underline{\mathcal{C}_2}(U - X)$;
(2) $\underline{\mathcal{C}_3}(X) = \{x \in U : N(x) \subseteq X\}$, $\overline{\mathcal{C}_3}(X) = \{x \in U : N(x) \bigcap X \neq \emptyset\}$;
(3) $\underline{\mathcal{C}_4}(X) = \{x \in U : \exists u(u \in N(x) \wedge N(u) \subseteq X)\}$, $\overline{\mathcal{C}_4}(X) = \{x \in U : \forall u(u \in N(x) \to N(u) \cap X \neq \emptyset)\}$;
(4) $\underline{\mathcal{C}_5}(X) = \{x \in U : \forall u(x \in N(u) \to N(u) \subseteq X)\}$, $\overline{\mathcal{C}_5}(X) = \bigcup\{N(x) : x \in U \wedge N(x) \bigcap X \neq \emptyset\}$;
(5) $\underline{\mathcal{C}_6}(X) = \{x \in U : \forall u(x \in N(u) \to u \in X)\}$, $\overline{\mathcal{C}_6}(X) = \bigcup\{N(x) : x \in X\}$.

Then $\underline{\mathcal{C}_i}$ *(resp.* $\overline{\mathcal{C}_i}$*) is called a covering lower (resp. upper) approximation operation and* $\underline{\mathcal{C}_i}(X)$ *(resp.* $\overline{\mathcal{C}_i}(X)$*) is called covering lower (resp. upper) approximation of* X. *Here,* $i = 2, 3, 4, 5, 6$.

Remark 1.3. *In [8],* $\underline{\mathcal{C}_i}$ *and* $\overline{\mathcal{C}_i}$ *are denoted by* $\underline{\mathcal{C}_{i-1}}$ *and* $\overline{\mathcal{C}_{i-1}}$ *respectively. Here,* $i = 2, 3, 4, 5, 6$.

K.Qin et al. gave the following theorem.

Theorem 1.4 ([8]). *Let* $(U, \mathcal{C})$ *be a covering approximation space. Then the following are equivalent.*

(1) $\{N(x) : x \in U\}$ *forms a partition of* U.
(2) $\overline{\mathcal{C}_5}(X) = \overline{\mathcal{C}_6}(X)$ *for each* $X \subseteq U$.
(3) $\overline{\mathcal{C}_5}(X) = \overline{\mathcal{C}_4}(X)$ *for each* $X \subseteq U$.
(4) $\overline{\mathcal{C}_3}(X) = \overline{\mathcal{C}_4}(X)$ *for each* $X \subseteq U$.
(5) $\overline{\mathcal{C}_6}(X) = \overline{\mathcal{C}_4}(X)$ *for each* $X \subseteq U$.
(6) $\overline{\mathcal{C}_3}(X) = \overline{\mathcal{C}_6}(X)$ *for each* $X \subseteq U$.
(7) $\underline{\mathcal{C}_5}(X) = \underline{\mathcal{C}_3}(X)$ *for each* $X \subseteq U$.

Recently, taking Theorem 1.4 into account, Z. Yun et al. [16] investigated the following question.

Question 1.5 ([16]). *Can we characterize the conditions under which* $\{N(x) : x \in U\}$ *forms a partition of* U *by using only a single covering approximation operator among* C_2-C_6*?*

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The following results were obtained.

Theorem 1.6 ([16]). *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $\{N(x) : x \in U\}$ forms a partition of U.
(2) $\overline{C_3}(\underline{\mathcal{C}_3}(X)) = \underline{\mathcal{C}_3}(X)$ for each $X \subseteq U$.
(3) $\overline{C_6}(\underline{\mathcal{C}_6}(X)) = \underline{\mathcal{C}_6}(X)$ for each $X \subseteq U$.
(4) $\underline{C_4}(X) \subseteq X$ for each $X \subseteq U$.

Theorem 1.7 ([16]). *Let $(U, \mathcal{C})$ be a covering approximation space.*

(1) If $\overline{C_2}(\underline{\mathcal{C}_2}(X)) = \underline{\mathcal{C}_2}(X)$ for each $X \subseteq U$, then $\{N(x) : x \in U\}$ forms a partition of U, not vice versa.

(2) If $\{N(x) : x \in U\}$ forms a partition of U, then $\overline{C_5}(\underline{\mathcal{C}_5}(X)) = \underline{\mathcal{C}_5}(X)$ for each $X \subseteq U$, not vice versa.

As an open problem, the following question is raised in the end of [16].

Question 1.8 ([16]). *How to give sufficient and necessary conditions for $\{N(x) : x \in U\}$ to form a partition of U by using only a single covering approximation operator C_i $(i = 2, 5)$?*

In this paper, we investigate Question 1.5 and Question 1.8 by S_r-covering approximation spaces. Here, S_r-covering approximation spaces was introduced by X.Ge in [1].

Definition 1.9 ([1]). *A covering approximation space $(U, \mathcal{C})$ is called an S_r-space (S_r-space is the abbreviation of S_r-covering approximation space) if $x \in K \in \mathcal{C}$ implies $D(x) \subset K$, where $D(x) = U - \bigcup(\mathcal{C} - \mathcal{C}_x)$.*

In this paper, we gives a "nice" characterization for S_r-space. By this result, we translate the condition for $\{N(x) : x \in U\}$ to form a partition of the universe of discourse U into S_r-space $(U, \mathcal{C})$ in Question 1.5 and Question 1.8. Furthermore, we obtain some simple characterizations for S_r-space $(U, \mathcal{C})$ by using only a single covering approximation operator and by using only elements of covering $\mathcal{C}$, which answer Question 1.5 and Question 1.8 and improve some results obtained in [16].

2 Preliminaries

For a covering approximation space $(U, \mathcal{C})$, we say that $\{N(x) : x \in U\}$ forms a partition of U if for every pair $x, y \in U$, $N(x) = N(y)$ or $N(x) \bigcap N(y) = \emptyset$. Before our discussion, we give some notations.

Note 2.1. *Let $(U, \mathcal{C})$ be a covering approximation space. Throughout this paper, we use the following notations, where $x \in U$, $X \subseteq U$ and $\mathcal{F} \subseteq 2^U$.*

(1) $\bigcap \mathcal{F} = \bigcap\{F : F \in \mathcal{F}\}$.
(2) $\bigcup \mathcal{F} = \bigcup\{F : F \in \mathcal{F}\}$.

(3) $\mathcal{C}_x = \{K : x \in K \in \mathcal{C}\}$.
(4) $N(x) = \bigcap \mathcal{C}_x$.
(5) $D(x) = U - \bigcup(\mathcal{C} - \mathcal{C}_x)$.

Remark 2.2. *It is clear that* $x \in N(x)$ *and* $x \in D(x)$. *Note that* $x \in K \in \mathcal{C}$ *if and only if* $K \in \mathcal{C}_x$, *we also replace* $x \in K \in \mathcal{C}$ *by* $K \in \mathcal{C}_x$ *in this paper.*

The following three lemmas are known.

Lemma 2.3 ([8, 9])**.** *Let* $(U, \mathcal{C})$ *be a covering approximation space and* $X, Y \subseteq U$. *Then the following hold.*

(1) $\underline{\mathcal{C}_i}(U) = U = \overline{\mathcal{C}_i}(U)$, $\underline{\mathcal{C}_i}(\emptyset) = \emptyset = \overline{\mathcal{C}_i}(\emptyset)$ *for* $i = 2, 3, 4, 5, 6$.
(2) $\underline{\mathcal{C}_i}(X) \subseteq X \subseteq \overline{\mathcal{C}_i}(X)$ *for* $i = 2, 3, 5, 6$.
(3) $X \subseteq Y \subseteq U \Longrightarrow \underline{\mathcal{C}_i}(X) \subseteq \underline{\mathcal{C}_i}(Y)$, $\overline{\mathcal{C}_i}(X) \subseteq \overline{\mathcal{C}_i}(Y)$ *for* $i = 2, 3, 4, 5, 6$.
(4) $\underline{\mathcal{C}_i}(X \bigcap Y) = \underline{\mathcal{C}_i}(X) \bigcap \underline{\mathcal{C}_i}(Y)$, $\overline{\mathcal{C}_i}(X \bigcup Y) = \overline{\mathcal{C}_i}(X) \bigcup \overline{\mathcal{C}_i}(Y)$ *for* $i = 3, 5, 6$.
(5) $\underline{\mathcal{C}_i}(X) = U - \overline{\mathcal{C}_i}(U - X)$, $\overline{\mathcal{C}_i}(X) = U - \underline{\mathcal{C}_i}(U - X)$ *for* $i = 2, 3, 4, 5, 6$.

Lemma 2.4 ([8])**.** *Let* $(U, \mathcal{C})$ *be a covering approximation space. Then the following are equivalent.*

(1) $\{N(x) : x \in U\}$ *forms a partition of* U.
(2) For every pair $x, y \in U$, $x \in N(y) \Longrightarrow y \in N(x)$.

Lemma 2.5 ([1])**.** *Let* $(U, \mathcal{C})$ *be a covering approximation space and* $x, y \in U$. *Then the following are equivalent.*

(1) $x \in N(y)$.
(2) $\mathcal{C}_y \subseteq \mathcal{C}_x$.
(3) $N(x) \subseteq N(y)$.
(4) $D(y) \subseteq D(x)$.
(5) $y \in D(x)$.

Proposition 2.6. *Let* $(U, \mathcal{C})$ *be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ *is an* S_r*-spaces.*
(2) $\{N(x) : x \in U\}$ *forms a partition of* U.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-spaces. Let $x, y \in U$ and $x \in N(y)$. Then $y \in D(x)$ by Lemma 2.5. For each $K \in \mathcal{C}_x$, $D(x) \subseteq K$, we have $y \in K$. This proves that $y \in N(x)$. By Lemma 2.4, $\{N(x) : x \in U\}$ forms a partition of U.

(2) $\Longrightarrow$ (1): Suppose that $\{N(x) : x \in U\}$ forms a partition of U. Let $K \in \mathcal{C}$ and $x \in K$. Then $N(x) \subseteq K$. If $y \in D(x)$, then $x \in N(y)$ by Lemma 2.5. By Lemma 2.4, $y \in N(x) \subseteq K$. This proves that $D(x) \subseteq K$. So $(U, \mathcal{C})$ is an S_r-space. □

Proposition 2.6 gives a "nice" characterization for S_r-space, which is help for us to further comprehend [1, Remark 1.2]).

3 The main results

Theorem 3.1. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\overline{C_2}(\{x\}) \subseteq K$ for each $x \in U$ and each $K \in \mathcal{C}_x$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $x \in U$ and $K \in \mathcal{C}_x$. Then $D(x) \subseteq K$. If $y \in \overline{C_2}(\{x\}) = U - \underline{C_2}(U - \{x\})$, then $y \notin \underline{C_2}(U - \{x\})$. So, for each $K' \in \mathcal{C}$, if $K' \subseteq U - \{x\}$ then $y \notin K'$. That is, for each $K' \in \mathcal{C}$, if $x \notin K'$ then $y \notin K'$, and hence $y \notin \bigcup(\mathcal{C} - \mathcal{C}_x)$. It follows that $y \in U - \bigcup(\mathcal{C} - \mathcal{C}_x) = D(x) \subseteq K$. This proves that $\overline{C_2}(\{x\}) \subseteq K$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$. Then $\overline{C_2}(\{x\}) \subseteq K$. If $y \in D(x) = U - \bigcup(\mathcal{C} - \mathcal{C}_x)$, then $y \notin \bigcup(\mathcal{C} - \mathcal{C}_x)$. So $y \notin K$ for each $K \in \mathcal{C} - \mathcal{C}_x$. That is, for each $K \in \mathcal{C}$, if $x \notin K$ then $y \notin K$. Note that $x \notin K$ if and only if $K \subseteq U - \{x\}$. Thus, $y \notin \underline{C_2}(U - \{x\})$. It follows that $y \in U - \underline{C_2}(U - \{x\}) = \overline{C_2}(\{x\}) \subseteq K$. This proves that $D(x) \subseteq K$. So $(U, \mathcal{C})$ is an S_r-space. □

Let $(U, \mathcal{C})$ be a covering approximation space. It is clear that if $\overline{C_2}(\underline{C_2}(X)) = \underline{C_2}(X)$ for each $X \subseteq U$. Then $\overline{C_2}(K) = K$ for each $K \in \mathcal{C}$. So the following corollary improves Theorem 1.7(1), and the proof is quite simple.

Corollary 3.2. *Let $(U, \mathcal{C})$ be a covering approximation space. If $\overline{C_2}(K) = K$ for each $K \in \mathcal{C}$, then $(U, \mathcal{C})$ is an S_r-space.*

Proof. Let $\overline{C_2}(K) = K$ for each $K \in \mathcal{C}$. If $x \in U$ and $K \in \mathcal{C}_x$, then $\overline{C_2}(\{x\}) \subseteq \overline{C_2}(K) = K$ from Lemma 2.3(3). By Theorem 3.1, $(U, \mathcal{C})$ is an S_r-space. □

Remark 3.3. *[16, Example 3.9] and Proposition 2.6 show that Corollary 3.2 can not be reversed.*

What are sufficient and necessary conditions such that $\overline{C_2}(K) = K$ for each $K \in \mathcal{C}$? The following proposition gives an answer.

Proposition 3.4. *Let $(U, \mathcal{C})$ be a covering approximation space. Then $\overline{C_2}(K) = K$ for each $K \in \mathcal{C}$ if and only if the following hold.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\overline{C_2}(K) = \bigcup\{\overline{C_2}(\{x\}) : x \in K\}$ for each $K \in \mathcal{C}$.

Proof. Necessity: Let $\overline{C_2}(K) = K$ for each $K \in \mathcal{C}$. By Corollary 3.2, $(U, \mathcal{C})$ is an S_r-space. Let $K \in \mathcal{C}$. By Lemma 2.3(3), $\overline{C_2}(\{x\}) \subseteq \overline{C_2}(K)$ for each $x \in K$. Thus $\bigcup\{\overline{C_2}(\{x\}) : x \in K\} \subseteq \overline{C_2}(K)$. On the other hand, by Lemma 2.3(2), $x \in \overline{C_2}(\{x\})$ for each $x \in K$, so $\overline{C_2}(K) = K \subseteq \bigcup\{\overline{C_2}(\{x\}) : x \in K\}$. Consequently, $\overline{C_2}(K) = \bigcup\{\overline{C_2}(\{x\}) : x \in K\}$.

Sufficiency: Suppose that (1) and (2) hold. Let $K \in \mathcal{C}$. By Theorem 3.1, $\overline{C_2}(\{x\}) \subseteq K$ for each $x \in K$. Thus, $\overline{C_2}(K) = \bigcup\{\overline{C_2}(\{x\}) : x \in K\} \subseteq K$. On the other hand, $K \subseteq \overline{C_2}(K)$ by Lemma 2.3(2). So $\overline{C_2}(K) = K$. □

Similarly, the following proposition is obtained, which gives sufficient and necessary conditions such that $\overline{C_2}(\underline{\mathcal{C}_2}(X)) = \underline{\mathcal{C}_2}(X)$ for each $X \subseteq U$. We omit its proof.

Proposition 3.5. *Let $(U, \mathcal{C})$ be a covering approximation space. Then $\overline{C_2}(\underline{\mathcal{C}_2}(X)) = \underline{\mathcal{C}_2}(X)$ for each $X \subseteq U$ if and only if the following hold.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\overline{C_2}(X) = \bigcup\{\overline{C_2}(\{x\}) : x \in X\}$ for each union X of elements of $\mathcal{C}$.

Lemma 3.6. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\overline{C_3}(\{x\}) \subseteq K$ for each $x \in U$ and each $K \in \mathcal{C}_x$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $x \in U$ and $K \in \mathcal{C}_x$. Then $D(x) \subseteq K$. If $y \in \overline{C_3}(\{x\}) = \{z \in U : N(z) \bigcap \{x\} \neq \emptyset\}$, then $N(y) \bigcap \{x\} \neq \emptyset$, so $x \in N(y)$. By Lemma 2.5, $y \in D(x) \subseteq K$. This proves that $\overline{C_3}(\{x\}) \subseteq K$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$. Then $\overline{C_3}(\{x\}) \subseteq K$. If $y \in D(x)$, then $x \in N(y)$ from Lemma 2.5, i.e., $N(y) \bigcap \{x\} \neq \emptyset$. It follows that $y \in \{z \in U : N(z) \bigcap \{x\} \neq \emptyset\} = \overline{C_3}(\{x\}) \subseteq K$. This proves that $D(x) \subseteq K$. So $(U, \mathcal{C})$ is an S_r-space. □

Theorem 3.7. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\overline{C_3}(K) = K$ for each $K \in \mathcal{C}$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $K \in \mathcal{C}$. By Lemma 3.6, $\overline{C_3}(\{x\}) \subseteq K$ for each $x \in K$. By Lemma 2.3(4), $\overline{C_3}(K) = \bigcup\{\overline{C_3}(\{x\}) : x \in K\} \subseteq K$. On the other hand, by Lemma 2.3(3), $K \subseteq \overline{C_3}(K)$. Consequently, $\overline{C_3}(K) = K$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$. Then $\overline{C_3}(K) = K$. By Lemma 2.3(3), $\overline{C_3}(\{x\}) \subseteq \overline{C_3}(K) = K$. By Lemma 3.6, $(U, \mathcal{C})$ is an S_r-space. □

The following shows that "$\underline{C_4}(X) \subseteq X$" in Theorem 1.6(4) can not be replaced by "$\underline{C_4}(X) = X$"

Example 3.8. *There exists a covering approximation space $(U, \mathcal{C})$ such that $(U, \mathcal{C})$ is an S_r-space and $\underline{C_4}(X) \neq X$ for some $X \subseteq U$.*

Proof. Let $U = \{a, b.c\}$ and $\mathcal{C} = \{\{a, b\}, \{c\}\}$. Then $(U, \mathcal{C})$ is a Pawlak approximation space. It is known that each Pawlak approximation space is an S_r-space (see [1, Remark 3.4]). Put $X = \{a, c\}$. It is not difficult to check that $\underline{C_4}(X) = \{c\}$. So $\underline{C_4}(X) \neq X$. □

However, we have the following.

Theorem 3.9. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\underline{C_4}(K) = K$ for each $K \in \mathcal{C}$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $K \in \mathcal{C}$. By Theorem 1.6 and Proposition 2.6, $\underline{C_4}(K) \subseteq K$. On the other hand, Let $x \in K$. Then $x \in N(x)$ and $N(x) \subseteq K$. By the definition of $\underline{C_4}(K)$, $x \in \underline{C_4}(K)$. This proves that $K \subseteq \underline{C_4}(K)$. Consequently, $\underline{C_4}(K) = K$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$, then $\overline{C_4}(K) \subseteq K$. If $y \in D(x)$, then $x \in N(y)$ from Lemma 2.5. Note that $N(x) \subseteq K$. So $y \in \{z \in U : \exists u(u \in N(z) \wedge N(u) \subseteq K)\} = \overline{C_4}(K) \subseteq K$. This proves that $D(x) \subseteq K$. So $(U, \mathcal{C})$ is an S_r-space. $\square$

Lemma 3.10. *Let $(U, \mathcal{C})$ be a covering approximation space and $X \subset U$. Then $\underline{C_5}(X) = X$ if and only if $\overline{C_5}(X) = X$.*

Proof. Necessity: Suppose that $\underline{C_5}(X) = X$. Let $y \in \overline{C_5}(X) = \bigcup\{N(x) : x \in U \wedge N(x) \bigcap X \neq \emptyset\}$. Then there is $z \in U$ such that $y \in N(z)$ and $N(z) \bigcap X \neq \emptyset$. Pick $v \in N(z) \bigcap X$, then $v \in X = \underline{\mathcal{C}_5}(X) = \{x \in U : \forall u(x \in N(u) \Longrightarrow N(u) \subseteq X)\}$. It follows that $N(z) \subseteq X$ since $v \in N(z)$. So $y \in N(z) \subseteq X$. This proves that $\overline{\mathcal{C}_5}(X) \subseteq X$. By Lemma 2.3(2), $X \subseteq \overline{\mathcal{C}_5}(X)$. Consequently, $\overline{\mathcal{C}_5}(X) = X$.

Sufficiency: Suppose that $\overline{\mathcal{C}_5}(X) = X$. By Lemma 2.3(2), $\underline{\mathcal{C}_5}(X) \subseteq X$. It suffices to prove that $X \subseteq \underline{\mathcal{C}_5}(X)$. If $X \not\subseteq \underline{\mathcal{C}_5}(X)$, then there is $y \in X$ such that $y \notin \underline{\mathcal{C}_5}(X) = \{x \in U : \forall u(x \in N(u) \Longrightarrow N(u) \subseteq X)\}$. So there is $v \in U$ such that $y \in N(v) \not\subseteq X$. Pick $z \in N(v)$ such that $z \notin X$. Note that $y \in N(v) \bigcap X$. So $N(v) \bigcap X \neq \emptyset$. Thus $z \in \bigcup\{N(x) : x \in U \wedge N(x) \bigcap X \neq \emptyset\} = \overline{\mathcal{C}_5}(X) = X$. This contradicts that $z \notin X$. $\square$

Lemma 3.11. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\overline{C_5}(\{x\}) \subseteq K$ for each $x \in U$ and each $K \in \mathcal{C}_x$.

Proof. (1) $\Longrightarrow$ (2). Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $x \in U$ and $K \in \mathcal{C}_x$, then $D(x) \subseteq K$. If $y \in \overline{C_5}(\{x\}) = \bigcup\{N(z) : z \in U \wedge x \in N(z)\}$, then there exists $z \in U$ such that $x \in N(z)$ and $y \in N(z)$. By Lemma 2.5, $z \in D(x) \subseteq K$, hence $N(z) \subseteq K$. It follows that $y \in N(z) \subseteq K$. This proves that $\overline{C_5}(\{x\}) \subseteq K$.

(2) $\Longrightarrow$ (1). Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$, then $\overline{C_5}(\{x\}) \subseteq K$. If $y \in D(x)$, then $x \in N(y)$ from Lemma 2.5. So $N(y) \subseteq \bigcup\{N(z) : z \in U \wedge x \in N(z)\} = \overline{C_5}(\{x\}) \subseteq K$. It follows that $y \in N(y) \subseteq K$. This proves that $D(x) \subseteq K$. So $(U, \mathcal{C})$ is an S_r-space. $\square$

Theorem 3.12. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ is an S_r-space.

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(2) $\overline{C_5}(K) = K$ *for each* $K \in \mathcal{C}$.
(3) $\underline{C_5}(K) = K$ *for each* $K \in \mathcal{C}$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $K \in \mathcal{C}$. By Lemma 3.11, $\overline{C_5}(\{x\}) \subseteq K$ for each $x \in K$. By Lemma 2.3(4), $\overline{C_5}(K) = \bigcup\{\overline{C_5}(\{x\}) : x \in K\} \subseteq K$. On the other hand, by Lemma 2.3(2), $K \subseteq \overline{C_5}(K)$. Consequently, $\overline{C_5}(K) = K$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$, then $\overline{C_5}(K) = K$. By Lemma 2.3(3), $\overline{C_3}(\{x\}) \subseteq \overline{C_5}(K) = K$. By Lemma 3.11, $(U, \mathcal{C})$ is an S_r-space.

(2) $\Longleftrightarrow$ (3): It holds by Lemma 3.10. □

Theorem 3.13. *Let* $(U, \mathcal{C})$ *be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ *is an* S_r*-space.*
(2) $\underline{C_6}(K) = K$ *for each* $K \in \mathcal{C}$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $K \in \mathcal{C}$. Then $\underline{C_6}(K) \subseteq K$ by Lemma 2.3(2). It suffices to prove that $K \subseteq \underline{C_6}(K)$. Let $x \in K$, then $D(x) \subseteq K$ since $(U, \mathcal{C})$ is an S_r-space. For each $u \in U$, if $x \in N(u)$, then $u \in D(x)$ by Lemma 2.5. It follows that $u \in K$. So $x \in \{z \in U : \forall u(z \in N(u) \to u \in K)\} = \underline{C_6}(K)$. This proves that $K \subseteq \underline{C_6}(K)$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$, then $\underline{C_6}(K) = K$. If $y \in D(x)$, then $x \in N(y)$ from Lemma 2.5. $x \in K = \underline{C_6}(K) = \{x \in U : \forall u(x \in N(u) \to u \in K)\}$, so $x \in N(u)$ implies $u \in K$ for each $u \in U$. It follows that $y \in K$ since $x \in N(y)$. This proves that $D(x) \subseteq K$. So $(U, \mathcal{C})$ is an S_r-space. □

4 Conclusions

This paper answers an open problem posed by Z.Yun et al. in [16]. We give some simple characterizations for S_r-space $(U, \mathcal{C})$ by using only a single covering approximation operator and by using only elements of covering. The main results are summarized as follows.

Theorem 4.1. *Let* $(U, \mathcal{C})$ *be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ *ia an* S_r*-space.*
(2) $\{N(x) : x \in U\}$ *forms a partition of* U.
(3) $\overline{C_2}(\{x\}) \subseteq K$ *for each* $x \in U$ *and each* $K \in \mathcal{C}_x$.
(4) $\overline{C_3}(K) = K$ *for each* $K \in \mathcal{C}$.
(5) $\underline{C_4}(K) = K$ *for each* $K \in \mathcal{C}$.
(6) $\overline{C_5}(K) = K$ *for each* $K \in \mathcal{C}$.
(7) $\underline{C_5}(K) = K$ *for each* $K \in \mathcal{C}$.
(8) $\underline{C_6}(K) = K$ *for each* $K \in \mathcal{C}$.

In the previous sections, covering approximation operators $\mathcal{C}_2$-$\mathcal{C}_6$ are used for our discussion. However, there are also other useful covering approximation operators, which play an important role in research of covering approximation spaces [7, 11, 17, 20, 21].

Definition 4.2 ([20]). *Let (U, C) be a covering approximation space and $x \in U$.*
$Md(x) = \{K : K \in \mathcal{C}_x \wedge (\forall S \in \mathcal{C}_x \wedge S \subseteq K \rightarrow K = S)\}$
is called the minimal description of x.

Definition 4.3 ([20]). *Let (U, C) be a covering approximation space and $X \subseteq U$. Put*

(1) $CL(X) = \bigcup\{K : K \in \mathcal{C} \wedge K \subseteq X\}$;
(2) $FH(X) = CL(X) \bigcup\{Md(x) : x \in X - CL(X)\}$;
(3) $SH(X) = \bigcup\{K : K \in \mathcal{C} \wedge K \bigcap X \neq \emptyset\}$;
(4) $TH(X) = \bigcup\{Md(x) : x \in X\}$;
(5) $RH(X) = CL(X) \bigcup\{K : K \in \mathcal{C} \wedge K \bigcap(X - CL(X)) \neq \emptyset\}$;
(6) $IH(X) = CL(X) \bigcup\{N(x) : x \in X - CL(X)\}$.

CL is called covering lower approximation operation. FH, SH, TH, RH and IH are called the first, the second, the third, the fourth, and the fifth covering upper approximation operations, respectively.

Can we characterize the conditions under which $(U, \mathcal{C})$ is an S_r-space by using only a single covering approximation operator in Definition 4.3? It is an interesting question and is still worthy to be considered in research of covering approximation spaces.

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Difference of Generalized Composition Operators from H^∞ to the Bloch Space

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[1]

Abstract

We characterized the difference of generalized composition operator on the bounded analytic function space to the Bloch space in the disk. The boundedness and compactness of it were investigated.

1 Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane, and $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. The algebra of all holomorphic functions with domain $\mathbb{D}$ will be denoted by $H(\mathbb{D})$.

The Bloch space $\mathcal{B}$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)\,|f'(z)| < \infty,$$

then $\|\cdot\|_{\mathcal{B}}$ is a complete semi-norm on $\mathcal{B}$, which is Möbius invariant.

The space $\mathcal{B}$ becomes a Banach space with the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}}.$$

Denote $H^\infty(\mathbb{D})$ by H^∞ ,the space of all bounded analytic functions in the unit disk with the norm

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

Let φ be an analytic self-map of $\mathbb{D}$, and $g \in H(\mathbb{D})$, the generalized composition operator C_φ^g induced by φ and g is defined by

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi,$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

The definition of the generalized composition was first introduced by S. Li, S. Stević in [9], and in the paper, the boundedness and compactness of the generalized composition operator on Zygmund spaces and Bloch type spaces were investigated by them.

In the past few decades, boundedness, compactness, isometries and essential norms of composition and closely related operators between various spaces of holomorphic functions have been studied by many authors, see, e.g., [1, 2, 6, 14, 18, 19, 21, 22]. Recently, many papers focused on studying the mapping properties of the difference of two composition operators, i.e.,

$$(C_\varphi - C_\psi)(f) = f\circ\varphi - f\circ\psi.$$

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Differences of composition operators were studies first on hardy space $H^2(\mathbb{D})$ (see,e.g[3]). In [13], MacCluer, Ohno and Zhao, characterized the compactness of the difference of two composition operators on $H^\infty(\mathbb{D})$ in terms of the Poincaré distance. A fewer years later, these results were extended to the setting of $H^\infty(B_n)$ by Toews [20] . In [23], Z. H. Zhou and L. Zhang discussed the differences of the products of integral type and composition operators from H^∞ to the Bloch space, more results ,for example, can be seen in [4, 5, 8, 15, 16, 17].

Building on those foundation, this paper continues the research of this part, and discusses the difference of two generalized composition operators from the bounded analytic function space to the Bloch space in the disk.

2 Notation and Lemmas

First, we will introduce some notation and state a couple of lemmas.

For $a \in \mathbb{D}$, the involution φ_a which interchanges the origin and point a, is defined by

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z}.$$

For z, w in $\mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z,w) = |\varphi_z(w)| = \left|\frac{z-w}{1-\overline{z}w}\right|,$$

and the hyperbolic metric is given by

$$\beta\left(z,w\right) = \inf_{\gamma} \int\limits_{\gamma} \frac{|d\xi|}{1-|\xi|^2} = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)},$$

where γ is any piecewise smooth curve in $\mathbb{D}$ from z to w.

The following lemma is well known in [24].

Lemma 1. *For all $z, w \in \mathbb{D}$, we have*

$$1-\rho^2(z,w) = \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2}.$$

A little modification of Lemma 1 in [7] shows the following lemma.

Lemma 2. *There exists a constant $C > 0$ such that*

$$\left|\left(1-|z|^2\right)f'(z) - \left(1-|w|^2\right)f'(w)\right| \le C\,\|f\|_{\mathcal{B}} \cdot \rho(z,w)$$

for all $z, w \in \mathbb{D}$ and $f \in \mathcal{B}$.

Lemma 3. *Assume that $f \in H^\infty(\mathbb{D})$, then for each $n \in N$, there is a positive constant C independent of f such that*

$$\sup_{z\in D}(|1-|z|)^n \left|f^{(n)}(z)\right| < C||f||_\infty.$$

Remark The Lemma 3 can be concluded from [11].

Throughout the remainder of this paper, we will denote $\frac{1-|z|^2}{1-|\varphi(z)|^2}$ by the φ^* and constants are denoted by C , they are positive and not necessarily the same in each appearance.

3 Main theorems

Theorem 1. *Let φ_1, φ_2 be analytic self-maps of the unit disk and $g_1, g_2 \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ *is bounded;*

(ii)
$$\sup_{z\in D} |\varphi_1^*(z)|\, |g_1(z)|\, \rho(\varphi_1(z), \varphi_2(z)) < \infty \tag{1}$$

$$\sup_{z\in D} |\varphi_2^*(z)|\, |g_2(z)|\, \rho(\varphi_1(z), \varphi_2(z)) < \infty \tag{2}$$

and
$$\sup_{z\in D} |\varphi_1^*(z) g_1(z) - \varphi_2^*(z) g_2(z)| < \infty. \tag{3}$$

Proof. We first prove $(ii) \Rightarrow (i)$. Assume that $(1), (2), (3)$ hold.

As the definition of C_φ^g, obviously, $\left|(C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2}) f(0) = 0\right|$

By Lemma 2 and Lemma3, for every $f \in H^\infty$, we have

$$\begin{aligned}
& ||C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2}||_{\mathcal{B}} \\
= \ & \sup_{z\in\mathbb{D}} (1-|z|^2)\, |f'(\varphi_1(z)) g_1(z) - f'(\varphi_2(z)) g_2(z)| \\
= \ & \sup_{z\in\mathbb{D}} \left|(1-|\varphi_1(z)|^2)\varphi_1^*(z) f'(\varphi_1(z)) g_1(z) - (1-|\varphi_2(z)|^2)\varphi_2^*(z) f'(\varphi_2(z)) g_2(z)\right| \\
\leq \ & \sup_{z\in\mathbb{D}} |\varphi_1^*(z) g_1(z)| \left|(1-|\varphi_1(z)|^2) f'(\varphi_1(z)) - (1-|\varphi_2(z)|^2) f'(\varphi_2(z))\right| \\
+ \ & \sup_{z\in\mathbb{D}} (1-|\varphi_2(z)|^2)\, |f'(\varphi_2(z))|\, |g_1(z)\varphi_1^*(z)) - g_2(z)\varphi_2^*(z))| \\
\leq \ & C \sup_{z\in\mathbb{D}} |\varphi_1^*(z) g_1(z)|\, \rho(\varphi_1(z), \varphi_2(z))\, \|f\|_{\mathcal{B}} \\
+ \ & \sup_{z\in\mathbb{D}} |g_1(z)\varphi_1^*(z)) - g_2(z)\varphi_2^*(z))|\, \|f\|_{\mathcal{B}} \\
\leq \ & C\, \|f\|_\infty .
\end{aligned}$$

That is $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2}$ is bounded.

Next we show that (i) implies (ii). We assume $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ is bounded.

For every $\omega \in \mathbb{D}$, we take the test function

$$f_{\varphi_1,\omega}(z) = \frac{\varphi_1(\omega) - z}{1 - \overline{\varphi_1(\omega)} z}.$$

We can obtain easily that $f_{\varphi_1,\omega} \in H^\infty$ and $||f_{\varphi_1,\omega}||_\infty \leq 1$.
Therefore, we have

$$\begin{aligned}
& C \geq ||(C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2}) f_{\varphi_1,\omega}||_{\mathcal{B}} \\
= \ & \sup_{z\in\mathbb{D}} (1-|z|^2) \left|f'_{\varphi_1,\omega}(\varphi_1(z)) g_1(z) - f'_{\varphi_1,\omega}(\varphi_2(z)) g_2(z)\right| \\
\geq \ & (1-|\omega|^2) \left|f'_{\varphi_1,\omega}(\varphi_1(\omega)) g_1(\omega) - f'_{\varphi_1,\omega}(\varphi_2(\omega)) g_2(\omega)\right| \\
= \ & \left|\varphi_1^*(\omega) g_1(\omega) - \frac{(1-|\varphi_1(\omega)|^2)(1-|\varphi_2(\omega)|^2)}{(1-\overline{\varphi_1(\omega)}\varphi_2(\omega))^2} \varphi_2^*(\omega) g_2(\omega)\right| \\
\geq \ & \left||\varphi_1^*(\omega) g_1(\omega)| - (1-\rho(\varphi_1(\omega), \varphi_2(\omega))^2\, |\varphi_2^*(\omega) g_2(\omega)|\right| .
\end{aligned}$$

This is

$$\left||\varphi_1^*(\omega)g_1(\omega)| - (1-\rho(\varphi_1(\omega),\varphi_2(\omega))^2\,|\varphi_2^*(\omega)g_2(\omega)|\right| \leq C. \tag{4}$$

Similarly,letting the test function $f_{\varphi_2,\omega}(z) = \frac{\varphi_2(\omega)-z}{1-\overline{\varphi_2(\omega)}z}$, we can obtain

$$\left||\varphi_2^*(\omega)g_2(\omega)| - (1-\rho(\varphi_1(\omega),\varphi_2(\omega))^2\,|\varphi_1^*(\omega)g_1(\omega)|\right| \leq C. \tag{5}$$

We take the test functions as follow:

$$f(z) = f^2_{\varphi_1,\omega}(z) = (\frac{\varphi_1(\omega)-z}{1-\overline{\varphi_1(\omega)}z})^2, g(z) = f^2_{\varphi_2,\omega}(z) = (\frac{\varphi_2(\omega)-z}{1-\overline{\varphi_2(\omega)}z})^2. \tag{6}$$

The following conclusions can be easily concluded

$$2(1-\rho(\varphi_1(\omega),\varphi_2(\omega))^2)\rho(\varphi_1(\omega),\varphi_2(\omega))\,|\varphi_2^*(\omega)g_2(\omega)| \leq C, \tag{7}$$

$$2(1-\rho(\varphi_1(\omega),\varphi_2(\omega))^2)\rho(\varphi_1(\omega),\varphi_2(\omega))\,|\varphi_1^*(\omega)g_1(\omega)| \leq C. \tag{8}$$

If $\rho(\varphi_1(\omega),\varphi_2(\omega)) \leq \frac{1}{2}$,then by (8), we have

$$|\varphi_1^*(z)g_1(z)|\,\rho(\varphi_1(z),\varphi_2(z)) < C.$$

If $\rho(\varphi_1(\omega),\varphi_2(\omega)) > \frac{1}{2}$,then by (7), we have

$$(1-\rho(\varphi_1(\omega),\varphi_2(\omega))^2)\,|\varphi_2^*(\omega)g_2(\omega)| \leq C,$$

then, $|\varphi_1^*(\omega))g_1(\omega)| \leq C$ is followed by (4), so

$$|\varphi_1^*(\omega)|\,|g_1(\omega)|\,\rho(\varphi_1(\omega),\varphi_2(\omega)) < C.$$

We can get (1) by use of the arbitrary of ω. Analogously, (2) was also can be obtained.

Finally, in order to prove the condition (3), using Lemma 2 and Lemma 3, we have

$$\begin{aligned}
& C \geq ||(C^{g_1}_{\varphi_1} - C^{g_2}_{\varphi_2})f_{\varphi_1,\omega}||_{\mathcal{B}} \\
\geq \quad & |g_1(\omega)\varphi_1^*(\omega) - g_2(\omega)\varphi_2^*(\omega)| \\
- \quad & |g_2(\omega)\varphi_2^*(\omega)| \left|1 - \frac{(1-|\varphi_1(\omega)|^2)(1-|\varphi_2(\omega)|^2)}{(1-\overline{\varphi_1(\omega)}\varphi_2(\omega))^2}\right| \\
\geq \quad & |g_1(\omega)\varphi_1^*(\omega) - g_2(\omega)\varphi_2^*(\omega)| \\
- \quad & |g_2(\omega)\varphi_2^*(\omega)| \left|(1-|\varphi_1(\omega)|^2)f'_{\varphi_1,\omega}(\varphi_1(\omega)) - (1-|\varphi_2(\omega)|^2)f'_{\varphi_1,\omega}(\varphi_2(\omega))\right| \\
\geq \quad & |g_1(\omega)\varphi_1^*(\omega) - g_2(\omega)\varphi_2^*(\omega)| - C\,|g_2(\omega)\varphi_2^*(\omega)|\,\rho(\varphi_1(\omega),\varphi_2(\omega)).
\end{aligned}$$

Then,

$$\sup_{z\in D} |\varphi_1^*(z)g_1(z) - \varphi_2^*(z)g_2(z)| < \infty.$$

This is completes the proof of this theorem. □

By the studying similarly to the proof of Theorem 3.2 in the paper [7], the following theorem can be obtained.

Theorem 2. *Let φ_1, φ_2 be analytic self-maps of the unit disk and $g_1, g_2 \in H(\mathbb{D})$, $C_{\varphi_1}^{g_1}, C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ are bounded but not compact, Then the following statements are equivalent.*

(i) $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ *is compact;*

(ii) Both (a) *and* (b) *hold:*

$$(a)\Gamma^*(\varphi_1) = \Gamma^*(\varphi_2) \neq \emptyset, then \Gamma^*(\varphi_1) \subset \Gamma(\varphi_1) \cap \Gamma(\varphi_2)$$

$$(b) For \{z_n\} \in \Gamma(\varphi_1) \cap \Gamma(\varphi_2),$$

$$\lim_{n\to\infty} |\varphi_1^*(z_n)|\, |g_1(z_n)|\, \rho(\varphi_1(z_n), \varphi_2(z_n)) = 0$$

$$\lim_{n\to\infty} |\varphi_2^*(z_n)|\, |g_2(z_n)|\, \rho(\varphi_1(z_n), \varphi_2(z_n)) = 0$$

and

$$\lim_{n\to\infty} |\varphi_1^*(z_n) g_1(z_n) - \varphi_2^*(z_n) g_2(z_n)| = 0$$

(iii)

$$\lim_{|\lambda|\to 1} ||(C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2})\varphi_\lambda||_{\mathcal{B}} = 0$$

and

$$\lim_{|\lambda|\to 1} ||(C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2})(\varphi_\lambda)^2||_{\mathcal{B}} = 0.$$

Here, $\Gamma(\varphi_1)$ is the set of sequence $\{z_n\}$ in $\mathbb{D}$ such that $|\varphi_1(z_n)| \to 1$. $\Gamma^*(\varphi_1)$ is the set of sequence $\{z_n\}$ in $\mathbb{D}$ such that $|\varphi_1(z_n)| \to 1$ and $\varphi_1^*(z_n) g_1(z_n)$ does not approach the 0.

Next, the other major theorem will be given

Theorem 3. *Let φ_1, φ_2 be analytic self-maps of the unit disk and $g_1, g_2 \in H(\mathbb{D})$, $C_{\varphi_1}^{g_1}, C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ are bounded, Then the following statements are equivalent.*

(i) $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ *is compact;*

(ii)

$$\lim_{|\varphi_1(z)|\to 1} |\varphi_1^*(z)|\, |g_1(z)|\, \rho(\varphi_1(z), \varphi_2(z)) = 0$$

$$\lim_{|\varphi_2(z)|\to 1} |\varphi_2^*(z)|\, |g_2(z)|\, \rho(\varphi_1(z), \varphi_2(z)) = 0$$

and

$$\lim_{|\varphi_1(z)|, |\varphi_2(z)|\to 1} |\varphi_1^*(z) g_1(z) - \varphi_2^*(z) g_2(z)| = 0.$$

Proof. We first prove $(i) \Rightarrow (ii)$.We assume that $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ is compact, then, $C_{\varphi_1}^{g_1}, C_{\varphi_2}^{g_2}$ are compact or noncompact.

If they are compact, the following conclusions are obtained obviously by the Theorem 2 in [12],

$$\lim_{|\varphi_1(z)|\to 1} |\varphi_1^*(z)|\, |g_1(z)| = 0, \lim_{|\varphi_2(z)|\to 1} |\varphi_2^*(z)|\, |g_2(z)| = 0,$$

then, the (ii) holds by them.

If they are all noncompact, for a sequence $\{z_n\}$, such that $|\varphi_1(z_n)| \to 1$, if

$$|\varphi_1^*(z_n)|\, |g_1(z_n)| \to 0,$$

then,

$$\lim_{n\to\infty} |\varphi_1^*(z_n)|\, |g_1(z_n)|\, \rho(\varphi_1(z_n), \varphi_2(z_n)) = 0;$$

if

$$\lim_{n\to\infty} |\varphi_1^*(z_n)|\, |g_1(z_n)| \neq 0,$$

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then $\{z_n\} \in \Gamma^*(\varphi_1)$. By Theorem 2,

$$\{z_n\} \in \Gamma^*(\varphi_1) \subset \Gamma(\varphi_1) \cap \Gamma(\varphi_2),$$

and

$$\lim_{n\to\infty} |\varphi_1^*(z_n)|\,|g_1(z_n)|\,\rho(\varphi_1(z_n), \varphi_2(z_n)) = 0.$$

Hence,

$$\lim_{|\varphi_1(z)|\to 1} |\varphi_1^*(z)|\,|g_1(z)|\,\rho(\varphi_1(z), \varphi_2(z)) = 0.$$

According to similarly proof, we can get

$$\lim_{|\varphi_2(z)|\to 1} |\varphi_2^*(z)|\,|g_2(z)|\,\rho(\varphi_1(z), \varphi_2(z)) = 0.$$

For $\{z_n\}$ such that $|\varphi_1(z_n)|, |\varphi_2(z_n)| \to 1$, using Theorem 2, we have

$$\lim_{n\to\infty} |\varphi_1^*(z_n)g_1(z_n) - \varphi_2^*(z_n)g_2(z_n)| = 0.$$

Due to the arbitrary of $\{z_n\}$, we have

$$\lim_{|\varphi_1(z)|,|\varphi_2(z)|\to 1} |\varphi_1^*(z)g_1(z) - \varphi_2^*(z)g_2(z)| = 0.$$

This is completes the proof of $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$ If the operators $C_{\varphi_1}^{g_1}, C_{\varphi_2}^{g_2}$ are all noncompact, (i) holds obviously by Theorem 2. If one of the operators $C_{\varphi_1}^{g_1}, C_{\varphi_2}^{g_2}$ is compact, we may also assume that $C_{\varphi_1}^{g_1}$ is compact, then by the Theorem 2 in [10], we have

$$\lim_{|\varphi_1(z)|\to 1} |\varphi_1^*(z)|\,|g_1(z)| = 0.$$

Let $\{z_n\}$ be an arbitrary sequence in $\mathbb{D}$, such that $|\varphi_2(z_n)| \to 1$ as $n \to \infty$.

If $|\varphi_1(z_n)|$ approach 1, since

$$\lim_{|\varphi_1(z)|,|\varphi_2(z)|\to 1} |\varphi_1^*(z)g_1(z) - \varphi_2^*(z)g_2(z)| = 0,$$

We obtain

$$\lim_{n\to\infty} |\varphi_2^*(z_n)|\,|g_2(z_n)| = 0.$$

If $|\varphi_1(z_n)|$ does not approach 1, then $\rho(\varphi_1(z), \varphi_2(z))$ does not approach 0, since,

$$\lim_{|\varphi_2(z)|\to 1} |\varphi_2^*(z)|\,|g_2(z)|\,\rho(\varphi_1(z), \varphi_2(z)) = 0.$$

We also obtain

$$\lim_{n\to\infty} |\varphi_2^*(z_n)|\,|g_2(z_n)| = 0.$$

Due to the arbitrary of $\{z_n\}$, we have

$$\lim_{|\varphi_2(z)|\to 1} |\varphi_2^*(z)|\,|g_2(z)| = 0.$$

Therefore, $C_{\varphi_2}^{g_2}$ is a compact operator, therefore, $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2}$ is compact.

This is completes the proof of this theorem. □

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Isometries among the generalized composition operators on Bloch type spaces

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[1]

Abstract

In this paper, we characterize the isometries among the generalized composition operators on Bloch type spaces in the disk.

1 Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane, and $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. The algebra of all holomorphic functions with domain $\mathbb{D}$ will be denoted by $H(\mathbb{D})$.

We recall that the Bloch type space $\mathcal{B}^\alpha$ $(\alpha > 0)$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty,$$

then $\|\cdot\|_{\mathcal{B}^\alpha}$ is a complete semi-norm on $\mathcal{B}^\alpha$, which is Möbius invariant.

It is well known that $\mathcal{B}^\alpha$ is a Banach space under the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}^\alpha}.$$

Let φ be an analytic self-map of $\mathbb{D}$, and $g \in H(\mathbb{D})$, the generalized composition operator C_φ^g induced by φ and g is defined by

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi,$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

The definition of generalized composition operator was first introduced by S. Li, S. Stević in [20], and in the paper, the boundedness and compactness of the generalized composition operator on Zygmund spaces and Bloch type spaces were investigated by them.

If we use the derivative of some function g to instead of g in operator C_φ^g, we can get a new integral operator L_g^φ, which is also called generalized composition operator. Let $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, the operator L_g^φ induced by φ and g is defined by

$$(L_g^\varphi f)(z) = \int_0^z f'(\varphi(\xi))g'(\xi)d\xi,$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

More results about boundedness, compactness, differences and essential norms of composition and closely related operators between various spaces of holomorphic functions have

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been studied by many authors, see, e.g., [12, 18, 19, 21, 25, 27]. Recently, many papers focused on studying isometries of the composition operators on various spaces of holomorphic functions.

Let X and Y be two Banach spaces, and recall that a linear isometry is a linear operator T from X to Y such that $\|Tf\|_Y = \|f\|_X$ for all $f \in X$.

In [3], Banach showed great interet in the form of an isometry on a specific Banach space. In most cases the isometries of a space of analytic functions on the disk or the ball have the canonical form of weighted composition operators, which is also true for most symmetric function spaces. For example, the surjective isometries of Hardy and Bergman spaces are certain weighted composition operators. See [13, 14, 15].

The description of all isometric composition operators is known for the Hardy space H^2 (see [8]). An analogous statement for the Bergman space A^2_α with standard radial weights has recently been obtained in [7], and there is a unified proof for all Hardy spaces and also for arbitrary Bergman spaces with reasonable radial weights [24]. In [9], Colonna gave a characterization of the isometric composition operators on the Bloch space in terms of the factorization of the symbol in H^∞, which shows that there is a very large class of isometries besides the rotations. By contrast, in [26], Zorboska showed that in the case $\alpha \neq 1$, the isometries of the composition operators on $\mathcal{B}^\alpha$ are the operators whose symbol is a rotation.

Continued the work of isometry, in 2008, Bonet, Lindström and Wolf [4] studied isometric weighted composition operators on weighted Banach spaces of type H^∞. Cohen and Colonna [6] discussed the spectrum of an isometric composition operators on the Bloch space of the polydisk. In 2009, Allen and Colonna [1] investigated the isometric composition operators on the Bloch space in $\mathcal{C}^n$. They [2] also discussed the isometries and spectra of multiplication operators on the Bloch space in the disk. Isometries of weighted spaces of holomorphic functions on unbounded domains were discussed by Boyd and Rueda in [5]. In 2010, Li and Zhou discussed the isometries on products of composition and integral operators on Bloch type space in [10].more results ,for example, can be seen in [11, 16, 17, 22, 23].

The paper continues the research of it, and discusses the isometries among the generalized composition operators on Bloch type space in the disk.

2 Notation and Lemmas

First, we will introduce some notations and state a couple of lemmas.

For $a \in \mathbb{D}$, the involution φ_a which interchanges the origin and point a, is defined by

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

For z, w in $\mathbb{D}$, the pseudohyperbolic distance between z and w is given by

$$\rho(z, w) = |\varphi_z(w)| = \left|\frac{z - w}{1 - \overline{z}w}\right|.$$

The following lemma is well known [25].

Lemma 1. *For all $z, w \in \mathbb{D}$, we have*

$$1 - \rho^2(z, w) = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{z}w|^2}.$$

For $\varphi \in S(D)$, the Schwarz-Pick lemma shows that $\rho(\varphi(z), \varphi(w)) \leq \rho(z, w)$, and if equality holds for some $z \neq w$, then φ is an automorphism of the disk. It is also well known that for $\varphi \in S(\mathbb{D})$, C_φ is always bounded on $\mathcal{B}$.

A little modification of Lemma 1 in [4] shows the following lemma.

Lemma 2. *There exists a constant $C > 0$ such that*

$$\left|\left(1-|z|^2\right)^{\alpha} f'(z) - \left(1-|w|^2\right)^{\alpha} f'(w)\right| \le C \|f\|_{\mathcal{B}^\alpha} \cdot \rho(z,w)$$

for all $z, w \in \mathbb{D}$ and $f \in \mathcal{B}^\alpha$.

Throughout the rest of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next.

3 Main theorems

Theorem 1. *Let φ be analytic self maps of the unit disk and $g \in H(\mathbb{D})$. Then the operator $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is an isometry in the semi-norm if and only if the following conditions hold:*

(A) $\sup\limits_{z\in D} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |g(z)| \le 1$;

(B) *For every $a \in \mathbb{D}$, there at least exists a sequence $\{z_n\}$ in $\mathbb{D}$, such that* $\lim\limits_{n\to\infty} \rho(\varphi(z_n), a) = 0$ *and* $\lim\limits_{n\to\infty} \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha} |g(z_n)| = 1$.

Proof. We prove the sufficiency first.

By condition (A), for every $f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned} ||C_\varphi^g f||_{\mathcal{B}^\beta} &= \sup_{z\in D} (1-|z|^2)^\beta |f'(\varphi(z))| |g(z)| \\ &= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |g(z)| (1-|\varphi(z)|^2)^\alpha |f'(\varphi(z))| \\ &\le \|f\|_{\mathcal{B}^\alpha}. \end{aligned}$$

Next we show that the property (B) implies $||C_\varphi^g f||_{\mathcal{B}^\beta} \ge ||f||_{\mathcal{B}^\alpha}$

Given any $f \in \mathcal{B}^\alpha$, then $||f||_{\mathcal{B}^\alpha} = \lim\limits_{m\to\infty} (1-|a_m|^2)^\alpha |f'(a_m)|$ for some sequence $\{a_m\} \subset D$. For any fixed m, by property (B), there is a sequence $\{z_k^m\} \subset \mathbb{D}$ such that

$$\rho(\varphi(z_k^m), a_m) \to 0 \text{ and } \frac{(1-|z_k^m|^2)^\beta}{(1-|\varphi(z_k^m)|^2)^\alpha} |g(z_k^m)| \to 1$$

as $k \to \infty$. By Lemma 2, for all m and k,

$$\left|(1-|\varphi(z_k^m)|^2)^\alpha f'(\varphi(z_k^m)) - (1-|a_m|^2)^\alpha f'(a_m)\right| \le C||f||_{\mathcal{B}^\alpha} \cdot \rho(\varphi(z_k^m), a_m).$$

Hence

$$(1-|\varphi(z_k^m)|^2)^\alpha |f'(\varphi(z_k^m))| \ge (1-|a_m|^2)^\alpha |f'(a_m)| - C||f||_{\mathcal{B}^\alpha} \cdot \rho(\varphi(z_k^m), a_m)$$

Therefore,

$$\begin{aligned} ||C_\varphi^g f||_{\mathcal{B}^\beta} &= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |g(z)| (1-|\varphi(z)|^2)^\alpha |f'(\varphi(z))| \\ &\ge \limsup_{k\to\infty} \frac{(1-|z_k^m|^2)^\beta}{(1-|\varphi(z_k^m)|^2)^\alpha} |g(z_k^m)| (1-|\varphi(z_k^m)|^2)^\alpha |f'(\varphi(z_k^m))| \\ &= (1-|a_m|^2)^\alpha |f'(a_m)|. \end{aligned}$$

The inequality $||C_\varphi^g f||_{\mathcal{B}^\beta} \ge ||f||_{\mathcal{B}^\alpha}$ follows by letting $m \to \infty$.

From the above discussions, we have $||C_\varphi^g f||_{\mathcal{B}^\beta} = ||f||_{\mathcal{B}^\alpha}$, which means that C_φ^g is an isometry operator in the semi-norm from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$.

Now we turn to the necessity.

For any $a \in \mathbb{D}$, we begin by taking test function

$$f_a(z) = \int_0^z \frac{(1-|a|^2)^\alpha}{(1-\bar{a}t)^{2\alpha}} dt. \tag{1}$$

It is clear that $f_a'(z) = \frac{(1-|a|^2)^\alpha}{(1-\bar{a}z)^{2\alpha}}$. Using Lemma 1, we have

$$(1-|z|^2)^\alpha |f_a'(z)| = \frac{(1-|z|^2)^\alpha (1-|a|^2)^\alpha}{|1-\bar{a}z|^{2\alpha}} = (1-\rho^2(a,z))^\alpha. \tag{2}$$

So

$$\|f_a\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f_a'(z)| \le 1. \tag{3}$$

On the other hand, since $(1-|a|^2)^\alpha |f_a'(a)| = \frac{(1-|a|^2)^{2\alpha}}{(1-|a|^2)^{2\alpha}} = 1$, we have $\|f_a\|_{\mathcal{B}^\alpha} = 1$. By isometry assumption, for any $a \in \mathbb{D}$, we have

$$\begin{aligned}
1 &= ||f_{\varphi(a)}||_{\mathcal{B}^\alpha} = ||C_\varphi^g f_{\varphi(a)}||_{\mathcal{B}^\beta} \\
&= \sup_{z\in D} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |g(z)| (1-|\varphi(z)|^2)^\alpha \left| f'_{\varphi(a)}(\varphi(z)) \right| \\
&\ge \frac{(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^\alpha} |g(a)|.
\end{aligned}$$

So (A) follows by the arbitrariness of a.

Since $||f_a||_{\mathcal{B}^\alpha} = ||C_\varphi^g f_a||_{\mathcal{B}^\beta} = 1$, there exists a sequence $\{z_m\} \subset \mathbb{D}$ such that

$$((1-|z_m|^2)^\beta \left| \frac{d(C_\varphi^g f_a)}{dz}(z_m) \right| = (1-|z_m|^2)^\beta |f_a'(\varphi(z_m))||g(z_m)| \to 1 \tag{4}$$

as $m \to \infty$.

It follows from (A) that

$$\begin{aligned}
& (1-|z_m|^2)^\beta |f_a'(\varphi(z_m))||g(z_m)| \\
= \;& \frac{(1-|z_m|^2)^\beta}{(1-|\varphi(z_m)|^2)^\alpha} |g(z_m)| (1-|\varphi(z_m)|^2)^\alpha |f_a'(\varphi(z_m))| && (5)\\
\le \;& (1-|\varphi(z_m)|^2)^\alpha |f_a'(\varphi(z_m))|. && (6)
\end{aligned}$$

Combining (4) and (6), it follows that

$$\begin{aligned}
1 &\le \liminf_{m\to\infty} (1-|\varphi(z_m)|^2)^\alpha |f_a'(\varphi(z_m))| \\
&\le \limsup_{m\to\infty} (1-|\varphi(z_m)|^2)^\alpha |f_a'(\varphi(z_m))| \le 1.
\end{aligned}$$

The last inequality follows by (2) since $\varphi(z_m) \in \mathbb{D}$.

Consequently,

$$\lim_{m\to\infty} (1-|\varphi(z_m)|^2)^\alpha |f_a'(\varphi(z_m))| = \lim_{m\to\infty} (1-\rho^2(\varphi(z_m),a))^\alpha = 1. \tag{7}$$

That is, $\lim_{m\to\infty} \rho(\varphi(z_m), a) = 0$.

Combining (4), (5) and (7), we know

$$\lim_{m\to\infty} \frac{(1-|z_m|^2)^\beta}{(1-|\varphi(z_m)|^2)^\alpha} |g(z_m)| = 1.$$

This completes the proof of this theorem. □

Corollary 1. *Let U denote unitary transformation in the unit disk, then $C_U^1 : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is an isometry in the semi-norm.*

□

If we use the derivative of some function g to instead of g in operator C_φ^g, by the above theorem. we can easily get the following result about the operator L_g^φ.

Theorem 2. *Let φ be analytic self maps of the unit disk and $g \in H(\mathbb{D})$. Then the operator $C_g^\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is an isometry in the semi-norm if and only if the following conditions hold:*

(C) $\sup\limits_{z\in D} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |g'(z)| \le 1$;

(D) *For every $a \in \mathbb{D}$, there at least exists a sequence $\{z_n\}$ in $\mathbb{D}$, such that* $\lim\limits_{n\to\infty} \rho(\varphi(z_n), a) = 0$ *and* $\lim\limits_{n\to\infty} \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha} |g'(z_n)| = 1$.

Remark If $\alpha = 1$, $\beta = 1$, then $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$ will be Bloch space $\mathcal{B}$. There are similar results on the Bloch space $\mathcal{B}$ corresponding to Theorems 1 and 2.

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Coupled Fixed Point Theorems for Generalized Symmetric Contractions in Partially Ordered Metric Spaces and applications

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Abstract

In the setting of partially ordered metric spaces, we introduce the notion of generalized symmetric g-Meir-Keeler type contractions and use the notion to establish the existence and uniqueness of coupled common fixed points. Our notion extends the notion of generalized symmetric Meir-Keeler contractions given by Berinde et. al. [V. Berinde, and M. Pacurar, Coupled fixed point theorems for generalized symmetric Meir-Keeler contractions in ordered metric spaces, Fixed Point Theory and Appl., 2012, 2012:115, doi:10.1186/1687-1812-2012-115] to a pair of mappings. We also give some applications of our main results.

AMS Subject Classification: 47H10, 46T99, 54H25 .

Key Words : partially ordered metric space, fixed point, generalized symmetric contractions, coupled fixed point.

1 Introduction

Banach [1] in his classical work gave the following contractive theorem:

Theorem 1.1. *Let (X,d) be a metric space and $T : X \to X$ be a self mapping. If (X,d) is complete and T is a contraction, that is, there exists a constant $k \in [0,1)$ such that*

$$d(Tx,Ty) \leq kd(x,y), \forall x,y \in X \tag{1.1}$$

then, T has a unique fixed point $u \in X$ and for any $x_0 \in X$, the Picard iteration $\{T^n(x_0)\}$ converges to u.

This contraction principle proved to be a very powerful tool in nonlinear analysis, and different authors have generalized it in many ways. One can refer to the works noted in references [2]- [17]. Meir and Keeler [9] generalized the contraction principle due to Banach by considering a more general contractive condition in their work as follows:

Theorem 1.2. *[9] Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that*

$$\epsilon \leq d(x,y) < \epsilon + \delta(\epsilon) \Rightarrow d(T(x),T(y)) < \epsilon \tag{1.2}$$

*Corresponding author

for all $x, y \in X$. Then T admits a unique fixed point $x_0 \in X$ and for all $x \in X$, the sequence $\{T^n(x)\}$ converges to x_0.

By extending the Banach contraction principle to partially ordered sets, Turinici [16] laid the foundation for a new trend in fixed point theory. Ran and Reurings [17] developed some applications of Turinici's theorem to matrix equations. The work of Bhaskar and Lakshmikantham [18] is worth mentioning, as they introduced the new notion of fixed points for mappings having domain the product space $X \times X$, which they called coupled fixed points, and thereby proved some coupled fixed point theorems for mappings satisfying the mixed monotone property in partially ordered metric spaces. As an application, they discussed the existence and uniqueness of a solution for a periodic boundary value problem. Lakshmikantham and Ciric [19] extended the notion of the mixed monotone property to the mixed g-monotone property and generalized the results of Bhaskar and Lakshmikantham [18] by establishing the existence of coupled coincidence points, using a pair of commutative maps.This proved to be a milestone in the development of fixed point theory with applications to partially ordered sets. Since then much work has been done in this direction by different authors. For more details the reader may consult [20]-[31].
Gordji et. al. [32], extended the results of Bhaskar and Lakshmikantham [18], and Samet [33] by introducing the concept of generalized g-Meir-Keeler type contractions. Abdeljawad et. al. [34] and Jain et. al. [36] proved some interesting results in partially ordered partial metric spaces and remarked that the metric space case of their results, proved recently in Gordji et. al. [32] has gaps. They claimed that some of the results proved by Gordji et. al.[32] cannot be true if obtained via nonstrongly minihedral cones. On the other hand, Berinde et. al. [35] with their outstanding new approach introduced the notion of generalized symmetric Meir-Keeler contractions and complemented the results due to Samet [33]. In this paper, we introduce the notion of generalized symmetric g-Meir-Keeler type contractions that extends the concept of generalized symmetric Meir-Keeler contractions given by Berinde et. al. [35] to a pair of mappings. Following Abdeljawad et. al. [34], we establish the existence and uniqueness of coupled common fixed points for mixed g-monotone mappings satisfying generalized symmetric conditions in partially ordered metric spaces. To validate our results we also give some applications. Before we proceed, we first summarize some basic results and definitions useful in our study.

Definition 1.3. [18] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y; that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \quad \text{implies} \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \quad \text{implies} \quad F(x, y_1) \geq F(x, y_2)$$

Definition 1.4. [18] An element $(x, y) \in X \times X$, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.5. [19] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if $F(x, y)$ is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument; that is, for any $x, y \in X$,
$x_1, x_2 \in X, g(x_1) \leq g(x_2)$ implies $F(x_1, y) \leq F(x_2, y)$
and
$y_1, y_2 \in X, g(y_1) \leq g(y_2)$ implies $F(x, y_1) \geq F(x, y_2)$

Definition 1.6. [19] An element $(x, y) \in X \in X$, is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 1.7. [19] An element $(x, y) \in X \in X$, is called a coupled common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 1.8. [19] Let X be a non-empty set and $F : X \times X \to X$ and $g : X \to X$. We say that F and g are commutative if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.

Later, Choudhury and Kundu[20] introduced the notion of compatibility in the context of coupled coincidence point problems and used this notion to improve the results noted in [19].

Definition 1.9. [20] The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if $\lim_{n\to\infty} d(g(F(x_n, y_n), F(gx_n, gy_n)) = 0$ and $\lim_{n\to\infty} d(g(F(y_n, x_n), F(gy_n, gx_n)) = 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$ for some $x, y \in X$.

Recently, Gordji et. al. [32] replaced the mixed g-monotone property with the mixed strict g-monotone property and extended the results of Bhaskar and Lakshmikantham [18].

Definition 1.10. [32] Let $(X, \leq)$ be a partially ordered set and $F : X\times X \to X$ and $g : X \to X$. We say F has the mixed strict g-monotone property if for any $x, y \in X$,
$x_1, x_2 \in X, g(x_1) < g(x_2)$ implies $F(x_1, y) < F(x_2, y)$
and
$y_1, y_2 \in X, g(y_1) < g(y_2)$ implies $F(x, y_1) > F(x, y_2)$

Here if we replace g with identity mapping in Definition 1.10, we get the definition of mixed strict monotone property of F.

Theorem 1.11. *[36] Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Suppose that there exists a real number $k \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k[d(x, u) + d(y, v)] \tag{1.3}$$

for all $x, y, u, v \in X$ with $x \geq u$, $y \leq v$. Suppose that either
(a) F is continuous
or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n > 0$;
(ii) if a non-decreasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n > 0$;
Then F has a coupled fixed point in X.

We now introduce our notion.

Definition 1.12. Let $(X, \leq)$ be a partially ordered set and d be a metric on X. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. We say that F is a generalized symmetric g-Meir-Keeler type contraction if, for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that , for all $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$ (or $g(x) \geq g(u)$ and $g(y) \leq g(v)$),

$$\epsilon \leq \frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))] < \epsilon + \delta(\epsilon)$$

implies

$$\frac{1}{2}[d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))] < \epsilon \tag{1.4}$$

If, in Definition 1.12, we replace g by the identity mapping, we obtain the definition of a generalized symmetric Meir-Keeler type contraction due to Berinde et. al. [35].

Definition 1.13. [35] Let $(X, \leq)$ be a partially ordered set and d be a metric on X. Let $F : X \times X \to X$ be the given mapping. We say that F is a generalized symmetric Meir-Keeler type contraction if for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that , for all $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$ (or $x \geq u$ and $y \leq v$),

$$\epsilon \leq \frac{1}{2}[d(x,u)+d(y,v)] < \epsilon+\delta(\epsilon)$$

implies

$$\frac{1}{2}[d(F(x,y),F(u,v))+d(F(y,x),F(v,u))] < \epsilon \tag{1.5}$$

Proposition 1.14. *Let $(X,d,\leq)$ be a partially ordered metric space and $F: X\times X \to X$ be a given mapping. If contractive condition (1.3) is satisfied for $0<k<1$, then F is a generalized symmetric Meir-Keeler type contraction.*

Proof. Assume that (1.3) is satisfied for $0<k<1$. For all $\epsilon>0$, it is easy to check that (1.5) is satisfied with $\delta(\epsilon)=\left(\frac{1}{k}-1\right)\epsilon$. □

Lemma 1.15. *Let $(X,\leq)$ be a partially ordered set and d be a metric on X. Let $F: X\times X\to X$ and $g: X\to X$ be two mappings. If F is a generalized symmetric g-Meir-Keeler type contraction, then we have*

$$d(F(x,y),F(u,v))+d(F(y,x),F(v,u)) < d(g(x),g(u))+d(g(y),g(v)) \tag{1.6}$$

for all $x,y,u,v\in X$ with $g(x)<g(u), g(y)\geq g(v)$ (or $g(x)\leq g(u), g(y)>g(v)$).

Proof. Without loss of generality, we may assume that $g(x)<g(u), g(y)\geq g(v)$ where $x,y,u,v\in X$. Then $d(g(x),g(u))+d(g(y),g(v))>0$. Since F is a generalized symmetric g-Meir- Keeler type contraction, for $\epsilon=(\frac{1}{2})[d(g(x),g(u))+d(g(y),g(v))]$, there exists a $\delta(\epsilon)>0$ such that , for all $x_0,y_0,u_0,v_0\in X$ with $g(x_0)<g(u_0)$ and $g(y_0)\geq g(v_0)$,

$$\epsilon \leq \frac{1}{2}[d(g(x_0),g(u_0))+d(g(y_0),g(v_0))] < \epsilon+\delta(\epsilon)$$

implies

$$\frac{1}{2}[d(F(x_0,y_0),F(u_0,v_0))+d(F(y_0,x_0),F(v_0,u_0))] < \epsilon$$

Then the result follows by choosing $x=x_0, y=y_0, u=u_0, v=v_0$; that is,

$$d(F(x,y),F(u,v))+d(F(y,x),F(v,u)) < d(g(x),g(u))+d(g(y),g(v))$$

□

2 Existence of Coupled Coincidence Points

We now establish our first main result.

Theorem 2.1. *Let $(X,\leq,d)$ be a partially ordered metric space. Suppose that X has the following properties:*
(i) if $\{x_n\}$ is a sequence such that $x_{n+1}>x_n$ for each $n=1,2,\ldots$ and $x_n\to x$, then $x_n<x$ for each $n=1,2,\ldots$.
(ii) if $\{y_n\}$ is a sequence such that $y_{n+1}<y_n$ for each $n=1,2,\ldots$ and $y_n\to y$, then $y_n>y$ for each $n=1,2,\ldots$.
Let $F: X\times X\to X$ and $g: X\to X$ be mappings such that $F(X\times X)\subseteq g(X)$ and $g(X)$ is a complete subspace of (X,d). Also, suppose that
(a) F has the mixed strict g-monotone property;
(b) F is a generalized symmetric g-Meir-Keeler type contraction;
(c) there exists $x_0,y_0\in X$ such that $g(x_0)<F(x_0,y_0)$ and $g(y_0)\geq F(y_0,x_0)$(or $g(x_0)\leq F(x_0,y_0)$ and $g(y_0)>F(y_0,x_0)$).
Then, there exist $x,y\in X$ such that $g(x)=F(x,y)$ and $g(y)=F(y,x)$.

Proof. Without loss of generality, we may assume that there exist $x_0, y_0 \in X$ such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0), g(y_1) = F(y_0, x_0)$. Again we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1), g(y_2) = F(y_1, x_1)$. Continuing this process, we construct sequences $\{gx_n\}$ and $\{gy_n\}$ such that

$$g(x_{n+1}) = F(x_n, y_n), g(y_{n+1}) = F(y_n, x_n), \forall n \geq 0 \tag{2.1}$$

Using conditions (a), (c) and mathematical induction, it is easy to see that

$$g(x_0) < g(x_1) < g(x_2) < \ldots < g(x_n) < g(x_{n+1}) < \ldots \tag{2.2}$$

and

$$g(y_{n+1}) < g(y_n) < \ldots < g(y_2) < g(y_1) < g(y_0). \tag{2.3}$$

Denote by

$$\delta_n := d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) \tag{2.4}$$

Using (2.1) of Lemma 1.15, and condition (b), we have

$$\begin{aligned} \delta_n &:= d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) \\ &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &< d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n)) = \delta_{n-1} \end{aligned} \tag{2.5}$$

Thus, the sequence $\{\delta_n\}$ is a decreasing sequence. Therefore there exists some $\delta^* \geq 0$ such that $\lim_{n\to\infty} \delta_n = \delta^*$.

We claim that $\delta^* = 0$. Suppose, to the contrary, that $\delta^* \neq 0$. Then there exists a positive integer m such that, for any $n \geq m$, we have

$$\epsilon \leq \frac{\delta_n}{2} = \frac{1}{2}[d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))] < \epsilon + \delta(\epsilon) \tag{2.6}$$

where $\epsilon = \delta^*/2$ and $\delta(\epsilon)$ is chosen by condition (b).

In particular, for n = m, we have

$$\epsilon \leq \frac{\delta_m}{2} = \frac{1}{2}[d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1}))] < \epsilon + \delta(\epsilon) \tag{2.7}$$

Then, by condition (b), it follows that

$$\frac{1}{2}[d(F(x_m, y_m), F(x_{m+1}, y_{m+1})) + d(F(y_m, x_m), F(y_{m+1}, x_{m+1}))] < \epsilon \tag{2.8}$$

and hence, from (2.1), we have

$$\frac{1}{2}[d(g(x_{m+1}), g(x_{m+2})) + d(g(y_{m+1}), g(y_{m+2}))] < \epsilon \tag{2.9}$$

a contradiction to (2.6) for $n = m + 1$. Thus we must have $\delta^* = 0$ and hence

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))] = 0 \tag{2.10}$$

We now prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Take an arbitrary $\epsilon > 0$. Then, by (2.10), it follows that there exists some $k \in \mathbb{N}$ such that

$$\frac{1}{2}[d(g(x_k), g(x_{k+1})) + d(g(y_k), g(y_{k+1}))] < \delta(\epsilon) \tag{2.11}$$

Without loss of generality, assume that k has been chosen so large that $\delta(\epsilon) \leq \epsilon$ and define the set

$$\begin{aligned} \wedge := \{(g(x), g(y)) : (x, y) \in X^2, d(g(x), g(x_k)) + d(g(y), g(y_k)) < 2(\epsilon + \delta(\epsilon)), \text{ and} \\ g(x) > g(x_k), g(y) \leq g(y_k)\} \end{aligned} \tag{2.12}$$

We claim that $(g(x), g(y)) \in \wedge$ implies that

$$(F(x,y), F(y,x)) \in \wedge \tag{2.13}$$

where $x, y \in X$.
Take $(g(x), g(y)) \in \wedge$. Then, using the triangle inequality and (2.11), we have

$$\begin{aligned}\frac{1}{2}[d(g(x_k), F(x,y)) &+ d(g(y_k), F(y,x))] \leq \frac{1}{2}[d(g(x_k), g(x_{k+1})) + d(g(x_{k+1}), F(x,y))] \\ &+ \frac{1}{2}[d(g(y_k), g(y_{k+1})) + d(g(y_{k+1}), F(y,x))] \\ &= \frac{1}{2}[d(g(x_k), g(x_{k+1})) + d(g(y_k), g(y_{k+1}))] \\ &+ \frac{1}{2}[d(g(x_{k+1}), F(x,y)) + d(g(y_{k+1}), F(y,x))] \\ &< \delta(\epsilon) + \frac{1}{2}[d(F(x,y), F(x_k, y_k)) + d(F(y,x), F(y_k, x_k))] \end{aligned} \tag{2.14}$$

We distinguish two cases.
First Case: $\frac{1}{2}[d(g(x_k), F(x,y)) + d(g(y_k), F(y,x))] \leq \epsilon$. By Lemma 1.15 and Definition of $\wedge$, the inequality (2.14) becomes

$$\begin{aligned}\frac{1}{2}[d(g(x_k), F(x,y)) &+ d(g(y_k), F(y,x))] \leq \delta(\epsilon) \\ &+ \frac{1}{2}[d(F(x,y), F(x_k, y_k)) + d(F(y,x), F(y_k, x_k))] \\ &< \delta(\epsilon) + \frac{1}{2}[d(g(x), g(x_k)) + d(g(y), g(y_k))] \\ &\leq \delta(\epsilon) + \epsilon \end{aligned} \tag{2.15}$$

Second Case: $\epsilon < \frac{1}{2}[d(g(x), g(x_k)) + d(g(y), g(y_k))] < \delta(\epsilon) + \epsilon$. In this case, we have

$$\epsilon < \frac{1}{2}[d(g(x), g(x_k)) + d(g(y), g(y_k))] < \delta(\epsilon) + \epsilon \tag{2.16}$$

Then, since $g(x) > g(x_k)$ and $g(y) \leq g(y_k)$, by condition (b), we have

$$\frac{1}{2}[d(F(x,y), F(x_k, y_k)) + d(F(y,x), F(y_k, x_k))] < \epsilon \tag{2.17}$$

Using (2.17) in (2.14), we get

$$\frac{1}{2}[d(g(x_k), F(x,y)) + d(g(y_k), F(y,x))] < \delta(\epsilon) + \epsilon \tag{2.18}$$

Since F satisfies the mixed strict g -monotone property and $(g(x), g(y)) \in \wedge$, it follows that

$$F(x,y) > g(x_k), F(y,x) > g(y_k) \tag{2.19}$$

Also, $F(X \times X) \subseteq g(X)$. Consequently, we have $(F(x,y), F(y,x)) \in \wedge$; that is (2.13) holds. By (2.11), we have $(g(x_{k+1}), g(y_{k+1})) \in \wedge$.Then, using (2.13), we have

$$\begin{aligned}(g(x_{k+1}), g(y_{k+1})) \in \wedge &\Rightarrow d(F(x_{k+1}, y_{k+1}), F(y_{k+1}, x_{k+1})) = (g(x_{k+2}), g(y_{k+2}) \in \wedge \\ &\Rightarrow d(F(x_{k+2}, y_{k+2}), F(y_{k+2}, x_{k+2})) = (g(x_{k+3}), g(y_{k+3}) \in \wedge \\ &\Rightarrow \ldots \Rightarrow (g(x_n), g(y_n)) \in \wedge \Rightarrow \ldots \end{aligned} \tag{2.20}$$

Then, for all $n > k$, we have $(g(x_n), g(y_n)) \in \wedge$. This implies that, for all $n, m > k$, we have

$$\begin{aligned} d(g(x_n), g(x_m)) &+ d(g(y_n), g(y_m)) \leq d(g(x_n), g(x_k)) + d(g(x_k), g(x_m)) \\ &+ d(g(y_n), g(y_k)) + d(g(y_k), g(y_m)) \\ &= [d(g(x_n), g(x_k)) + d(g(y_n), g(y_k))] + [d(g(x_k), g(x_m)) + d(g(y_k), g(y_m))] \\ &\leq 4(\epsilon + \delta(\epsilon)) \leq 8\epsilon \end{aligned}$$

Therefore, the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy. Since $(g(X), d)$ is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} d(g(x_n), g(x)) = 0, \lim_{n\to\infty} d(g(y_n), g(y)) = 0 \tag{2.21}$$

Since the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are monotone increasing and monotone decreasing, respectively, by conditions (i) and (ii), we have

$$g(x_n) < g(x), \quad g(y_n) > g(y) \tag{2.22}$$

for each $n \geq 0$. Therefore, by (2.22) and Lemma 1.15, along with condition (b), we obtain

$$\begin{aligned} d(g(x_{n+1}), F(x, y)) &+ d(g(y_{n+1}), F(y, x)) \\ &= d(F(x_n, y_n), F(x, y)) + d(F(y_n, x_n), F(y, x)) \end{aligned}$$

$$< d(g(x_n), g(x)) + d(g(y_n), g(y)) \tag{2.23}$$

Letting $n \to \infty$ in (2.23) and using (2.21), we get

$$d(g(x), F(x, y)) + d(g(y), F(y, x)) \leq \lim_{n\to\infty} [d(g(x_n), g(x)) + d(g(y_n), g(y))] \tag{2.24}$$

which yields $F(x, y) = g(x)$, $F(y, x) = g(y)$. This completes the proof. □

Corollary 2.2. *Let $(X, \leq, d)$ be a partially ordered metric space. Suppose that (X, d) is complete and has the following properties:*
(i) if $\{x_n\}$ is a sequence such that $x_{n+1} > x_n$ for each $n = 1, 2, \ldots$ and $x_n \to x$, then $x_n < x$ for each $n = 1, 2, \ldots$.
(ii) if $\{y_n\}$ is a sequence such that $y_{n+1} < y_n$ for each $n = 1, 2, \ldots$ and $y_n \to y$, then $y_n > y$ for each $n = 1, 2, \ldots$.
Let $F : X \times X \to X$ be a mapping. Also, suppose that
(d) F has the mixed strict monotone property;
(e) F is a generalized symmetric Meir-Keeler type contraction;
(f) there exists $x_0, y_0 \in X$ such that $x_0 < F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$(or $x_0 \leq F(x_0, y_0)$ and $y_0 > F(y_0, x_0)$).
Then, there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Remark 2.3. If, in Theorem 2.1 condition (c) is replaced by the following condition:
(g) there exist $x_0, y_0 \in X$ such that $g(x_0) > F(x_0, y_0)$ and $g(y_0) \leq F(y_0, x_0)$ (or $g(x_0) \geq F(x_0, y_0)$ and $g(y_0) < F(y_0, x_0)$),
then we also get the existence of some $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.
And, if in Corollary 2.2, condition (f) is replaced by the following condition:
(h) there exist $x_0, y_0 \in X$ such that $x_0 > F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$
(or $x_0 \geq F(x_0, y_0)$ and $y_0 < F(y_0, x_0)$),
then we also get the existence of some $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Remark 2.4. Corollary 2.2, along with Remark 2.3, improves on the result of Berinde et. al. ([35], Theorem 2) by removing the continuity assumption on the mixed monotone operator F.

3 Existence and Uniqueness of Coupled Fixed Points

In this section we prove the existence and uniqueness of coupled fixed points. Before we proceed, we need to consider the following.
For a partially ordered set $(X, \leq)$, we endow $X \times X$ with the following order $\leq_g$

$$(u,v) \leq_g (x,y) \Rightarrow g(u) < g(x), g(y) \leq g(v), \forall (x,y),(u,v) \in X \times X \tag{3.1}$$

In this case, we say that (u,v) and (x,y) are g -comparable if either $(u,v) \leq_g (x,y)$ or $(x,y) \leq_g (u,v)$. If $g = I_X$, then we simply say that (u,v) and (x,y) are comparable and denote this fact by $(u,v) \leq (x,y)$.

Lemma 3.1. *Let $F : X \times X \to X$ and $g : X \to X$ be compatible maps and suppose there exists an element $(x,y) \in X \times X$ such that $g(x) = F(x,y)$ and $g(y) = F(y,x)$. Then $gF(x,y) = F(g(x), g(y))$ and $gF(y,x) = F(g(y), g(x))$.*

Proof. Since the pair (F, g) is compatible, it follows that

$$\lim_{n\to\infty} d(gF(x_n,y_n), F(g(x_n), g(y_n))) = 0$$

and

$$\lim_{n\to\infty} d(gF(y_n,x_n), F(g(y_n), g(x_n))) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} g(x_n) = a$, $\lim_{n\to\infty} F(y_n,x_n) = \lim_{n\to\infty} g(y_n) = b$ for some $a, b \in X$.
Taking $x_n = x, y_n = y$ and using the fact that $g(x) = F(x,y), g(y) = F(y,x)$, it follows immediately that $d(gF(x,y), F(g(x), g(y))) = 0$ and $d(gF(y,x), F(g(y), g(x))) = 0$.
Hence, $gF(x,y) = F(g(x), g(y))$ and $gF(y,x) = F(g(y), g(x))$. □

Theorem 3.2. *In Theorem 2.1, assume, in addition, that, for all non g-comparable points $(x,y), (x^*,y^*) \in X \times X$, there exists a point $(a,b) \in X \times X$ such that $(F(a,b), F(b,a))$ is comparable to both $(g(x), g(y))$ and $(g(x^*), (y^*))$. Also assume that F and g are compatible. Then,F and g have a unique coupled common fixed point; that is, there exists a point $(u,v) \in X \times X$ such that*

$$u = g(u) = F(u,v), v = g(v) = F(v,u) \tag{3.2}$$

Proof. From Theorem 2.1 it follows that the set of coupled coincidence points of F and g is non-empty. We shall first show that, if (x,y) and (x^*,y^*) are coupled coincidence points, that is, if $g(x) = F(x,y)$, $g(y) = F(y,x)$ and $g(x^*) = F(x^*,y^*)$, $g(y^*) = F(y^*,x^*)$, then

$$g(x) = g(x^*) \text{and} g(y) = g(y*) \tag{3.3}$$

For this, we distinguish the following two cases.
First Case. (x,y) is g-comparable to (x^*,y^*) with respect to the ordering in $X \times X$, where

$$F(x,y) = g(x), F(y,x) = g(y), F(x^*,y^*) = g(x^*), F(y^*,x^*) = g(y^*) \tag{3.4}$$

Without loss of generality, we may assume that

$$g(x) = F(x,y) < F(x^*,y^*) = g(x^*), g(y) = F(y,x) \geq F(y^*,x^*) = g(y^*) \tag{3.5}$$

Using Lemma 1.15 we have

$$0 < d(g(x), g(x^*)) + d(g(y^*), g(y)) = d(F(x,y), F(x^*,y^*)) + d(F(y^*,x^*), F(y,x))$$

$$< d(g(x), g(x^*)) + d(g(y^*), g(y))$$

a contradiction. Therefore, we have $(g(x), g(y)) = (g(x^*), g(y^*))$. Hence (3.3) holds.
Second Case. (x,y) is not g-comparable to (x^*,y^*).

By assumption, there exists a point $(a,b) \in X \times X$ such that $(F(a,b), F(b,a))$ is comparable to both $(g(x), g(y))$ and $(g(x^*), g(y^*))$. Then we have

$$g(x) = F(x,y) < F(a,b), \quad F(x^*,y^*) = g(x^*) < F(a,b), \tag{3.6}$$

and

$$g(y) = F(y,x) \geq F(b,a), \quad F(y^*,x^*) = g(y^*) \geq F(b,a), \tag{3.7}$$

Further, setting $x = x_0, y = y_0, a = a_0, b = b_0$ and $x^* = x_0^*, y^* = y_0^*$ as in the proof of Theorem 2.1, we obtain

$$\begin{aligned} g(x_{n+1}) &= F(x_n, y_n), \quad g(y_{n+1}) = F(y_n, x_n), \forall n = 0,1,2,\dots \\ g(a_{n+1}) &= F(a_n, b_n), \quad g(b_{n+1}) = F(b_n, a_n), \forall n = 0,1,2,\dots \\ g(x_{n+1}^*) &= F(x_n^*, y_n^*), \quad g(y_{n+1}^*) = F(y_n^*, x_n^*), \forall n = 0,1,2,\dots \end{aligned} \tag{3.8}$$

Since $(F(x,y), F(y,x)) = (g(x), g(y)) = (g(x_1), g(y_1))$ is comparable with $(F(a,b), F(b,a)) = (g(a_1), g(b_1))$, we have $g(x) < g(a_1)$ and $g(y) \geq g(b_1)$.Using the fact that F has the mixed strict g -monotone property, $g(x) < g(a_n)$ and $g(b_n) < g(y)$ for all $n \geq 2$. Thus, by Lemma 1.15, we have

$$\begin{aligned} 0 < d(g(x), g(a_{n+1})) + d(g(y), g(b_{n+1})) &= d(F(x,y), F(a_n, b_n)) + d(F(y,x), F(b_n, a_n)) \\ &< d(g(x), g(a_n)) + d(g(y), g(b_n)) \end{aligned} \tag{3.9}$$

Let $\alpha_n = d(g(x), g(a_n)) + d(g(y), g(b_n))$. Then, by (3.9), it follows that $\{\alpha_n\}$ is a decreasing sequence, and hence converges to some $\alpha \geq 0$. We claim that $\alpha = 0$. Suppose, to the contrary, that $\alpha > 0$. Then there exists a positive integer p such that, for $n \geq p$, we have

$$\epsilon \leq \frac{a_n}{2} = \frac{1}{2}[d(g(x), g(a_n)) + d(g(y), g(b_n))] < \epsilon + \delta(\epsilon), \tag{3.10}$$

where $\epsilon = \frac{\alpha}{2}$ and $\delta(\epsilon)$ is chosen by condition (b) of Theorem 2.1. In particular, for $n = p$,

$$\epsilon \leq \frac{a_p}{2} = \frac{1}{2}[d(g(x), g(a_p)) + d(g(y), g(b_p))] < \epsilon + \delta(\epsilon), \tag{3.11}$$

Then, by condition (b) of Theorem 2.1, we have

$$\frac{1}{2}[d(F(x,y), F(a_p, b_p)) + d(F(y,x), F(b_p, a_p))] < \epsilon, \tag{3.12}$$

and hence

$$\frac{1}{2}[d(g(x), g(a_{p+1})) + d(g(y), g(b_{p+1}))] < \epsilon, \tag{3.13}$$

a contradiction to (3.10) for $n = p + 1$. Thus $\alpha = 0$, and hence

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} [d(g(x), g(a_n)) + d(g(y), g(b_n))] = 0 \tag{3.14}$$

Similarly, it follows that

$$\lim_{n\to\infty} [d(g(x^*), g(a_n)) + d(g(y^*), g(b_n))] = 0 \tag{3.15}$$

Using the triangle inequality, we get

$$\begin{aligned} d(g(x), g(x^*)) + d(g(y), g(y^*)) &\leq d(g(x), g(a_n)) + d(g(a_n), g(x^*)) \\ &\quad + d(g(y), g(b_n)) + d(g(b_n), g(y^*)) \\ &= [d(g(x), g(a_n)) + d(g(y), g(b_n))] \\ &\quad + [d(g(x^*), g(a_n)) + d(g(y^*), g(b_n))] \to 0 \end{aligned} \tag{3.16}$$

as $n \to \infty$.
Hence, it follows that $d(g(x), g(x^*)) = 0$ and $d(g(y), g(y^*)) = 0$. Therefore, (3.3) holds immediately. Thus, in both the cases, we have proved that (3.3) holds.
Now, since $g(x) = F(x, y), g(y) = F(y, x)$ and the pair (F, g) is compatible, by Lemma 3.1, it follows that

$$g(g(x)) = gF(x, y) = F(gx, gy) \qquad \text{and} \qquad g(g(y)) = gF(y, x) = F(gy, gx). \tag{3.17}$$

Denote $g(x) = z, g(y) = w$. Then by (3.17),

$$g(z) = F(z, w) \qquad \text{and} \qquad g(w) = F(w, z). \tag{3.18}$$

Thus (z, w) is a coupled coincidence point.
Then by (3.3) with $x* = z$ and $y* = w$, it follows that $g(z) = g(x)$ and $g(w) = g(y)$, that is,

$$g(z) = z \qquad \text{and} \qquad g(w) = w. \tag{3.19}$$

By (3.18) and (3.19),
$z = g(z) = F(z, w)$ and $w = g(w) = F(w, z)$. Therefore,(z, w) is a coupled common fixed point of F and g.
To prove uniqueness, assume that (p, q) is another coupled common fixed point of F and g. Then by (3.3) we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. This completes the proof. □

Corollary 3.3. *Suppose that all the hypotheses of Corollary 2.2 hold, and further, for all $(x, y), (x^*, y^*) \in X \times X$, there exists a point $(a, b) \in X \times X$ that is comparable to (x, y) and (x^*, y^*). Then F has a unique coupled fixed point.*

4 Results of Integral Type

Inspired by the work of Suzuki [37], we prove the following result, which will be useful in developing some applications of the main results proved in Section 2.

Theorem 4.1. *Let $(X, d, \leq)$ be a partially ordered metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two given mappings. Assume that there exists a function $\theta : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:*
(I) $\theta(0) = 0$ and $\theta(t) > 0$ for any $t > 0$;
(II) θ is increasing and right continuous;
(III) for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that, for all $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$,

$$\begin{aligned} \epsilon \leq \theta(\frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))]) < \epsilon + \delta(\epsilon) \\ \textit{implies} \qquad \theta(\frac{1}{2}[d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))]) < \epsilon \end{aligned} \tag{4.1}$$

Then F is a generalized symmetric g-Meir-Keeler type contraction.

Proof. For any $\epsilon > 0$ it follows from (I) that $\theta(\epsilon) > 0$, and so there exists an $\alpha > 0$ such that, for all $u, v, u^*, v^* \in X$ with $g(u) \leq g(u^*)$ and $g(v) \geq g(v^*)$,

$$\begin{aligned} \theta(\epsilon) \leq \theta(\frac{1}{2}[d(g(u), g(u^*)) + d(g(v), g(^*v))]) < \theta(\epsilon) + \alpha \\ \text{implies} \qquad \theta(\frac{1}{2}[d(F(u, v), F(u^*, v^*)) + d(F(v, u), F(v^*, u^*))]) < \theta(\epsilon) \end{aligned} \tag{4.2}$$

By the right continuity of θ , there exists $\delta > 0$ such that $\theta(\epsilon+\delta) < \theta(\epsilon)+\alpha$.
For any $x, y, u, v \in X$ such that $g(x) \leq g(u), g(y) \geq g(v)$ and

$$\epsilon \leq \frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))] < \epsilon + \delta. \tag{4.3}$$

Then, since θ is an increasing function, we get the following:

$$\theta(\epsilon) \leq \theta(\frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))]) < \theta(\epsilon + \alpha) < \theta(\epsilon) + \alpha. \tag{4.4}$$

By (4.2), we have

$$\theta(\frac{1}{2}[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))]) < \theta(\epsilon)$$

and hence,

$$\frac{1}{2}[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))]) < \epsilon.$$

Therefore, it follows that F is a generalized symmetric g-Meir-Keeler type contraction. This completes the proof. □

The following result is an immediate consequence of Theorems 2.1 and 4.1.

Corollary 4.2. *Let $(X, d, \leq)$ be a partially ordered metric space. Given $F : X \times X \to X$ and $g : X \to X$ such that $F(X \times X) \subset g(X), g(X)$ is a complete subspace and the following hypotheses hold:*
(IV) F has the mixed strict g-monotone property;
(V) for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \int_0^{(1/2)[d(g(x),g(u))+d(g(y),g(v))]} \phi(t)dt < \epsilon + \delta(\epsilon)$$
$$\textit{implies} \qquad \int_0^{(1/2)[d(F(x,y),F(u,v))+d(F(y,x),F(v,u))]} \phi(t)dt < \epsilon \tag{4.5}$$

for all $gx \leq gu$ and $gy \geq gv$, where $\phi : [0, +\infty) \to [0, +\infty)$ is a locally integrable function satisfying $\int_0^s \phi(t)dt > 0$ for all $s > 0$;
(VI) there exist $x_0, y_0 \in X$ such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Assume that the hypotheses (i) and (ii) given in Theorem 2.1 hold. Then,F and g have a coupled coincidence point.

Corollary 4.3. *Let $(X, d, \leq)$ be a partially ordered metric space. Given $F : X \times X \to X$ and $g : X \to X$ such that $F(X \times X) \subset g(X), g(X)$ is a complete subspace and the following hypotheses hold:*
(VII) F has the mixed g-monotone property;
(VIII) for all $gx \leq gu$ and $gy \leq gv$

$$\int_0^{(1/2)[d(g(x),g(u))+d(g(y),g(v))]} \phi(t)dt \leq k \int_0^{(1/2)[d(F(x,y),F(u,v))+d(F(y,x),F(v,u))]} \phi(t)dt \tag{4.6}$$

where $k \in (0,1)$ and ϕ is a locally integrable function from $[0, +\infty)$ into itself satisfying $\int_0^s \phi(t)dt > 0$ for all $s > 0$;
(IX) there exist $x_0, y_0 \in X$ such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$.
Assume that the hypotheses (i) and (ii) given in Theorem 2.1 hold. Then,F and g have a coupled coincidence point.

Proof. For each $\epsilon > 0$, take $\delta(\epsilon) = (\frac{1}{k} - 1)\epsilon$ and apply Corollary 4.2. □

5 Applications to Integral Equations

As an application of the results proved in Sections 2 and 3, we study the existence of solutions for the following system of integral equations:

$$\begin{aligned} x(t) &= \int_a^b (K_1(t,s)+K_2(t,s))(f(s,x(s))+g(s,y(s)))ds+h(t) \\ y(t) &= \int_a^b (K_1(t,s)+K_2(t,s))(f(s,y(s))+g(s,x(s)))ds+h(t) \end{aligned} \tag{5.1}$$

where $t \in I = [a,b]$.
Let $\Phi : [0,+\infty) \to [0,+\infty)$ denote the class of functions $\phi : [0,) \to [0,)$ which satisfies the following conditions:
(i) ϕ is increasing;
(ii) for each $x \geq 0$, there exists a $k \in (0,1)$ such that $\phi(x) \leq (\frac{k}{2})x$
We assume that K_1, K_2, f, g satisfy the following conditions.

Assumption 5.1. (i) $K_1(t,s) \geq 0$ and $K_2(t,s) \leq 0$ for all $t,s \in [a,b]$;
(ii) There exist $\lambda, \mu > 0$ and $\phi \in \Phi$ such that for all $x,y \in \mathbb{R}, x > y$,

$$0 < f(t,x) - f(t,y) \leq \lambda\phi(x-y) \tag{5.2}$$

and

$$-\mu\phi(x-y) \leq g(t,x) - g(t,y) < 0; \tag{5.3}$$

(iii)

$$(\lambda+\mu)\sup_{t\in I}\int_a^b (K_1(t,s)-K_2(t,s))ds \leq 1; \tag{5.4}$$

Definition 5.2. An element $(\alpha,\beta) \in X \times X$ with $X = C(I,\mathbb{R})$ is called a coupled lower and upper solution of the integral equation (5.1) if for all $t \in I$,

$$\alpha(t) < \int_a^b (K_1(t,s)(f(s,\alpha(s))+g(s,\beta(s)))ds + \int_a^b K_2(t,s))(f(s,\beta(s))+g(s,\alpha(s)))ds+h(t)$$

and

$$\beta(t) \geq \int_a^b (K_1(t,s)(f(s,\beta(s))+g(s,\alpha(s)))ds + \int_a^b K_2(t,s))(f(s,\alpha(s))+g(s,\beta(s)))ds+h(t)$$

Theorem 5.3. *Consider the integral equation (5.1) with $K_1, K_2 \in C(I,\mathbb{R}), f,g \in C(I\times\mathbb{R},\mathbb{R})$ and $h \in C(I,\mathbb{R})$. Suppose that there exists a coupled lower and upper solution (α,β) of (5.1) with $\alpha \leq \beta$ and that Assumption 5.1 is satisfied. Then the integral equation (5.1) has a solution.*

Proof. Consider the natural order relation on $X = C(I,\mathbb{R})$; that is, for $x,y \in C(I,\mathbb{R})$

$$x \leq y \Rightarrow x(t) \leq y(t), \forall t \in I$$

It is well known that X is a complete metric space with respect to the sup metric

$$d(x,y) = \sup_{t\in I}|x(t)-y(t)|, x,y \in C(I,\mathbb{R}).$$

Suppose that $\{u_n\}$ is a strictly increasing sequence in X that converges to a point $u \in X$. Then for every $t \in I$, the sequence of real numbers

$$u_1(t) < u_2(t) < \ldots < u_n(t) < \ldots$$

converges to $u(t)$. Therefore, for all $t \in I, n \in \mathbb{N}, u_n(t) < u(t)$. Hence, $u_n < u$ for all n. Similarly, it can be verified that, if for all $t \in I, v(t)$ is a limit of a strictly decreasing sequence

$v_n(t)$ in X, then $v(t) < v_n(t)$ for all n and hence $v < v_n$. Therefore conditions (i) and (ii) of Corollary 2.1 hold.
Also, $X \times X = C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is a partially ordered set under the following order relation in $X \times X$

$$(x, y), (u, v) \in X \times X, (x, y) \leq (u, v) \Rightarrow x(t) \leq u(t) \quad \text{and} \quad y(t) \geq v(t), \forall t \in I.$$

For any $x, y \in X$, $\max\{x(t), y(t)\}$ and $\min\{x(t), y(t)\}$, for each $t \in I$, are in X and are the upper and lower bounds of x, y, respectively. Therefore, for every $(x, y), (u, v) \in X \times X$, there exists a $(\max\{x, u\}, \min\{y, v\}) \in X \times X$ that is comparable to (x, y) and (u, v).
Define $F : X \times X \to X$ by
$F(x, y)(t) = \int_a^b K_1(t, s)(f(s, x(s)) + g(s, y(s)))ds + \int_a^b K_2(t, s)(f(s, y(s)) + g(s, x(s)))ds + h(t)$ for all $t \in [a, b]$. We now show that F has the mixed strict monotone property. For $x_1(t) < x_2(t)$ for all $t \in [a, b]$ we have

$$\begin{aligned} F(x_1, y)(t) \quad - \quad & F(x_2, y)(t) = \int_a^b K_1(t, s)(f(s, x_1(s)) + g(s, y(s)))ds \\ + \quad & \int_a^b K_2(t, s)(f(s, y(s)) + g(s, x_1(s)))ds + h(t) \\ - \quad & \int_a^b K_1(t, s)(f(s, x_2(s)) + g(s, y(s)))ds \\ - \quad & \int_a^b K_2(t, s)(f(s, y(s)) + g(s, x_2(s)))ds - h(t) \\ = \quad & \int_a^b K_1(t, s)(f(s, x_1(s)) - f(s, x_2(s)))ds \\ + \quad & \int_a^b K_2(t, s)(g(s, x_1(s)) - g(s, x_2(s)))ds < 0 \end{aligned}$$

by Assumption 5.1. Hence $F(x_1, y)(t) < F(x_2, y)(t), \forall t \in I$; that is, $F(x_1, y) < F(x_2, y)$. Similarly, if $y_1 > y_2$, that is, $y_1(t) > y_2(t)$, for all $t \in [a, b]$, we have

$$\begin{aligned} F(x, y_1)(t) \quad - \quad & F(x, y_2)(t) = \int_a^b K_1(t, s)(f(s, x(s)) + g(s, y_1(s)))ds \\ + \quad & \int_a^b K_2(t, s)(f(s, y_1(s)) + g(s, x(s)))ds + h(t) \\ - \quad & \int_a^b K_1(t, s)(f(s, x(s)) + g(s, y_2(s)))ds \\ - \quad & \int_a^b K_2(t, s)(f(s, y_2(s)) + g(s, x(s)))ds - h(t) \\ = \quad & \int_a^b K_1(t, s)(g(s, y_1(s)) - g(s, y_2(s)))ds \\ + \quad & \int_a^b K_2(t, s)(f(s, y_1(s)) - f(s, y_2(s)))ds < 0 \end{aligned}$$

by Assumption 5.1. Hence $F(x, y_1)(t) < F(x, y_2)(t), \forall t \in I$; that is, $F(x, y_1) < F(x, y_2)$. Therefore F satisfies mixed strict monotone property.
Next, we verify that F satisfies (1.3). For $x \geq u, y \leq v$, that is, $x(t) \geq u(t), y(t) \leq v(t)$ for all

$t \in I$, we have

$$\begin{aligned}
F(x,y)(t) - F(u,v)(t) &= \int_a^b K_1(t,s)(f(s,x(s)) + g(s,y_1(s)))ds \\
&+ \int_a^b K_2(t,s)(f(s,y(s)) + g(s,x(s)))ds \\
&- \int_a^b K_1(t,s)(f(s,u(s)) + g(s,v(s)))ds \\
&- \int_a^b K_2(t,s)(f(s,v(s)) + g(s,u(s)))ds \\
&= \int_a^b K_1(t,s)(f(s,x(s)) - f(s,u(s)) - g(s,y(s)) - g(s,v(s)))]ds \\
&+ \int_a^b K_2(t,s)[(f(s,y(s)) - f(s,v(s))) - g(s,x(s)) - g(s,u(s)))ds \\
&= \int_a^b K_1(t,s)[(f(s,x(s)) - f(s,u(s)) - (g(s,v(s)) - g(s,y(s)))]ds \\
&- \int_a^b K_2(t,s)[f(s,v(s)) - f(s,y(s)) - (g(s,x(s)) - g(s,u(s))]ds \\
&\leq \int_a^b K_1(t,s)[\lambda\phi(x(s) - u(s)) + \mu\phi(v(s) - y(s))]ds \\
&- \int_a^b K_2(t,s)[\lambda\phi(v(s) - y(s)) + \mu\phi(x(s) - u(s))]ds \qquad (5.5)
\end{aligned}$$

Since the function ϕ is increasing and $x \geq u$ and $y \leq v$, we have

$$\phi(x(s) - u(s)) \leq \phi(\sup_{t \in I} |x(t) - u(t)|) = \phi(d(x,u))$$

and

$$\phi(v(s) - y(s)) \leq \phi(\sup_{t \in I} |v(t) - y(t)|) = \phi(d(v,y))$$

Hence, using (5.5) and the fact that $K_2(t,s) \leq 0$, we obtain

$$\begin{aligned}
|F(x,y)(t) - F(u,v)(t)| &\leq \int_a^b K_1(t,s)[\lambda\phi(d(x,u)) + \mu\phi(d(v,y))]ds \\
&- \int_a^b K_2(t,s)[\lambda\phi(d(v,y)) + \mu\phi(d(x,u))]ds \qquad (5.6)
\end{aligned}$$

Since all of the quantities on the right hand side of (5.5) are non-negative, inequality (5.6) is satisfied.
Similarly, we can show that

$$\begin{aligned}
|F(y,x)(t) - F(v,u)(t)| &\leq \int_a^b K_1(t,s)[\lambda\phi(d(v,y)) + \mu\phi(d(x,u))]ds \\
&- \int_a^b K_2(t,s)[\lambda\phi(d(x,u)) + \mu\phi(d(v,y))]ds \qquad (5.7)
\end{aligned}$$

Summing (5.6) and (5.7), dividing by 2 , and then taking the supremum with respect to t we get, by using (5.4) that

$$\begin{aligned}&\frac{d(F(x,y)+F(u,v))+d(F(y,x)+F(v,u))}{2}\\&\quad\leq(\lambda+\phi)\sup_{t\in I}\int_a^b(K_1(t,s)-K_2(t,s))ds.\frac{\phi(d(v,y))+\phi(d(x,u))}{2}\\&\quad\leq\frac{\phi(d(v,y))+\phi(d(x,u))}{2}\end{aligned}$$

Since ϕ is increasing,

$$\phi(d(x,u))\leq\phi(d(x,u)+d(v,y)),\quad \phi(d(v,y))\leq\phi(d(x,u)+d(v,y))$$

and hence

$$\frac{\phi(d(v,y))+\phi(d(x,u))}{2}\leq\phi(d(x,u)+d(v,y))\leq(\frac{k}{2})[d(x,u)+d(v,y)]$$

by the definition of ϕ. Thus

$$\frac{d(F(x,y)+F(u,v))+d(F(y,x)+F(v,u))}{2}\leq(\frac{k}{2})[d(x,u)+d(v,y)]$$

which reduces to the symmetric contractive condition (1.3).
Then, by Proposition 1.14, F is a generalized symmetric Meir-Keeler type contraction.
Finally, let (α,β) be a coupled lower and upper solution of the integral equation (5.1), then we have $\alpha(t)<F(\alpha,\beta)(t)$ and $\beta(t)\geq F(\beta,\alpha)(t)$ for all $t\in[a,b]$, that is, $\alpha<F(\alpha,\beta)$ and $\beta\geq F(\beta,\alpha)$.
Therefore, Corollaries 2.2 and 3.2 yield that F has a unique coupled fixed point (x,y) and hence the system (5.1) has a unique solution. □

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Pointwise Superconvergence Patch Recovery for the Gradient of the Linear Tetrahedral Element

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We consider the finite element approximation to the solution of a self-adjoint, second-order elliptic boundary value problem in three dimensions over a fully uniform mesh of piecewise linear tetrahedral elements. First, the supercloseness of the gradients between the piecewise linear finite element solution u_h and the linear interpolation u_I is derived by using a weak estimate and an estimate of the discrete derivative Green's function. We then analyze a superconvergence patch recovery scheme for the gradient of the finite element solution, showing that the recovered gradient of u_h is superconvergent to the gradient of the true solution u.

1 Introduction

Superconvergence of the gradient for the finite element approximation is a phenomenon whereby the convergent order of the derivatives of the finite element solutions exceeds the optimal global rate. Up to now, superconvergence is still an active research topic; see, for example, Babuška and Strouboulis [1], Chen [2], Chen and Huang [3], Lin and Yan [4], Wahlbin [5] and Zhu and Lin [6] for overviews of this field. Nevertheless, how to obtain the superconvergent numerical solution is an issue to researchers. In general, it needs to use post-processing techniques to get recovered gradients with high order accuracy from the finite element solution. Usual post-processing techniques include Interpolation technique, Projection technique, Average technique, Extrapolation technique, Superconvergence Patch Recovery (SPR) technique introduced by Zienkiewicz and Zhu [7–9] and Polynomial Patch Recovery (PPR) technique raised by Zhang and Naga [10]. In previous works, for the linear tetrahedral element, Chen and Wang [11] obtained the recovered gradient with $\mathcal{O}(h^2)$ order accuracy in the average sense of the L^2-norm by using SPR. Using the L^2-projection technique, in the average sense of the L^2-norm, Chen [12] got the recovered gradient with $\mathcal{O}(h^{1+min(\sigma,\frac{1}{2})})$ order accuracy. Goodsell [13] derived by using the average technique the pointwise superconvergence estimate of the

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recovered gradient with $\mathcal{O}(h^{2-\varepsilon})$ order accuracy.

Unlike the results in [11–13], this article will show a pointwise superconvergence estimate with $\mathcal{O}(h^2|\ln h|^{\frac{4}{3}})$ order accuracy for the recovered gradient by using SPR. In this article, we shall use the letter C to denote a generic constant which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

2 Model Problem and Finite Element Space

Suppose $\Omega \subset R^3$ is a rectangular block with boundary, $\partial\Omega$, consisting of faces parallel to the x-, y-, and z-axes. We consider the self-adjoint, variable coefficients second-order elliptic problem

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{3} \partial_j(a_{ij}\partial_i u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Here we assume f is smooth enough, and $A = (a_{ij})$ is a 3×3 symmetric matrix function in $(L^\infty(\Omega))^{3\times 3}$ and uniformly positive definite. Set $\partial_1 u = \frac{\partial u}{\partial x}$, $\partial_2 u = \frac{\partial u}{\partial y}$, and $\partial_3 u = \frac{\partial u}{\partial z}$. Thus, the variational formulation of (2.1) is

$$a(u\,,\,v) = (f\,,\,v) \quad \forall\, v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a(u\,,\,v) \equiv \int_\Omega \sum_{i,j=1}^{3} a_{ij}\partial_i u \partial_j v \, dxdydz$$

and

$$(f\,,\,v) = \int_\Omega f v \, dxdydz.$$

To discretize the problem (2.2), one proceeds as follows. The domain Ω is firstly partitioned into cubes of side h, and each of these is then subdivided into six tetrahedra (see Fig. 1). We denote by $\mathcal{T}^h$ this partition.

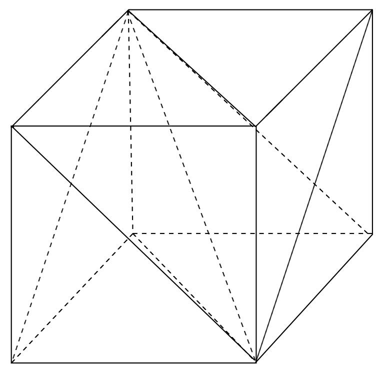

Figure 1: A tetrahedral partition

For this fully uniform mesh of tetrahedral elements, let $S_0^h(\Omega) \subset H_0^1(\Omega)$ be the piecewise linear tetrahedral finite element space, and $u_I \in S_0^h(\Omega)$ the Lagrange interpolant to the solution u of (2.2).

Discretizing (2.2) using S_0^h as approximating space means finding $u_h \in S_0^h$ such that $a(u_h\,,\, v) = (f\,,\, v)$ for all $v \in S_0^h$. Here u_h is a finite element approximation to u. Thus we have the Galerkin orthogonality relation

$$a(u - u_h\,,\, v) = 0 \;\; \forall\, v \in S_0^h(\Omega). \tag{2.3}$$

To derive the main result of this article, for every $Z \in \Omega$, we need to introduce the discrete derivative Green's function $\partial_{Z,\ell} G_Z^h \in S_0^h(\Omega)$ defined by

$$a(v, \partial_{Z,\ell} G_Z^h) = \partial_\ell v(Z) \;\forall\, v \in S_0^h(\Omega). \tag{2.4}$$

Here, for any direction $\ell \in R^3$, $|\ell| = 1$, $\partial_{Z,\ell} G_Z^h$ and $\partial_\ell v(Z)$ stand for the following onesided directional derivatives, respectively.

$$\partial_{Z,\ell} G_Z^h = \lim_{|\Delta Z| \to 0} \frac{G_{Z+\Delta Z}^h - G_Z^h}{|\Delta Z|},\; \partial_\ell v(Z) = \lim_{|\Delta Z| \to 0} \frac{v(Z + \Delta Z) - v(Z)}{|\Delta Z|},\; \Delta Z = |\Delta Z|\ell.$$

Remark 1. Since $\Delta Z = |\Delta Z|\ell$, that is, ΔZ is of the same direction as ℓ. Thus, provided that the direction ℓ is given, the above limits exist. Hence, no matter what direction is given, the above definition has good meaning.

3 Gradients Recovered by SPR and Superconvergence

SPR is a gradient recovery method introduced by Zienkiewicz and Zhu. This method is now widely used in engineering practices for its robustness in a posterior error estimation and its efficiency in computer implementation.

For $v \in S_0^h(\Omega)$, we denote by R_h the SPR-recovery operator and begin by defining the point values of $R_h v$ at the element nodes. After the recovered gradient values of all nodes are obtained, we give a linear interpolation by using these values, namely SPR-recovery gradient $R_h v$. Obviously $R_h v \in S_0^h(\Omega)$.

Let us firstly assume N is a interior node of the partition $\mathcal{T}^h$, and denote by ω the element patch around N containing 24 tetrahedra. Under the local coordinate system centered N, we let Q_i be the barycenter of a tetrahedron $e_i \subset \omega$, $i = 1, 2, \cdots, 24$. SPR uses the discrete least-squares fitting to seek linear vector function $\mathbf{p} \in (P_1(\omega))^3$, such that

$$\sum_{i=1}^{24} [\mathbf{p}(Q_i) - \nabla v(Q_i)] q(Q_i) = \mathbf{0} \;\;\; \forall q \in P_1(\omega), \tag{3.1}$$

where $v \in S_0^h(\Omega)$. The existence and uniqueness of the minimizer in (3.1) can be found in [14].

We define $R_hv(N) = \mathbf{p}(\mathbf{0})$. Then the following Lemma 3.1 and Lemma 3.2 hold.

Lemma 3.1. Let ω be the element patch around an interior node N, and $u \in W^{3,\infty}(\omega)$. For $u_I \in S_0^h(\Omega)$ the interpolant to u, we have

$$|\nabla u(N) - R_h u_I(N)| \le Ch^2 \|u\|_{3,\infty,\omega}.$$

Lemma 3.2. The recovery operator R_h satisfies

$$R_h v(N) = \frac{1}{24}\sum_{i=1}^{24} \nabla v(Q_i).$$

Lemma 3.3. For $v \in S_0^h(\Omega)$, we have the weak estimate

$$|a(u - u_I, v)| \le Ch^2 \|u\|_{3,\infty,\Omega} |v|_{1,1,\Omega}.$$

Lemma 3.4. For $\partial_{Z,\ell} G_Z^h$ the discrete derivative Green's function defined in (2.4), we have the following estimate

$$\left|\partial_{Z,\ell} G_Z^h\right|_{1,1} \le C|\ln h|^{\frac{4}{3}}.$$

Remark 2. The proofs of Lemma 3.1 and Lemma 3.2 can be seen in [11], Lemma 3.3 in [13], and Lemma 3.4 in [15].

Theorem 3.1. For u_I and u_h, the linear interpolant and the linear tetrahedral finite element approximation to u, respectively. Thus we have the following supercloseness estimate

$$|u_h - u_I|_{1,\infty,\Omega} \le Ch^2 |\ln h|^{\frac{4}{3}} \|u\|_{3,\infty,\Omega}.$$

Proof. For every $Z \in \Omega$ and any direction ℓ, from (2.3) and (2.4),

$$\partial_\ell (u_h - u_I)(Z) = a(u_h - u_I,\, \partial_{Z,\ell} G_Z^h) = a(u - u_I,\, \partial_{Z,\ell} G_Z^h).$$

Hence, using Lemma 3.3,

$$|\partial_\ell (u_h - u_I)(Z)| \le Ch^2 \|u\|_{3,\infty,\Omega} \left|\partial_{Z,\ell} G_Z^h\right|_{1,1,\Omega},$$

which combined with Lemma 3.4 completes the proof of Theorem 3.1.

Theorem 3.2. For $u_I \in S_0^h(\Omega)$ the linear interpolant to u, the solution of (2.2), and R_h the gradient recovered operator by SPR, we have the superconvergent estimate

$$|\nabla u - R_h u_I|_{0,\infty,\Omega} \le Ch^2 \|u\|_{3,\infty,\Omega}.$$

Proof. Denote by $F : \hat{e} \to e$ an affine transformation. Obviously, there exists an element $e \in \mathcal{T}^h$, using the triangle inequality and the Sobolev Embedding Theorem [16], and considering Lemma 3.2, such that

$$\begin{aligned}
|\nabla u - R_h u_I|_{0,\,\infty,\,\Omega} &= |\nabla u - R_h u_I|_{0,\,\infty,\,e} \\
&\le Ch^{-1} |\nabla \hat{u} - R_h \hat{u}_I|_{0,\,\infty,\,\hat{e}} \\
&\le Ch^{-1} \left[|\nabla \hat{u}|_{0,\,\infty,\,\hat{e}} + |R_h \hat{u}_I|_{0,\,\infty,\,\hat{e}} \right] \\
&\le Ch^{-1} \left[|\nabla \hat{u}|_{0,\,\infty,\,\hat{\chi}} + |\hat{u}_I|_{1,\,\infty,\,\hat{\chi}} \right] \\
&\le Ch^{-1} \|\hat{u}\|_{3,\,\infty,\,\hat{\chi}},
\end{aligned}$$

where $\hat{\chi}$ is a small patch of elements surrounding the tetrahedron, $\hat{e}$. Due to the fact that, for $\hat{u}$ quadratic over $\hat{\chi}$,

$$\nabla\hat{u} - R_h\hat{u_I} = 0 \ \text{ in } \hat{e},$$

so, from the Bramble-Hilbert Lemma [17],

$$|\nabla u - R_h u_I|_{0,\,\infty,\,\Omega} \leq Ch^{-1}\,|\hat{u}|_{3,\,\infty,\,\hat{\chi}} \leq Ch^2\,|u|_{3,\,\infty,\,\Omega}\,,$$

which completes the proof of Theorem 3.2.

Theorem 3.3. For $u_h \in S_0^h(\Omega)$ the linear finite element approximation to u, the solution of (2.2), and R_h the gradient recovered operator by SPR, we have the superconvergent estimate

$$|\nabla u - R_h u_h|_{0,\infty,\Omega} \leq Ch^2|\ln h|^{\frac{4}{3}}\|u\|_{3,\infty,\Omega}.$$

Proof. Using the triangle inequality, we have

$$\begin{aligned}
|\nabla u - R_h u_h|_{0,\,\infty,\,\Omega} &\leq |R_h(u_h - u_I)|_{0,\,\infty,\,\Omega} \\
&\quad + |\nabla u - R_h u_I|_{0,\,\infty,\,\Omega} \\
&\leq |u_h - u_I|_{1,\,\infty,\,\Omega} \\
&\quad + |\nabla u - R_h u_I|_{0,\,\infty,\,\Omega}\,,
\end{aligned}$$

which combined with Theorems 3.1 and 3.2 completes the proof of Theorem 3.3.

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Hyers-Ulam stability of quadratic functional equations in paranormed spaces

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Abstract. Lin [18, 19] introduced and investigated the following quadratic functional equations

$$cf\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - (n+c-1)x_j\right) \tag{0.1}$$

$$= (n+c-1)\left(f(x_1) + c\sum_{i=2}^{n} f(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right),$$

$$Q\left(\sum_{i=1}^{n} d_i x_i\right) + \sum_{1\le i<j\le n} d_i d_j Q(x_i - x_j) = \left(\sum_{i=1}^{n} d_i\right)\left(\sum_{i=1}^{n} d_i Q(x_i)\right). \tag{0.2}$$

In this paper, we prove the Hyers-Ulam stability of the above quadratic functional equations in paranormed spaces.

1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [7] and Steinhaus [31] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [8, 16, 20, 21, 29]). This notion was defined in normed spaces by Kolk [17].

We recall some basic facts concerning Fréchet spaces.

Definition 1.1. [33] Let X be a vector space. A paranorm $P : X \to [0,\infty)$ is a function on X such that

(1) $P(0) = 0$;

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(2) $P(-x) = P(x)$;

(3) $P(x + y) \leq P(x) + P(y)$ (triangle inequality)

(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a *paranormed space* if P is a *paranorm* on X.

The paranorm is called *total* if, in addition, we have

(5) $P(x) = 0$ implies $x = 0$.

A *Fréchet space* is a total and complete paranormed space.

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [23] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

In 1990, Th.M. Rassias [24] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [9] following the same approach as in Th.M. Rassias [23], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [9], as well as by Th.M. Rassias and Šemrl [28] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$ (cf. the books of P. Czerwik [5], D.H. Hyers, G. Isac and Th.M. Rassias [12]).

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [30] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [4] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 6], [13]–[15], [22], [25]–[27]).

Throughout this paper, assume that (X, P) is a Fréchet space and that $(Y, \|\cdot\|)$ is a Banach space.

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the quadratic functional equation (0.1) in paranormed spaces. In Section 3, we prove the Hyers-Ulam stability of the quadratic functional equation (0.2) in paranormed spaces.

2. Hyers-Ulam stability of the functional equation (0.1) in paranormed spaces

For a given mapping f, we define

$$\begin{aligned} Df(x_1, x_2, \cdots, x_n) &= cf\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - (n + c - 1)x_j\right) \\ &-(n + c - 1)\left(f(x_1) + c\sum_{i=2}^{n} f(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right). \end{aligned}$$

In this section, we prove the Hyers-Ulam stability of the functional equation $Df(x_1, \cdots, x_n) = 0$ in paranormed spaces.

Throughout this section, assume that $v := 2 - n - c$ is an integer greater than one.

Note that $P(vx) \leq vP(x)$ for all $x \in Y$.

Theorem 2.1. *Let r, θ be positive real numbers with $r > 2$. Let $f : Y \to X$ be a mapping such that*

$$P(Df(x_1, \cdots, x_n)) \leq \theta \sum_{i=1}^{n} \|x_i\|^r \tag{2.1}$$

for all $x_1, \cdots, x_n \in Y$. Then there exists a unique quadratic mapping $R : Y \to X$ such that

$$P(f(x) - R(x)) \leq \frac{\theta}{v^r - v^2} \|x\|^r \tag{2.2}$$

for all $x \in Y$.

Proof. Putting $x_2 = \frac{x}{v}$ and $x_1 = x_3 = x_4 = \cdots = x_n = 0$ in (2.1), we get

$$P\left(f(x) - v^2 f\left(\frac{x}{v}\right)\right) \leq \frac{\theta\|x\|^r}{v^r}$$

for all $x \in Y$

Hence

$$P\left(v^{2l} f\left(\frac{x}{v^l}\right) - v^{2m} f\left(\frac{x}{v^m}\right)\right) \leq \sum_{j=l}^{m-1} \frac{\theta\|x\|^r}{v^{(r-2)j+r}} \tag{2.3}$$

holds for all non-negative integers l and m with $m > l$ and all $x \in Y$. It follows from (2.3) that the sequence $\{v^{2k} f(\frac{x}{v^k})\}$ is a Cauchy sequence for all $x \in Y$. Since X is complete, the sequence $\{v^{2k} f(\frac{x}{v^k})\}$ converges. So the mapping $R : Y \to X$ can be defined as

$$R(x) := \lim_{k\to\infty} v^{2k} f\left(\frac{x}{v^k}\right)$$

for all $x \in Y$.

By (2.1),

$$\begin{aligned} P(DR(x_1, \cdots, x_n)) &= \lim_{k\to\infty} P\left(v^{2k} DR\left(\frac{x_1}{v^k}, \cdots, \frac{x_n}{v^k}\right)\right) \leq \lim_{k\to\infty} v^{2k} P\left(DR\left(\frac{x_1}{v^k}, \cdots, \frac{x_n}{v^k}\right)\right) \\ &\leq \lim_{k\to\infty} v^{2k}\theta\left(\sum_{i=1}^{n} ||\frac{x_i}{v^k}||^r\right) = \lim_{k\to\infty} \frac{\theta\left(\sum_{i=1}^{n} ||x_i|||^r\right)}{v^{(r-2)k}} = 0 \end{aligned}$$

for all $x_1, \cdots, x_n \in Y$. So $DR(x_1, \cdots, x_n) = 0$. So the mapping $R : Y \to X$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.3), we get (2.2). So there exists a quadratic mapping $R : Y \to X$ satisfying (2.2).

Now, let $T : Y \to X$ be another quadratic mapping satisfying (2.2). Then we have

$$\begin{aligned} P(R(x) - T(x)) &= P\left(v^{2q} R\left(\frac{x}{v^q}\right) - v^{2q} T\left(\frac{x}{v^q}\right)\right) \\ &\leq P\left(v^{2q}\left(R\left(\frac{x}{v^q}\right) - f\left(\frac{x}{v^q}\right)\right)\right) + P\left(v^{2q}\left(T\left(\frac{x}{v^q}\right) - f\left(\frac{x}{v^q}\right)\right)\right) \\ &\leq \frac{2\theta}{v^r - v^2}\|x\|^r \cdot \frac{v^{2q}}{v^{rq}}, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in Y$. So we can conclude that $R(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of R. Thus the mapping $R : Y \to X$ is the unique quadratic mapping satisfying (2.2). □

Theorem 2.2. *Let r, θ be positive real numbers with $r < 2$. Let $f : X \to Y$ be a mapping such that*

$$||Df(x_1, \cdots, x_n)|| \leq \theta \sum_{i=1}^{n} P(x_i)^r \tag{2.4}$$

for all $x_1, \cdots, x_n \in X$*. Then there exists a unique quadratic mapping* $R : X \to Y$ *such that*

$$||f(x) - R(x)|| \leq \frac{\theta}{v^2 - v^r} P(x)^r \tag{2.5}$$

for all $x \in X$*.*

Proof. Putting $x_2 = x$ and $x_1 = x_3 = x_4 = \cdots = x_n = 0$ in (2.4), we get

$$||f(vx) - v^2 f(x)|| \leq \theta P(x)^r$$

and so

$$\left\| f(x) - \frac{1}{v^2} f(vx) \right\| \leq \theta \frac{1}{v^2} P(x)^r$$

for all $x \in X$

Hence

$$\left\| \frac{1}{v^{2l}} f\left(v^l x\right) - \frac{1}{v^{2m}} f\left(v^m x\right) \right\| \leq \sum_{j=l}^{m-1} \theta \frac{1}{v^2} \frac{v^{rj} P(x)^r}{v^{2j}} \tag{2.6}$$

holds for all non-negative integers l and m with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{\frac{1}{v^{2k}} f(v^k x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{v^{2k}} f(v^k x)\}$ converges. So the mapping $R : X \to Y$ can be defined as

$$R(x) := \lim_{k\to\infty} \frac{1}{v^{2k}} f\left(v^k x\right)$$

for all $x \in X$.

By (2.4),

$$||DR(x_1, \cdots, x_n)|| = \lim_{k\to\infty} \left\| \frac{1}{v^{2k}} DR(v^k x_1, \cdots, v^k x_n) \right\| \leq \lim_{k\to\infty} \frac{1}{v^{2k}} \|DR(v^k x_1, \cdots, v^k x_n)\|$$
$$\leq \lim_{k\to\infty} \frac{v^{rk}}{v^{2k}} \theta \sum_{i=1}^{n} P(x_i)^r = 0$$

for all $x_1, \cdots, x_n \in X$. So $DR(x_1, \cdots, x_n) = 0$. So the mapping $R : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.5). So there exists a quadratic mapping $R : X \to Y$ satisfying (2.5).

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.5). Then we have

$$\begin{aligned} ||R(x) - T(x)|| &= \left\| \frac{1}{v^{2q}} R\left(v^q x\right) - \frac{1}{v^{2q}} T\left(v^q x\right) \right\| \\ &\leq \left\| \frac{1}{v^{2q}} \left(R\left(v^q x\right) - f\left(v^q x\right)\right) \right\| + \left\| \frac{1}{v^{2q}} \left(T\left(v^q x\right) - f\left(v^q x\right)\right) \right\| \\ &\leq \frac{2\theta}{v^2 - v^r} P(x)^r \cdot \frac{v^{rq}}{v^{2q}}, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $R(x) = T(x)$ for all $x \in X$. This proves the uniqueness of R. Thus the mapping $R : X \to Y$ is the unique quadratic mapping satisfying (2.5). □

3. Hyers-Ulam stability of the functional equation (0.2) in paranormed spaces

For a given mapping f, we define

$$DQ(x_1,\cdots,x_n) := Q\left(\sum_{i=1}^{n} d_i x_i\right) + \sum_{1\le i<j\le n} d_i d_j Q(x_i - x_j) - \sum_{i=1}^{n} d_i \left(\sum_{i=1}^{n} d_i Q(x_i)\right).$$

In this section, we prove the Hyers-Ulam stability of the functional equation $DQ(x_1,\cdots,x_n) = 0$ in paranormed spaces.

Throughout this section, assume that $d := \sum_{j=1}^{n} d_j$ is an integer greater than one.

Note that $P(dx) \le dP(x)$ for all $x \in Y$.

Theorem 3.1. *Let r,θ be positive real numbers with $r > 2$. Let $Q : Y \to X$ be a mapping such that*

$$P(DQ(x_1,\cdots,x_n)) \le \theta \sum_{i=1}^{n} \|x_i\|^r \tag{3.1}$$

for all $x_1,\cdots,x_n \in Y$. Then there exists a unique quadratic mapping $R : Y \to X$ such that

$$P(Q(x) - R(x)) \le \frac{n\theta}{d^r - d^2}\|x\|^r \tag{3.2}$$

for all $x \in Y$.

Proof. Putting $x_1 = \cdots = x_n = \frac{x}{d}$ in (3.1), we get

$$P\left(Q(x) - d^2 Q\left(\frac{x}{d}\right)\right) \le \frac{\theta(n\|x\|^r)}{d^r}$$

for all $x \in X$

Hence

$$P\left(d^{2l}Q\left(\frac{x}{d^l}\right) - d^{2m}Q\left(\frac{x}{d^m}\right)\right) \le \sum_{j=l}^{m-1} \frac{\theta(n\|x\|^r)}{d^{(r-2)j+r}} \tag{3.3}$$

holds for all non-negative integers l and m with $m > l$ and all $x \in Y$. It follows from (3.3) that the sequence $\{d^{2k}Q(\frac{x}{d^k})\}$ is a Cauchy sequence for all $x \in Y$. Since X is complete, the sequence $\{d^{2k}Q(\frac{x}{d^k})\}$ converges. So the mapping $R : Y \to X$ can be defined as

$$R(x) := \lim_{k\to\infty} d^{2k} Q\left(\frac{x}{d^k}\right)$$

for all $x \in Y$.

By (3.1),

$$\begin{aligned} P(DR(x_1,\cdots,x_n)) &= \lim_{k\to\infty} P\left(d^{2k} DR\left(\frac{x_1}{d^k},\cdots,\frac{x_n}{d^k}\right)\right) \le \lim_{k\to\infty} d^{2k} P\left(DR\left(\frac{x_1}{d^k},\cdots,\frac{x_n}{d^k}\right)\right) \\ &\le \lim_{k\to\infty} d^{2k}\theta\left(\sum_{i=1}^{n}\left\|\frac{x_i}{d^k}\right\|^r\right) = \lim_{k\to\infty} \frac{\theta\left(\sum_{i=1}^{n} \|x_i\|\|^r\right)}{d^{(r-2)k}} = 0 \end{aligned}$$

for all $x_1,\cdots,x_n \in Y$. So $DR(x_1,\cdots,x_n) = 0$. So the mapping $R : Y \to X$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.2). So there exists a quadratic mapping $R : Y \to X$ satisfying (3.2).

Now, let $T : Y \to X$ be another quadratic mapping satisfying (3.2). Then we have

$$\begin{aligned} P(R(x) - T(x)) &= P\left(d^{2q}R\left(\frac{x}{d^q}\right) - d^{2q}T\left(\frac{x}{d^q}\right)\right) \\ &\le P\left(d^{2q}\left(R\left(\frac{x}{d^q}\right) - Qf\left(\frac{x}{d^q}\right)\right)\right) + P\left(d^{2q}\left(T\left(\frac{x}{d^q}\right) - Q\left(\frac{x}{d^q}\right)\right)\right) \\ &\le \frac{2n\theta}{d^r - d^2}\|x\|^r \cdot \frac{d^{2q}}{d^{rq}}, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in Y$. So we can conclude that $R(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of R. Thus the mapping $R : Y \to X$ is the unique quadratic mapping satisfying (3.2). □

Theorem 3.2. *Let r, θ be positive real numbers with $r < 2$. Let $Q : X \to Y$ be a mapping such that*

$$||DQ(x_1, \cdots, x_n)|| \le \theta \sum_{i=1}^{n} P(x_i)^r \tag{3.4}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique quadratic mapping $R : X \to Y$ such that

$$||Q(x) - R(x)|| \le \frac{n\theta}{d^2 - d^r} P(x)^r$$

for all $x \in X$.

Proof. Putting $x_1 = \cdots = x_n = x$ in (3.4), we get

$$||Q(dx) - d^2 Q(x)|| \le n\theta P(x)^r$$

and so

$$\left\| Q(x) - \frac{1}{d^2} Q(dx) \right\| \le \frac{n\theta}{d^2} P(x)^r$$

for all $x \in X$

Hence

$$\left\| \frac{1}{d^{2l}} Q\left(d^l x\right) - \frac{1}{d^{2m}} Q\left(d^m x\right) \right\| \le \frac{n\theta}{d^2} \sum_{j=l}^{m-1} \frac{d^{rj}}{d^{2j}} P(x)^r \tag{3.5}$$

holds for all non-negative integers l and m with $m > l$ and all $x \in X$. It follows from (3.5) that the sequence $\{\frac{1}{d^{2k}} Q(d^k x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{d^{2k}} Q(d^k x)\}$ converges. So the mapping $R : X \to Y$ can be defined as

$$R(x) := \lim_{k \to \infty} \frac{1}{d^{2k}} Q\left(d^k x\right)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.1. □

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Union soft sets applied to commutative BCI-ideals

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Abstract. The aim of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, the notion of union soft commutative BCI-ideals is introduced, and related properties are investigated. Relations between a union soft commutative BCI-ideal and a (closed) union soft BCI-ideal are displayed. Conditions for a union soft BCI-ideal to be a union soft commutative BCI-ideal are established. Characterizations of a union soft commutative BCI-ideal are considered, and a new union soft commutative BCI-ideal from an old one is constructed.

1. Introduction

The real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [26]. In response to this situation Zadeh [27] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [28]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [23]. Maji et al. [19] and Molodtsov [23] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [23] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory

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and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [19] described the application of soft set theory to a decision making problem. Maji et al. [18] also studied several operations on the theory of soft sets. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun and Park [17] studied applications of soft sets in ideal theory of BCK/BCI-algebras. Jun et al. [14, 15] introduced the notion of intersectional soft sets, and considered its applications to BCK/BCI-algebras. Also, Jun [10] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [1, 3, 5, 6, 7, 9, 11, 12, 13, 16, 24, 25, 29] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we discuss applications of the union soft sets in a commutative BCI-ideals of BCI-algebras. We introduce the notion of union soft commutative BCI-ideals, and investigated related properties. We consider relations between a union soft commutative BCI-ideal and a (closed) union soft BCI-ideal. We provide conditions for a union soft BCI-ideal to be a union soft commutative BCI-ideal, and establish characterizations of a union soft commutative BCI-ideal. We construct a new union soft commutative BCI-ideal from an old one.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$. In a BCI-algebra X, the following hold:

(b1) $(\forall x, y \in X)\ (x * (x * (x * y)) = x * y)$,
(b2) $(\forall x, y \in X)\ (0 * (x * y) = (0 * x) * (0 * y))$.

A BCI-algebra X is said to be *commutative* (see [22]) if

$$(\forall x, y \in X)\,(x \leq y \ \Rightarrow \ x = y * (y * x))\,. \tag{2.1}$$

Proposition 2.1. *A BCI-algebra X is commutative if and only if it satisfies:*

$$(\forall x, y \in X)\,(x * (x * y) = y * (y * (x * (x * y))))\,. \tag{2.2}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCI-algebra X is called a *BCI-ideal* of X if it satisfies:

$$0 \in I, \tag{2.3}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \ \Rightarrow \ x \in I)\,. \tag{2.4}$$

A BCI-ideal I of a BCI-algebra X satisfies:

$$(\forall x \in X)(\forall y \in I)\,(x \leq y \ \Rightarrow \ x \in I) \tag{2.5}$$

A BCI-ideal I of a BCI-algebra X is said to be *closed* if it satisfies:

$$(\forall x \in X)\,(x \in I \ \Rightarrow \ 0 * x \in I)$$

A subset I of a BCI-algebra X is called a *commutative BCI-ideal* (briefly, *c-BCI-ideal*) of X (see [20]) if it satisfies (2.3) and

$$(x * y) * z \in I,\ z \in I \ \Rightarrow \ x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in I \tag{2.6}$$

for all $x, y, z \in X$.

Proposition 2.2 ([20])**.** *A BCI-ideal I of a BCI-algebra X is commutative if and only if $x*y \in I$ implies $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in I$.*

Proposition 2.3 ([20])**.** *Let I be a closed BCI-ideal of a BCI-algebra X. Then I is commutative if and only if it satisfies:*

$$(\forall x, y \in X)\,(x * y \in I \ \Rightarrow \ x * (y * (y * x)) \in I)$$

Observe that every c-BCI-ideal is a BCI-ideal, but the converse is not true (see [20]).

We refer the reader to the books [8, 21] for further information regarding BCK/BCI-algebras.

A soft set theory is introduced by Molodtsov [23], and Çağman et al. [4] provided new definitions and various results on soft set theory.

In what follows, let U be an initial universe set and E be a set of parameters. We say that the pair (U, E) is a *soft universe.* Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.4 ([4, 23])**.** A soft set $\mathscr{F}_A$ over U is defined to be the set of ordered pairs

$$\mathscr{F}_A := \{(x, f_A(x)) : x \in E,\ f_A(x) \in \mathscr{P}(U)\}\,,$$

where $f_A : E \to \mathscr{P}(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.

The function f_A is called the approximate function of the soft set $\mathscr{F}_A$. The subscript A in the notation f_A indicates that f_A is the approximate function of $\mathscr{F}_A$.

In what follows, denote by $S(U)$ the set of all soft sets over U.

Let $\mathscr{F}_A \in S(U)$ and let $\tau \subseteq U$. Then the *τ-exclusive set* of $\mathscr{F}_A$ is defined to be the set

$$e(\mathscr{F}_A;\tau) := \{x \in A \mid f_A(x) \subseteq \tau\}.$$

Obviously, we have the following properties:

(1) $e(\mathscr{F}_A;U) = A$.
(2) $f_A(x) = \cap\{\tau \subseteq U \mid x \in e(\mathscr{F}_A;\tau)\}$.
(3) $(\forall \tau_1, \tau_2 \subseteq U)\,(\tau_1 \subseteq \tau_2 \;\Rightarrow\; e(\mathscr{F}_A;\tau_1) \subseteq e(\mathscr{F}_A;\tau_2))$.

3. Union soft c-BCI-ideals

Definition 3.1 ([10])**.** Let $(U,E) = (U,X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft BCI-ideal* (briefly, *U-soft BCI-ideal*) over U if the approximate function f_A of $\mathscr{F}_A$ satisfies:

$$(\forall x \in A)\,(f_A(0) \subseteq f_A(x)), \tag{3.1}$$

$$(\forall x, y \in A)\,(f_A(x) \subseteq f_A(x * y) \cup f_A(y)). \tag{3.2}$$

Definition 3.2. Let $(U,E) = (U,X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft commutative BCI-ideal* (briefly, *U-soft c-BCI-ideal*) over U if the approximate function f_A of $\mathscr{F}_A$ satisfies (3.1) and

$$f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \subseteq f_A((x * y) * z) \cup f_A(z) \tag{3.3}$$

for all $x, y, z \in A$.

Example 3.3. Let $(U,E) = (U,X)$ where $X = \{0, a, 1, 2, 3\}$ is a BCI-algebra with the following Cayley table:

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Let τ_1, τ_2 and τ_3 be subsets of U such that $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0,\tau_1), (a,\tau_2), (1,\tau_3), (2,\tau_3), (3,\tau_3)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft c-BCI-ideal over U.

Theorem 3.4. *Let $(U,E) = (U,X)$ where X is a BCI-algebra. Then every U-soft c-BCI-ideal is a U-soft BCI-ideal.*

Proof. Let $\mathscr{F}_A$ be a U-soft c-BCI-ideal over U where A is a subalgebra of E. Taking $y = 0$ in (3.3) and using (a1) and (III) imply that

$$\begin{aligned} f_A(x) = f_A(x * 0) &= f_A(x * ((0 * (0 * x)) * (0 * (0 * (x * 0))))) \\ &\subseteq f_A((x * 0) * z) \cup f_A(z) = f_A(x * z) \cup f_A(z) \end{aligned}$$

for all $x, z \in A$. Therefore $\mathscr{F}_A$ is a U-soft BCI-ideal over U. □

The following example shows that the converse of Theorem 3.4 is not true.

Example 3.5. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3, 4\}$ is a BCI-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Let τ_1, τ_2 and τ_3 be subsets of U such that $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (1, \tau_2), (2, \tau_3), (3, \tau_3), (4, \tau_3)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft BCI-ideal over U. But it is not a U-soft c-BCI-ideal over U since

$$f_E(2 * ((3 * (3 * 2)) * (0 * (0 * (2 * 3))))) = \tau_3 \nsubseteq \tau_1 = f_E((2 * 3) * 0) \cup f_E(0).$$

We provide conditions for a U-soft BCI-ideal to be a U-soft c-BCI-ideal.

Theorem 3.6. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(1) *$\mathscr{F}_A$ is a U-soft c-BCI-ideal over U.*
(2) *$\mathscr{F}_A$ is a U-soft BCI-ideal over U and its approximate function f_A satisfies:*

$$(\forall x, y \in A)\left(f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \subseteq f_A(x * y)\right). \tag{3.4}$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. Then $\mathscr{F}_A$ is a U-soft BCI-ideal over U (see Theorem 3.4). If we take $z = 0$ in (3.3) and use (a1) and (3.1), then we have (3.4).

Conversely, let $\mathscr{F}_A$ be a U-soft BCI-ideal over U such that its approximate function f_A satisfies (3.4). Then $f_A(x * y) \subseteq f_A((x * y) * z) \cup f_A(z)$ for all $x, y, z \in A$ by (3.2), which implies from (3.4) that

$$f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \subseteq f_A((x * y) * z) \cup f_A(z)$$

for all $x, y, z \in A$. Therefore $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. □

Definition 3.7 ([10])**.** Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. A U-soft BCI-ideal $\mathscr{F}_A$ over U is said to be *closed* if the approximate function f_A of $\mathscr{F}_A$ satisfies:

$$(\forall x \in A)\,(f_A(0 * x) \subseteq f_A(x))\,. \tag{3.5}$$

Lemma 3.8 ([10])**.** *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$.*

(1) *If $\mathscr{F}_A$ is a U-soft BCI-ideal over U, then the approximate function f_A satisfies the following condition:*

$$(\forall x, y, z \in A)\,(x * y \leq z \;\Rightarrow\; f_A(x) \subseteq f_A(y) \cup f_A(z))\,. \tag{3.6}$$

(2) *If the approximate function f_A of $\mathscr{F}_A$ satisfies* (3.1) *and* (3.6)*, then $\mathscr{F}_A$ is a U-soft BCI-ideal over U.*

Theorem 3.9. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A$ be a closed U-soft BCI-ideal over U. Then the following are equivalent:*

(1) *$\mathscr{F}_A$ is a U-soft c-BCI-ideal over U.*
(2) *The approximate function f_A of $\mathscr{F}_A$ satisfies:*

$$(\forall x, y \in A)\,(f_A(x * (y * (y * x))) \subseteq f_A(x * y))\,. \tag{3.7}$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. Note that

$$\begin{aligned}
&(x * (y * (y * x))) * (x * ((y * (y * x)) * (0 * (0 * (x * y)))))\\
&\quad \leq ((y * (y * x)) * (0 * (0 * (x * y)))) * (y * (y * x))\\
&\quad = ((y * (y * x)) * (y * (y * x))) * (0 * (0 * (x * y)))\\
&\quad = 0 * (0 * (0 * (x * y))) = 0 * (x * y)
\end{aligned}$$

for all $x, y \in A$. Using Lemma 3.8(1), (3.4) and (3.5), we have

$$\begin{aligned}
&f_A(x * (y * (y * x)))\\
&\quad \subseteq f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \cup f_A(0 * (x * y))\\
&\quad \subseteq f_A(x * y) \cup f_A(0 * (x * y)) = f_A(x * y)
\end{aligned}$$

for all $x, y \in A$. Now suppose that the approximate function f_A of $\mathscr{F}_A$ satisfies (3.7). Since

$$\begin{aligned}
&(x * ((y * (y * x)) * (0 * (0 * (x * y))))) * (x * (y * (y * x)))\\
&\quad \leq (y * (y * x)) * ((y * (y * x)) * (0 * (0 * (x * y))))\\
&\quad \leq 0 * (0 * (x * y)),
\end{aligned}$$

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it follows from Lemma 3.8(1), (3.5) and (3.7) that

$$\begin{aligned} &f_A(x*((y*(y*x))*(0*(0*(x*y))))) \\ &\quad\subseteq f_A(x*(y*(y*x)))\cup f_A(0*(0*(x*y))) \\ &\quad\subseteq f_A(x*y)\cup f_A(0*(0*(x*y))) = f_A(x*y) \end{aligned}$$

for all $x, y \in A$. By Theorem 3.6, $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. □

Theorem 3.10. *Let $(U, E) = (U, X)$ where X is a commutative BCI-algebra. Then every closed U-soft BCI-ideal is a U-soft c-BCI-ideal.*

Proof. Let $\mathscr{F}_A$ be a closed U-soft BCI-ideal over U where A is a subalgebra of E. Using (a3), (b1), (I), (III) and Proposition 2.1, we have

$$\begin{aligned} (x*(y*(y*x)))*(x*y) &= (x*(x*y))*(y*(y*x)) \\ &= (y*(y*(x*(x*y))))*(y*(y*x)) \\ &= (y*(y*(y*x)))*(y*(x*(x*y))) \\ &= (y*x)*(y*(x*(x*y))) \\ &\leq (x*(x*y))*x = 0*(x*y) \end{aligned}$$

It follows from Lemma 3.8(1) and (3.5) that

$$f_A(x*(y*(y*x)))\subseteq f_A(x*y)\cup f_A(0*(x*y)) = f_A(x*y),$$

for all $x, y \in A$. Therefore, by Theorem 3.9, $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. □

Using the notion of τ-exclusive sets, we consider a characterization of a U-soft c-BCI-ideal.

Lemma 3.11 ([10])**.** *Let $(U, E) = (U, X)$ where X is a BCI-algebra, Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent.*

(1) *$\mathscr{F}_A$ is a U-soft BCI-ideal over U.*
(2) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a BCI-ideal of A for any $\tau \subseteq U$.*

Theorem 3.12. *Let $(U, E) = (U, X)$ where X is a BCI-algebra, Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent.*

(1) *$\mathscr{F}_A$ is a U-soft c-BCI-ideal over U.*
(2) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a c-BCI-ideal of A for any $\tau \subseteq U$.*

Proof. Assume that $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. Then $\mathscr{F}_A$ is a U-soft BCI-ideal over U by Theorem 3.4. Hence $e(\mathscr{F}_A;\tau)$ is a BCI-ideal of A for all $\tau \subseteq U$ by Lemma 3.11. Let $\tau \subseteq U$ and $x, y \in A$ be such that $x * y \in e(\mathscr{F}_A;\tau)$. Then $f_A(x * y) \subseteq \tau$, and so

$$f_A(x*((y*(y*x))*(0*(0*(x*y)))))\subseteq f_A(x*y)\subseteq\tau$$

by Theorem 3.6. Thus

$$x*((y*(y*x))*(0*(0*(x*y))))\in e(\mathscr{F}_A;\tau).$$

It follows from Proposition 2.2 that $e(\mathscr{F}_A;\tau)$ is a c-BCI-ideal of A.

Conversely, suppose that the nonempty τ-exclusive set of $\mathscr{F}_A$ is a c-BCI-ideal of A for any $\tau \subseteq U$. Then $e(\mathscr{F}_A;\tau)$ is a BCI-ideal of A for all $\tau \subseteq U$. Hence $\mathscr{F}_A$ is a U-soft BCI-ideal over U by Lemma 3.11. Let $x, y \in A$ be such that $f_A(x*y) = \tau$. Then $x*y \in e(\mathscr{F}_A;\tau)$, and so

$$x*((y*(y*x))*(0*(0*(x*y)))) \in e(\mathscr{F}_A;\tau)$$

by Proposition 2.2. Hence

$$f_A\left(x*((y*(y*x))*(0*(0*(x*y))))\right) \subseteq \tau = f_A(x*y).$$

It follows from Theorem 3.6 that $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. □

The c-BCI-ideals $e(\mathscr{F}_A;\tau)$ in Theorem 3.12 are called the *exclusive c-BCI-ideals* of $\mathscr{F}_A$.

Theorem 3.13. *Let $(U,E) = (U,X)$ where X is a BCI-algebra. Let $\mathscr{F}_E, \mathscr{G}_E \in S(U)$ such that*

(i) $(\forall x \in E)\left(f_E(x) \subseteq g_E(x)\right)$,
(ii) *$\mathscr{F}_E$ and $\mathscr{G}_E$ are U-soft BCI-ideals over U.*

If $\mathscr{F}_E$ is closed and $\mathscr{G}_E$ is a U-soft c-BCI-ideal over U, then $\mathscr{F}_E$ is also a U-soft c-BCI-ideal over U.

Proof. Assume that $\mathscr{F}_E$ is closed and $\mathscr{G}_E$ is a U-soft c-BCI-ideal over U. Let τ be a subset of U such that $e(\mathscr{F}_E;\tau) \neq \emptyset \neq e(\mathscr{G}_E;\tau)$. Then $e(\mathscr{F}_E;\tau)$ and $e(\mathscr{G}_E;\tau)$ are BCI-ideals of E and obviously $e(\mathscr{F}_E;\tau) \supseteq e(\mathscr{G}_E;\tau)$. Let $x \in e(\mathscr{F}_E;\tau)$. Then $f_E(x) \subseteq \tau$, and so $f_E(0*x) \subseteq f_E(x) \subseteq \tau$ since $\mathscr{F}_E$ is closed. Thus $0*x \in e(\mathscr{F}_E;\tau)$, and thus $e(\mathscr{F}_E;\tau)$ is a closed BCI-ideal of E. Since $\mathscr{G}_E$ is a U-soft c-BCI-ideal over U, it follows from Theorem 3.12 that $e(\mathscr{G}_E;\tau)$ is a c-BCI-ideal of E. Let $x, y \in E$ be such that $x*y \in e(\mathscr{F}_E;\tau)$. Then $0*(x*y) \in e(\mathscr{F}_E;\tau)$. Since $(x*(x*y))*y = 0 \in e(\mathscr{G}_E;\tau)$, it follows from Proposition 2.2 that

$$\begin{aligned}
&(x*(x*y))*(y*(y*(x*(x*y)))) \\
&\quad = (x*(x*y))*((y*(y*(x*(x*y))))*(0*(0*((x*(x*y))*y)))) \\
&\quad \in e(\mathscr{G}_E;\tau) \subseteq e(\mathscr{F}_E;\tau)
\end{aligned}$$

so from (a3) that

$$(x*(y*(y*(x*(x*y)))))*(x*y) \in e(\mathscr{F}_E;\tau).$$

Hence $x*(y*(y*(x*(x*y)))) \in e(\mathscr{F}_E;\tau)$ by (2.4). Note that

$$\begin{aligned}
&(x*(y*(y*x)))*(x*(y*(y*(x*(x*y))))) \\
&\quad \leq (y*(y*(x*(x*y))))*(y*(y*x)) \\
&\quad \leq (y*x)*(y*(x*(x*y))) \\
&\quad \leq (x*(x*y))*x = 0*(x*y) \in e(\mathscr{F}_E;\tau).
\end{aligned}$$

Using (2.5) and (2.4), we have $x*(y*(y*x)) \in e(\mathscr{F}_E;\tau)$. Hence $e(\mathscr{F}_E;\tau)$ is a c-BCI-ideal of E by Proposition 2.3. Therefore $\mathscr{F}_E$ is a U-soft c-BCI-ideal over U by Theorem 3.12. □

Theorem 3.14. *Let $(U, E) = (U, X)$ and $\mathscr{F}_A \in S(U)$ where X is a BCI-algebra and A is a subalgebra of E. For a subset τ of U, define a soft set $\mathscr{F}_A^*$ over U by*

$$f_A^* : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} f_A(x) & \text{if } x \in e(\mathscr{F}_A; \tau), \\ U & \text{otherwise.} \end{cases}$$

If $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U, then so is $\mathscr{F}_A^*$.

Proof. If $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U, then $e(\mathscr{F}_A; \tau)$ is a c-BCI-ideal of A for any $\tau \subseteq U$. Hence $0 \in e(\mathscr{F}_A; \tau)$, and so $f_A^*(0) = f_A(0) \subseteq f_A(x) \subseteq f_A^*(x)$ for all $x \in A$. Let $x, y, z \in A$. If $(x * y) * z \in e(\mathscr{F}_A; \tau)$ and $z \in e(\mathscr{F}_A; \tau)$, then $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in e(\mathscr{F}_A; \tau)$ and so

$$\begin{aligned} &f_A^*(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \\ &\quad = f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \\ &\quad \subseteq f_A((x * y) * z) \cup f_A(z) = f_A^*((x * y) * z) \cup f_A^*(z). \end{aligned}$$

If $(x * y) * z \notin e(\mathscr{F}_A; \tau)$ or $z \notin e(\mathscr{F}_A; \tau)$, then $f_A^*((x * y) * z) = U$ or $f_A^*(z) = U$. Hence

$$f_A^*(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \subseteq U = f_A^*((x * y) * z) \cup f_A^*(z).$$

This shows that $\mathscr{F}_A^*$ is a U-soft c-BCI-ideal over U. □

Theorem 3.15. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then any c-BCI-ideal of E can be realized as an exclusive c-BCI-ideal of some U-soft c-BCI-ideal over U.*

Proof. Let A be a c-BCI-ideal of E. For any subset $\tau \subsetneq U$, let $\mathscr{F}_A$ be a soft set over U defined by

$$f_A : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in A, \\ U & \text{if } x \notin A. \end{cases}$$

Obviously, $f_A(0) \subseteq f_A(x)$ for all $x \in E$. For any $x, y, z \in E$, if $(x * y) * z \in A$ and $z \in A$ then $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in A$. Hence

$$f_A((x * y) * z) \cup f_A(z) = \tau = f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))).$$

If $(x * y) * z \notin A$ or $z \notin A$ then $f_A((x * y) * z) = U$ or $f_A(z) = U$. It follows that

$$f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \subseteq U = f_A((x * y) * z) \cup f_A(z).$$

Therefore $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U, and clearly $e(\mathscr{F}_A; \tau) = A$. This completes the proof. □

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SOME INEQUALITIES WHICH HOLD FOR STARLIKE LOG-HARMONIC MAPPINGS OF ORDER α

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Abstract

Let $H(D)$ be the linear space of all analytic functions defined on the open disc $D = \{z| \quad |z| < 1\}$. A log-harmonic mappings is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = w\frac{\overline{f}}{f}f_z$$

where $w(z) \in H(D)$ is second dilatation such that $|w(z)| < 1$ for all $z \in D$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)}$$

where $h(z)$ and $g(z)$ are analytic function in D. On the other hand, if f vanishes at $z = 0$ but it is not identically zero then f admits following representation

$$f(z) = z\,|z|^{2\beta}\,h(z)\overline{g(z)}$$

where $Re\beta > -\frac{1}{2}$, h and g are analytic in D, $g(0) = 1$, $h(0) \neq 0$. Let $f = z\,|z|^{2\beta}\,h\overline{g}$ be a univalent log-harmonic mapping.

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We say that f is a starlike log-harmonic mapping of order α if

$$\frac{\partial(\arg f(re^{i\theta}))}{\partial\theta} = Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > \alpha\ ,\ 0 \leq \alpha < 1.\ (\forall z \in U)$$

and denote by $S^*_{lh}(\alpha)$ the set of all starlike log-harmonic mappings of order α.

The aim of this paper is to define some inequalities of starlike log-harmonic functions of order α $(0 \leq \alpha \leq 1)$.

I. Introduction

Let Ω be the family of functions $\phi(z)$ regular in the unit disc D and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in D$.

Next, denote by $P(\alpha)(0 \leq \alpha < 1)$ the family of functions

$$p(z) = 1 + p_1 z + ...$$

regular in D and such that $p(z)$ in $P(\alpha)$ if and only if

$$p(z) = \frac{1 + (1-2\alpha)\ \phi(z)}{1 - \phi(z)}$$

for some functions $z \in \Omega$ and every $z \in D$.

Let $S_1(z)$ and $S_2(z)$ be analytic functions in the open unit disc, with $S_1(0) = S_2(0)$, if $S_1(z) = S_2(\phi(z))$ then we say that $S_1(z)$ is subordinate to $S_2(z)$, where $\phi(z) \in \Omega([4])$, and we write $S_1(z) \prec S_2(z)$.

Let $H(D)$ be the linear space of all analytic functions defined on the open disc $D = \{z|\,|z| < 1\}$. A log-harmonic mappings is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = w\frac{\overline{f}}{f}f_z$$

where $w(z) \in H(D)$ is second dilatation such that $|w(z)| < 1$ for all $z \in D$.

It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)}$$

where $h(z)$ and $g(z)$ are analytic function in D.

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On the other hand, if f vanishes at $z=0$ but it is not identically zero then f admits following representation

$$f(z) = z\left|z\right|^{2\beta} h(z)\overline{g(z)}$$

where $Re\beta > -\frac{1}{2}$, h and g are analytic in D, $g(0)=1$, $h(0) \neq 0$.

Let $f = z\left|z\right|^{2\beta} h\overline{g}$ be a univalent log-harmonic mapping. We say that f is a starlike logharmonic mapping of order α if

$$\frac{\partial(\arg f(re^{i\theta}))}{\partial\theta} = Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > \alpha \;,\; 0 \leq \alpha < 1. \;\; (\forall z \in U)$$

and denote by $S^*_{lh}(\alpha)$ the set of all starlike log-harmonic mappings of order α([3]).

If $\alpha = 0$, we get the class of starlike log-harmonic mappings. Also, let

$$ST(\alpha) = \left\{ f \in S^*_{lh}(\alpha)\, and\, f \in H(U) \right\}.$$

If $f \in S^*_{lh}(0)$ then $F(\varsigma) = \log(f(e^{\varsigma}))$ is univalent and harmonic on the half plane $\{\varsigma \mid \; Re\{\varsigma\} < 0\}$. It is known that F is closely related with the theory of nonparametric minimal surfaces over domains of the form $-\infty < u < u_0(v)$, $u_0(v+2\pi) = u_0(v)$, see ([1],[2]).

In this paper, we obtain Marx-Strohhacker Inequality and new distortion theorems using the subordination prinsiple for the starlike log-harmonic mappings of order α, previously studied by Z. Abdulhadi and Y. Abu Muhanna [3] who obtained the representation theorem and a different distortion theorem for the same class.

II. Main Results

Theorem 2.1. Let $f(z) = zh(z)\overline{g(z)}$ be an analytic logaritmic harmonic function in the open unit disc U. If $f(z)$ satisfies the condition

$$z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \prec \frac{2(1-\alpha)z}{1-z} = F(z) \tag{1}$$

then $f \in S^*_{lh}(\alpha)$.

Proof. We define the function by

$$\frac{h}{g} = (1-\phi(z))^{-2(1-\alpha)} \tag{2}$$

where $(1-\phi(z))^{-2(1-\alpha)}$ has the value 1 at $z = 0$. Then $w(z)$ is analytic and $\phi(0) = 0$. If we take the logarithmic derivative of (2) and the after brief calculations, we get

$$z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \prec \frac{2(1-\alpha)z\phi'(z)}{1-\phi(z)}$$

Now it is easy to realize that the subordination (1) is equivalent to $|\phi(z)| < 1$ for all $z \in U$. Indeed assume the contrary: then there is a $z_1 \in U$ such that $|\phi(z_1)| = 1$, so by I.S. Jack Lemma $z_1\phi'(z_1) = k\phi(z_1)$ for some $k \geq 1$ and for such $z_1 \in U$, we have

$$z_1\frac{h'(z_1)}{h(z_1)} - z_1\frac{g'(z_1)}{g(z_1)} \prec \frac{2(1-\alpha)k\phi(z_1)}{1-\phi(z_1)} = F(\phi(z_1)) \notin F(U)$$

but this contradicts (1); so our assumption is wrong, i.e, $|\phi(z)| < 1$ for all $z \in u$. By using condition (1) we get

$$1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} = \frac{1+(1-2\alpha)\phi(z)}{1-\phi(z)} \quad . \tag{3}$$

The equality (3) shows that $f(z) \in S^*_{lh}(\alpha)$.

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Corollary 2.2. For the starlike logharmonic functions of order α, we have Marx-Strohhacker Inequality is

$$\left|1-\left(\frac{g}{h}\right)^{\frac{1}{2(1-\alpha)}}\right|<1$$

g and $h$are analytic in u and $0\notin hg(u)$.

Proof. Using theorem 2.1 we have

$$(1-\phi(z))^{\frac{1-2\alpha+1}{-1}}=\frac{h}{g}\ \Rightarrow(1-\phi(z))^{-2(1-\alpha)}=\frac{h}{g}\ \Rightarrow\frac{1}{(1-\phi(z))^{2(1-\alpha)}}=\frac{h}{g}\Rightarrow\frac{1}{1-\phi(z)}=\left(\frac{h}{g}\right)^{\frac{1}{2(1-\alpha)}}\Rightarrow$$
$$1-\phi(z)=\left(\frac{g}{h}\right)^{\frac{1}{2(1-\alpha)}}\ \Rightarrow1-\left(\frac{g}{h}\right)^{\frac{1}{2(1-\alpha)}}=\phi(z)\ \Rightarrow\left|1-\left(\frac{g}{h}\right)^{\frac{1}{2(1-\alpha)}}\right|=|\phi(z)|<1.$$

Theorem 2.3. If $f\in S^{*}_{lh}(\alpha)$ then

$$\frac{1}{(1+r)^{2(1-\alpha)}}\leq\left|\frac{h}{g}\right|<\frac{1}{(1-r)^{2(1-\alpha)}} \tag{4}$$

Proof. The set of the values of the function $\left(\frac{2(1-\alpha)z}{(1-z)}\right)$ is the closed disc with the centre C and the radius ρ, where

$$C=C(r)=\left(\frac{2(1-\alpha)r^2}{1-r^2},0\right)\quad,\quad\rho=\rho(r)=\frac{2(1-\alpha)r}{1-r^2}.$$

Using the subordination, we can write

$$\left|\left(z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}\right)-\frac{2(1-\alpha)r^2}{1-r^2}\right|\leq\frac{2(1-\alpha)r}{1-r^2}. \tag{5}$$

Therefore we have

$$-\frac{2(1-\alpha)r}{1+r}\leq Re\left(z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}\right)\leq\frac{2(1-\alpha)r}{1-r}. \tag{6}$$

On the other hand

$$Re\left(z\frac{h'}{h}\right)-Re\left(z\frac{g'}{g}\right)=r\frac{\partial}{\partial r}\left(\log|h|-\log|g|\right). \tag{7}$$

If we consider the relations (5), (6), (7) together we obtain

$$-\frac{2(1-\alpha)}{1+r}\leq\frac{\partial}{\partial r}\left(\log|h|-\log|g|\right)\leq\frac{2(1-\alpha)}{1-r} \tag{8}$$

After the integrating we obtain (4).

Theorem 2.4. If $f \in S^*_{lh}(\alpha)$ then

$$\frac{|b_1| - |a_1|r}{|a_1| - |b_1|r}(1-r)^{2(1-\alpha)} \leq \frac{|g'(z)|}{|h'(z)|} \leq \frac{|b_1| + |a_1|r}{|a_1| + |b_1|r}(1+r)^{2(1-\alpha)}. \tag{9}$$

Proof. Using theorem 2.3 we can write

$$(1-r)^{2(1-\alpha)} \leq \frac{|g(z)|}{|h(z)|} \leq (1+r)^{2(1-\alpha)} \tag{10}$$

On the other hand, since f is solution of the non-linear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = w\frac{\overline{f}}{f}f_z$$

then we obtain

$$w(z) = \frac{\frac{g'(z)}{h'(z)}}{\frac{g(z)}{h(z)}} = \frac{b_1}{a_1} + \dots \tag{11}$$

Now we define the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)}, z \in D. \tag{12}$$

Therefore $\phi(z)$ satisfies the condition of Schwarz lemma. Using the estimate the Schwarz lemma $|\phi(z)| \leq r$,which given

$$|\phi(z)| = \left|\frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)}\right| \leq r \tag{13}$$

The inequality (13) can be written in the following form

$$\left|\frac{w(z) - \frac{b_1}{a_1}}{1 - \frac{\overline{b_1}}{\overline{a_1}}w(z)}\right| \leq r \Rightarrow \left|w(z) - \frac{b_1}{a_1}\right| \leq r\left|1 - \frac{\overline{b_1}}{\overline{a_1}}w(z)\right| \tag{14}$$

The inequality (14) is equivalent

$$\left|w(z) - \frac{(1-r^2)\left|\frac{b_1}{a_1}\right|}{1 - (\frac{b_1}{a_1})^2 r^2}\right| \leq \frac{\left(1 - \left|\frac{b_1}{a_1}\right|^2\right) r}{1 - \left|\frac{b_1}{a_1}\right|^2 r^2} \tag{15}$$

6

The equality holds in the inequality (15) only for the function

$$w(z) = \frac{\frac{g'(z)}{h'(z)}}{\frac{g(z)}{h(z)}} \tag{16}$$

From the inequality (15) we have

$$\frac{\left|\frac{b_1}{a_1}\right| - r}{1 - \left|\frac{b_1}{a_1}\right| r} \leq |w(z)| \leq \frac{\left|\frac{b_1}{a_1}\right| + r}{1 + \left|\frac{b_1}{a_1}\right| r} \tag{17}$$

Considering the relation (10), (17) together, end after the simple calculations,

$$\frac{|b_1| - |a_1| r}{|a_1| - |b_1| r} \left|\frac{g(z)}{h(z)}\right| \leq \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{|b_1| + |a_1| r}{|a_1| + |b_1| r} \left|\frac{g(z)}{h(z)}\right| \tag{18}$$

using inequality (4) in the inequality (18) we get (8).

7

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N-differentiation composition operators from weighted Banach spaces of holomorphic function to weighted Bloch spaces

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[1]

Abstract

In this paper, we characterize nth differentiation composition operators from weighted Banach space of holomorphic function to weighted Bloch space, and give some necessary and sufficient conditions for the boundedness and compactness of the operators.

1 Introduction

Let $H(\mathbb{D})$ and $S(\mathbb{D})$ denote the class of analytic functions and analytic self-maps on the unit disk $\mathbb{D}$ of the complex plane of $\mathbb{C}$, respectively. Let v and w be strictly positive continuous and bounded functions (weight) on $\mathbb{D}$.

Weighted Banach spaces of holomorphic functions is defined by

$$H_v^\infty = \{f \in H(\mathbb{D}) : \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\},$$

endowed with the weighted sup-norm $\|.\|_v$.

An $f \in H(\mathbb{D})$ belongs to weighted Bloch spaces $\mathcal{B}_w$ if

$$b_w(f) = \sup_{z\in\mathbb{D}} w(z)|f'(z)| < \infty.$$

The quantity $b_w(f)$ defines a seminorm on $\mathcal{B}_w$, while a natural norm is given by

$$\|f\|_{\mathcal{B}_w} = |f(0)| + b_w(f).$$

This norm makes $\mathcal{B}_w$ into a Banach space.

By $\mathcal{B}_{w,0}$ we denote the little weighted Bloch space, the subspace of $\mathcal{B}_w$, consisting of all $f \in \mathcal{B}_w$ such that

$$\lim_{|z|\to 1} w(z)|f'(z)| = 0.$$

Each ϕ in $S(\mathbb{D})$ induces through composition a linear composition operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D}), f \mapsto f \circ \phi$. And n-differentiating composition operator is a linear operator defined by

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$D^n_\phi : H(\mathbb{D}) \to H(\mathbb{D}), f \mapsto f^{(n)}(\phi)$. We are interested in D^n_ϕ acting from weighted Banach spaces of holomorphic functions to weighted Bloch spaces.

In the setting of weighted spaces the so-called associated weight plays an important role. For a weight v its associated weight $\widetilde{v}$ is defined follows:

$$\widetilde{v}(z) = \frac{1}{\sup\{|f(z)| : f \in H^\infty_v, \|f\|_v \le 1\}} = \frac{1}{\|\delta_z\|_{H^\infty_v}},$$

where δ_z denotes the point evaluation of z. By [1] the associated weight $\widetilde{v}$ is continuous, $\widetilde{v} \ge v > 0$ and for every $z \in \mathbb{D}$ we can find $f_z \in H^\infty_v$ with $\|f_z\|_v \le 1$ such that $|f_z(z)| = \frac{1}{\widetilde{v}(z)}$.

We say that a weight v is radial if $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. A positive continuous function v is called normal if there exist $\delta \in [0, 1)$ and $s, t(0 < s < t)$ such that for every $z \in \mathbb{D}$ with $|z| \in [\delta, 1)$,

$$\frac{v(|z|)}{(1-|z|)^s} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{|z|\to 1} \frac{v(|z|)}{(1-|z|)^s} = 0;$$

$$\frac{v(|z|)}{(1-|z|)^t} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{|z|\to 1} \frac{v(|z|)}{(1-|z|)^t} = \infty.$$

A radial, non-increasing weight is called typical if $\lim_{|z|\to 1} v(z) = 0$. When studying the structure and isomorphism classes of the space H^∞_v (see [6, 7]), Lusky introduced the following condition $(L1)$ (renamed after the author) for radial weights:

$$(L1) \inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{1-2^{-n}} > 0,$$

which will play a great role in this article. Moreover, radial weights with (L1) (for example, see [2]) are essential, that is, we can find a constant $k > 0$ such that

$$v(z) \le \widetilde{v}(z) \le kv(z) \text{ for every } z \in \mathbb{D}.$$

Now, let $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, z \in \mathbb{D}$, be the Möbius transformation that interchanges a and 0. We will use the fact that derivative of φ_a is given by

$$\varphi'_a(z) = -\frac{1-|a|^2}{(1-\bar{a}z)^2} \text{ for every } z \in \mathbb{D}.$$

Our aim in this note is to characterize boundedness and compactness of operator D^ϕ_n from weighted Banach spaces of holomorphic functions to weighted Bloch spaces in terms of the involved weights as well as the inducing map. For $n = 0$ and $n = 1$, as corollaries we get a characterization of boundedness and compactness of C_ϕ and $C_\phi D$ that act from weighted Banach spaces of holomorphic functions to weighted Bloch spaces.

Throughout this paper, we will use the symbol C to denote a finite positive number, and it may differ from one occurrence to the other.

2 Background and Some Lemmas

Now let us state a couple of lemmas, which are used in the proof of the main results in the next sections. The first lemma is taken from [9].

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Lemma 1. *Let v be a radial weight satisfying condition (L1). There is a constant $C > 0$ (depending only on the weight v) such that for all $f \in H_v^\infty$,*

$$|f^{(n)}(z)| \leq C \frac{\|f\|_v}{v(z)(1-|z|^2)^n}, \tag{1}$$

for every $z \in \mathbb{D}$ and $n \in \mathbb{N}$.

Proof. We will prove the theorem by mathematical induction.

For $n = 1$, see Lemma 2 in [9].

If (1) is true for $n-1$. Then for n, let $u(z) = v(z)(1-|z|^2)^{n-1}$, since

$$|f^{(n-1)}(z)| \leq C \frac{\|f\|_v}{v(z)(1-|z|^2)^{n-1}},$$

then $f^{(n-1)} \in H_u^\infty$.

For $f^{(n-1)}$ using the result of $n = 1$ the lemma is proved. □

The following result is well-known (see, e.g. [3, 8])

Lemma 2. *Suppose that w is a normal weight and v is a radial weight satisfying (L1). Then the operator $D_\phi^n : H_v^\infty \to \mathcal{B}_w$(or $\mathcal{B}_{w,0}$) is compact if and only if whenever $\{f_m\}$ is a bounded sequence in H_v^∞ with $f_m \to 0$ uniformly on compact subsets of $\mathbb{D}$, and then $\|D_\phi^n f_m\|_{\mathcal{B}_w} \to 0$.*

The following lemma can be proved similarly to Lemma 1 in [4] (see, also [5]). It will be useful to give a criterion for compactness in $\mathcal{B}_{w,0}$.

Lemma 3. *Assume w is normal. A closed set K in $\mathcal{B}_{w,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f \in K} w(z)|f'(z)| = 0. \tag{2}$$

3 The Boundedness of $D_\phi^n : H_v^\infty \to \mathcal{B}_w$(or $\mathcal{B}_{w,0}$)

In this section we formulate and prove results regarding the boundedness of the operator $D_\phi^n : H_v^\infty \to \mathcal{B}_w$(*or* $\mathcal{B}_{w,0}$).

Theorem 1. *Suppose that w be arbitrary weight, v be a radial weight satisfying condition (L1), then $D_\phi^n : H_v^\infty \to \mathcal{B}_w$ is bounded if and only if*

$$\sup_{z \in \mathbb{D}} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} < \infty, \tag{3}$$

Proof. First, we assume that the operator $D_\phi^n : H_v^\infty \to \mathcal{B}_w$ is bounded. Fix a point $a \in \mathbb{D}$, and consider the function

$$f_a(z) = \varphi_a^{n+1}(z) g_a(z) \text{ for every } z \in \mathbb{D},$$

where g_a is a function in the unit ball of H_v^∞ such that $g_a(a) = \frac{1}{\tilde{v}(a)}$. Then

$$\|f_a\|v = \sup_{z \in \mathbb{D}} v(z)|f_a(z)| \leq \sup_{z \in \mathbb{D}} v(z)|g_a(z)| \leq 1.$$

It is easy to check that

$$(\varphi_a^{n+1})^{(k)}(a) = 0, \quad k = 0, 1, ..., n;$$

$$(\varphi_a^{n+1})^{(n+1)}(a) = \frac{(-1)^{n+1}(n+1)!}{(1-|a|^2)^{n+1}}.$$

So

$$f_a^{(n+1)}(a) = \sum_{k=0}^{n+1} C_{n+1}^k (\varphi_a^{n+1})^{(k)}(a) g_a^{(n+1-k)}(a) = \frac{(-1)^{n+1}(n+1)!}{(1-|a|^2)^{n+1}\widetilde{v}(a)}.$$

Then by the boundedness of $D_\phi^n : H_v^\infty \to \mathcal{B}_w$, we have

$$\begin{aligned} \infty &> \|D_\phi^n f_{\phi(a)}\|_{\mathcal{B}_w} \geq \sup_{z\in\mathbb{D}} w(z)|f_{\phi(a)}^{(n+1)}(\phi(z))\phi'(z)| \\ &\geq w(a)|f_{\phi(a)}^{(n+1)}(\phi(a))\phi'(a)| = \frac{(n+1)!w(a)|\phi'(a)|}{(1-|\phi(a)|^2)^{n+1}\widetilde{v}(\phi(a))}. \end{aligned}$$

Since v has (L1), the weights v and $\widetilde{v}$ are equivalent then $\widetilde{v}$ can be replaced by v, and combine with the arbitrariness of $a \in \mathbb{D}$, we obtain (3).

Conversely, an application of Lemma 1 yields

$$w(z)|f^{(n+1)}(\phi(z))\phi'(z)| \leq C\frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}}\|f\|_v, \tag{4}$$

and

$$|f^{(n)}(\phi(0))| \leq C\frac{\|f\|_v}{v(\phi(0))(1-|\phi(0)|^2)^n}.$$

Combine with this and taking the supremum in (4) over $\mathbb{D}$, then employing condition (3), we see that $D_\phi^n : H_v^\infty \to \mathcal{B}_w$ must be bounded. □

By the similar proof of Theorem 1 we see that the following result is true.

Theorem 2. *Suppose that w be arbitrary weight, v be a radial weight satisfying condition (L1), then $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$ is bounded if and only if*

$$\lim_{|z|\to 1} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} = 0. \tag{5}$$

Especially, for $n = 0$, we obtain necessary and sufficient conditions for the boundedness of the operators $C_\phi : H_v^\infty \to \mathcal{B}_w$(or $\mathcal{B}_{w,0}$).

Corollary 1. *Suppose that w be arbitrary weight, v be a radial weight satisfying condition (L1), then the following statements hold:*

(i) $C_\phi : H_v^\infty \to \mathcal{B}_w$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)} < \infty.$$

(ii) $C_\phi : H_v^\infty \to \mathcal{B}_{w,0}$ is bounded if and only if

$$\lim_{|z|\to 1} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)} = 0.$$

For $n = 1$, D_ϕ^n is the operator $C_\phi D$, then we have the following corollary .

Corollary 2. *Suppose that w be arbitrary weight, v be a radial weight satisfying condition (L1), then the following statements hold:*

(i) $C_\phi D : H_v^\infty \to \mathcal{B}_w$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^2} < \infty.$$

(ii) $C_\phi D : H_v^\infty \to \mathcal{B}_{w,0}$ is bounded if and only if

$$\lim_{|z|\to 1} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^2} = 0.$$

4 The Compactness of $D_\phi^n : H_v^\infty \to \mathcal{B}_w$(or $\mathcal{B}_{w,0}$)

In this section, we turn our attention to the question of compactness.

Theorem 3. *Suppose that w be arbitrary weight, v be a radial weight satisfying condition (L1). Then $D_\phi^n : H_v^\infty \to \mathcal{B}_w$ is compact if and only if*

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} = 0. \tag{6}$$

Proof. First, we assume that the operator $D_\phi^n : H_v^\infty \to \mathcal{B}_w$ is compact. Let $\{z_m\}_m \subset \mathbb{D}$ be a sequence with $|\phi(z_m)| \to 1$ such that

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} = \lim_{m\to\infty} \frac{w(z_m)|\phi'(z_m)|}{v(\phi(z_m))(1-|\phi(z_m)|^2)^{n+1}}.$$

By passing to a subsequence and still denoted by $\{z_m\}_m$, we assume that there is $N \in \mathbb{N}$, such that $|\phi(z_m)|^m \geq \frac{1}{2}$ for every $m \geq N$. For every $m \in \mathbb{N}$, we consider functions

$$f_m(z) = z^m \varphi_{\phi(z_m)}^{n+1}(z) g_{\phi(z_m)}(z) \ \text{ for every } \ z \in \mathbb{D},$$

where $g_{\phi(z_m)}$ is a function in the unit ball of H_v^∞ such that $|g_{\phi(z_m)}(\phi(z_m))| = \frac{1}{\widetilde{v}(\phi(z_m))}$. Again since v has (L1), $\widetilde{v}$ may be replaced by v. Obviously, $\{f_m\}_m \subset H_v^\infty$ is a bounded sequence that tends to zero uniformly on the compact subsets of $\mathbb{D}$. Hence by Lemma 2, we have that $\|D_\phi^n f_m\|_{\mathcal{B}_w} \to 0$. Moreover,

$$(z^m \varphi_{\phi(z_m)}^{n+1})^{(k)}(\phi(z_m)) = 0, \quad k = 0, 1, ..., n;$$

$$(z^m \varphi_{\phi(z_m)}^{n+1})^{(n+1)}(\phi(z_m)) = \frac{(-1)^{n+1}(n+1)!\phi^m(z_m)}{(1-|\phi(z_m)|^2)^{n+1}}.$$

Since

$$f_m^{(n+1)}(\phi(z_m)) = \sum_{k=0}^{n+1} C_{n+1}^k (z^m \varphi_{\phi(z_m)}^{n+1})^{(k)} g_{\phi_{z_m}}^{(n+1-k)}(\phi(z_m)).$$

Therefore $|f_m^{(n+1)}(\phi(z_m))| = \frac{(n+1)!|\phi(z_m)|^m}{\widetilde{v}(\phi(z_m))(1-|\phi(z_m)|^2)^{n+1}}$, and for $m \geq N$

$$\begin{aligned} 0 \ \leftarrow \ & \|D_\phi^n f_m\|_{\mathcal{B}_w} \geq w(z_m)|f_m^{(n+1)}(\phi(z_m))\phi'(z_m)| \\ = \ & \frac{(n+1)!w(z_m)|\phi'(z_m)||\phi(z_m)|^m}{\widetilde{v}(\phi(z_m))(1-|\phi(z_m)|^2)^{n+1}} \\ \geq \ & \frac{1}{2} \frac{w(z_m)|\phi'(z_m)|}{v(\phi(z_m))(1-|\phi(z_m)|^2)^{n+1}}, \end{aligned}$$

and the claim follows.

Conversely, suppose that (6) holds. Let $\{f_m\}_m \subset H_v^\infty$ be a bounded sequence which converges to zero uniformly on the compact subsets of $\mathbb{D}$, we may assume that $\|f_m\|_v \leq 1$ for every $m \in \mathbb{N}$. By Lemma 2 we have to show that

$$\|D_\phi^n f_m\|_{\mathcal{B}_w} \to 0 \text{ if } m \to \infty.$$

Let us fix $\varepsilon > 0$. By hypothesis there is $0 < r < 1$ such that

$$\frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} < \frac{\varepsilon}{2C} \text{ if } |\phi(z)| > r,$$

where C is the constant given in Lemma 1. Thus, if $|\phi(z)| > r$, by Lemma 1,

$$w(z)|\phi'(z)||f_m^{(n+1)}(\phi(z))| \leq C\frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}}\|f_m\|_v < \frac{\varepsilon}{4}. \tag{7}$$

Now, we can find $M > 0$ such that

$$\sup_{|\phi(z)|\leq r} w(z)|\phi'(z)| \leq M. \tag{8}$$

Moreover, since $\{f_m\}_m$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $m \to \infty$. Cauchy's integral formula gives that $\{f_m^{(n+1)}\}_m$ also converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $m \to \infty$. So there is $N_1 \in \mathbb{N}$ such that

$$\sup_{|\phi(z)|\leq r} |f_m^{(n+1)}(\phi(z))| \leq \frac{\varepsilon}{4M} \text{ for every } m \geq N_1. \tag{9}$$

Also, $\{f_m^{(n)}(\phi(0))\}_m$ converges to 0 as $m \to \infty$, then there exists $N_2 > 0$ such that $|f_m^{(n)}(\phi(0))| < \frac{\varepsilon}{2}$ for every $m > N_2$. Finally, together with (7) (8) and (9) we can conclude that

$$\begin{aligned}
\|D_\phi^n f_m\|_{\mathcal{B}_w} &= |f_m^{(n)}(\phi(0))| + \sup_{z\in\mathbb{D}} w(z)|\phi'(z)||f_m^{(n+1)}(\phi(z))| \\
&\leq |f_m^{(n)}(\phi(0))| + \sup_{|\phi(z)|\leq r} w(z)|\phi'(z)| \sup_{|\phi(z)|\leq r} |f_m^{(n+1)}(\phi(z))| \\
&\quad + \sup_{|\phi(z)|>r} w(z)|\phi'(z)||f_m^{(n+1)}(\phi(z))| \\
&< \varepsilon,
\end{aligned}$$

for every $m \geq N$, where $N := \max\{N_1, N_2\}$. Hence the claim follows. □

Theorem 4. *Suppose that w be a normal weight, v be a radial weight satisfying condition (L1). Then $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} = 0. \tag{10}$$

Proof. Suppose that $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$ is compact. Then $D_\phi^n : H_v^\infty \to \mathcal{B}_w$ is compact. Hence, by Theorem 3 we see that (6) holds. Then for every $\varepsilon > 0$ there exists a $r \in (0,1)$ such that

$$\frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} < \varepsilon \text{ if } r < |\phi(z)| < 1.$$

On the other hand, since $h(z) = \frac{z^{n+1}}{(n+1)!} \in H_v^\infty$, from the compactness of $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$, it follows that $\phi \in \mathcal{B}_{w,0}$. Then there exists a $\rho \in (r, 1)$ such that

$$w(z)|\phi'(z)| < \varepsilon \inf_{t\in[0,r]} v(t)(1-|t|^2)^{n+1} \ \text{ if } \ \rho < |z| < 1, \tag{11}$$

Therefore, when $\rho < |z| < 1$ and $r < |\phi(z)| < 1$, we have that

$$\frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} < \varepsilon. \tag{12}$$

If $\rho < |z| < 1$ and $|\phi(z)| \leq r$, combine with (11), we have that

$$\frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} \leq \frac{w(z)|\phi'(z)|}{\inf\limits_{t\in[0,r]} v(t)(1-|t|^2)^{n+1}} < \varepsilon. \tag{13}$$

Inequalities (12) and (13) imply (10) holds.

Conversely, assume that (10) holds. Then (3) holds, which along with (4) implies that the set $D_\phi^n(\{f \in H_v^\infty : \|f\|_v \leq 1\})$ is bounded in $\mathcal{B}_{w,0}$. By Lemma 3 we see that $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_v \leq 1} w(z)|f^{(n+1)}(\phi(z))\phi'(z)| = 0.. \tag{14}$$

Taking the supremum in (4) over the unit ball of H_v^∞, then letting $|z| \to 1$, we obtain (14), from which the compactness of $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$ follows. □

Noticing the results of Theorem 2 and Theorem 4, we conclude that the boundedness and compactness of the operator $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$ is equivalent. Similarly, for $n = 0$, we obtain necessary and sufficient conditions for the compactness of the operators $C_\phi : H_v^\infty \to \mathcal{B}_w(or\ \mathcal{B}_{w,0})$.

Corollary 3. *Suppose that w be a normal weights, v be a radial weight satisfying condition (L1). Then the following statements hold:*

(i) $C_\phi : H_v^\infty \to \mathcal{B}_w$ is compact if and only if

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)} = 0.$$

(ii) $C_\phi : H_v^\infty \to \mathcal{B}_{w,0}$ is compact if and only if

$$\lim_{|z|\to 1} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)} = 0.$$

And for $n = 1$, D_ϕ^n is the operator $C_\phi D$.

Corollary 4. *Suppose that w be a normal weights, v be a radial weight satisfying condition (L1). Then the following statements hold:*

(i) $C_\phi D : H_v^\infty \to \mathcal{B}_w$ is compact if and only if

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^2} = 0.$$

(ii) $C_\phi D : H_v^\infty \to \mathcal{B}_{w,0}$ is compact if and only if

$$\lim_{|z|\to 1} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^2} = 0.$$

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FUZZY n-JORDAN $*$-DERIVATIONS ON INDUCED FUZZY C^*-ALGEBRAS

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Abstract. Using the fixed point method, we prove the fuzzy version of the Hyers-Ulam stability of n-Jordan $*$-derivations on induced fuzzy C^*-algebras associated with the following functional equation

$$f\left(\frac{b-a}{3}\right)+f\left(\frac{a-3c}{3}\right)+f\left(\frac{3a-b+3c}{3}\right)=f(a).$$

1. Introduction and Preliminaries

The stability of functional equations originated from a question of Ulam [36] concerning the stability of group homomorphisms in 1940. More precisely, he proposed the following problem: Given a group $\mathcal{G}$, a metric group $(\mathcal{G}', d)$ and $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : \mathcal{G} \to \mathcal{G}'$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in \mathcal{G}$, then there exists a homomorphism $T : \mathcal{G} \to \mathcal{G}'$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in \mathcal{G}$? In 1941, Hyers [16] gave a partial solution of the Ulam's problem for the case of approximate additive mappings under the assumption that $\mathcal{G}$ and $\mathcal{G}'$ are Banach spaces. In 1950, Aoki [1] generalized the Hyers' theorem for approximately additive mappings. In 1978, Th. M. Rassias [33] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 9, 11, 12, 13, 14, 19, 30, 31, 34, 35]).

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1. ([4, 10]) *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \ge n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*

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(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [17] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 27, 28, 32]).

Katsaras [18] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematics have defined fuzzy normed on a vector space from various points of view [15, 21, 23, 24, 25, 29, 37]. In particular, Bag and Samanta [3] following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [20]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [2].

We use the definition of fuzzy normed spaces given in [3, 23, 24] to investigate a fuzzy version of the Hyers-Ulam stability of n-Jordan $*$-derivations in induced fuzzy C^*-algebras associated with the following functional equation

$$f\left(\frac{b-a}{3}\right) + f\left(\frac{a-3c}{3}\right) + f\left(\frac{3a-b+3c}{3}\right) = f(a).$$

Definition 1.2. ([3, 23, 24, 25]) Let $\mathcal{X}$ be a complex vector space. A function $N : \mathcal{X} \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $s, t \in \mathbb{R}$,

N_1**:** $N(x, t) = 0$ for $t \leq 0$
N_2**:** $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$
N_3**:** $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \in \mathbb{C} - \{0\}$
N_4**:** $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$
N_5**:** $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$
N_6**:** for $x \neq 0$, $N(x, .)$ is continuous on $\mathbb{R}$.

The pair $(\mathcal{X}, N)$ is called a *fuzzy normed vector space*.

Definition 1.3. ([3, 23, 24, 25]) Let $(\mathcal{X}, N)$ be a fuzzy normed vector space.

(1) A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be *convergent* if there exists an $x \in \mathcal{X}$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. in this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.

(2) A sequence $\{x_n\}$ in $\mathcal{X}$ is called *Cauchy* if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : \mathcal{X} \to \mathcal{Y}$ between fuzzy normed vector space $\mathcal{X}$, $\mathcal{Y}$ is continuous at point $x_0 \in \mathcal{X}$ if for each sequence $\{x_n\}$ converging to x_0 in $\mathcal{X}$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : \mathcal{X} \to \mathcal{Y}$ is continuous at each $x \in \mathcal{X}$, then $f : \mathcal{X} \to \mathcal{Y}$ is said to be *continuous* on $\mathcal{X}$ (see [2]).

Definition 1.4. Let $\mathcal{X}$ be a $*$-algebra and $(\mathcal{X}, N)$ a fuzzy normed space.

(1) The fuzzy normed space $(\mathcal{X}, N)$ is called a fuzzy normed $*$-algebra if

$$N(xy, st) \geq N(x, s) \cdot N(y, t) \qquad \& \qquad N(x^*, t) = N(x, t)$$

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(2) A complete fuzzy normed $*$-algebra is called a fuzzy Banach $*$-algebra.

Example 1.5. Let $(\mathcal{X}, \|\cdot\|)$ be a normed $*$-algebra. let

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|}, & t>0,\ x\in\mathcal{X} \\ 0, & t\le 0,\ x\in\mathcal{X}. \end{cases}$$

Then $N(x,t)$ is a fuzzy norm on $\mathcal{X}$ and $(\mathcal{X}, N(x,t))$ is a fuzzy normed $*$-algebra.

Definition 1.6. Let $(\mathcal{X}, \|\cdot\|)$ be a C^*-algebra and N a fuzzy norm on $\mathcal{X}$.
(1) The fuzzy normed $*$-algebra $(\mathcal{X}, N)$ is called an induced fuzzy normed $*$-algebra
(2) The fuzzy Banach $*$-algebra $(\mathcal{X}, N)$ is called an induced fuzzy C^*-algebra.

Definition 1.7. Let $(\mathcal{X}, N)$ be an induced fuzzy normed $*$-algebra. Then a $\mathbb{C}$-linear mapping $D : (\mathcal{X}, N) \to (\mathcal{X}, N)$ is called a *fuzzy n-Jordan $*$-derivation* if

$$D(a^n) = D(a)a^{n-1} + aD(a)a^{n-2} + \ldots + a^{n-2}D(a)a + a^{n-1}D(a) \quad \& \quad D(a^*) = D(a)^*$$

for all $a \in \mathcal{X}$.

Throughout this paper, assume that $(\mathcal{X}, N)$ is an induced fuzzy C^*-algebra.

2. Main results

Lemma 2.1. *Let $(\mathcal{Z}, N)$ be a fuzzy normed vector space and let $f : \mathcal{X} \to \mathcal{Z}$ be a mapping such that*

$$N\left(f\left(\frac{y-x}{3}\right) + f\left(\frac{x-3z}{3}\right) + f\left(\frac{3x-y+3z}{3}\right), t\right) \ge N\left(f(x), \frac{t}{2}\right) \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$ and all $t > 0$. Then f is additive, i.e., $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathcal{X}$.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$N(3f(0), t) = N\left(f(0), \frac{t}{3}\right) \ge N\left(f(0), \frac{t}{2}\right)$$

for all $t > 0$. By N_5 and N_6, $N(f(0), t) = 1$ for all $t > 0$. It follows from N_2 that $f(0) = 0$.

Letting $y = x = 0$ in (2.1), we get

$$N(f(0) + f(-z) + f(z), t) \ge N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from N_2 that $f(-z) + f(z) = 0$ for all $z \in \mathcal{X}$. So

$$f(-z) = -f(z)$$

for all $z \in \mathcal{X}$.

Letting $x = 0$ and replacing y, z by $3y, -z$, respectively, in (2.1), we get

$$N(f(y) + f(z) + f(-y-z), t) \ge N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from N_2 that

$$f(y) + f(z) + f(-y-z) = 0 \tag{2.2}$$

for all $y, z \in \mathcal{X}$. Thus

$$f(y+z) = f(y) + f(z)$$

for all $y, z \in \mathcal{X}$, as desired. □

Using the fixed point method, we prove the Hyers-Ulam stability of fuzzy n-Jordan $*$-derivations on induced fuzzy C^*-algebras.

Theorem 2.2. *Let $\varphi : \mathcal{X}^3 \to [0,\infty)$ be a function such that there exists an $L < \frac{3}{3^n}$ with*

$$\varphi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) \le \frac{L}{3}\varphi(x,y,z) \tag{2.3}$$

for all $x,y,z \in \mathcal{X}$. Let $f : \mathcal{X} \to \mathcal{X}$ be a mapping such that

$$N\left(\mu f\left(\frac{y-x}{3}\right) + \mu f\left(\frac{x-3z}{3}\right) + \mu f\left(\frac{3x-y+3z}{3}\right) - f(\mu x), t\right) \ge \frac{t}{t+\varphi(x,y,z)}, \tag{2.4}$$

$$N(f(w^n) - f(w)w^{n-1} - wf(w)w^{n-2} - \cdots - w^{n-2}f(w)w - w^{n-1}f(w) + f(v^*) - f(v)^*, t) \ge \frac{t}{t+\varphi(w,v,0)} \tag{2.5}$$

for all $x,y,z,w,v \in \mathcal{X}$, all $t>0$ and all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Then $D(x) = N - \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$ exists for each $x \in \mathcal{X}$ and defines a fuzzy n-Jordan $*$-derivation $D : \mathcal{X} \to \mathcal{X}$ such that

$$N(f(x) - D(x), t) \ge \frac{(1-L)t}{(1-L)t + \varphi(x,2x,0)} \tag{2.6}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. Letting $\mu = 1$, $y = 2x$ and $z = 0$ in (2.4), we get

$$N\left(3f\left(\frac{x}{3}\right) - f(x), t\right) \ge \frac{t}{t+\varphi(x,2x,0)} \tag{2.7}$$

for all $x \in \mathcal{X}$.

Consider the set

$$S := \{g : \mathcal{X} \to \mathcal{X}\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\alpha \in \mathbb{R}_+ : N(g(x) - h(x), \alpha t) \ge \frac{t}{t+\varphi(x,2x,0)}, \ \forall x \in \mathcal{X}, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see the proof of [22, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 3g\left(\frac{x}{3}\right)$$

for all $x \in \mathcal{X}$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t+\varphi(x,2x,0)}$$

for all $x \in \mathcal{X}$ and all $t > 0$. Hence

$$\begin{aligned} N\left(Jg(x) - Jh(x), L\varepsilon t\right) &= N\left(3g\left(\frac{x}{3}\right) - 3h\left(\frac{x}{3}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{3}\right) - h\left(\frac{x}{3}\right), \frac{L}{3}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{3}}{\frac{Lt}{3} + \varphi\left(\frac{x}{3}, \frac{2x}{3}, 0\right)} \geq \frac{\frac{Lt}{3}}{\frac{Lt}{3} + \frac{L}{3}\varphi\left(x, 2x, 0\right)} \\ &= \frac{t}{t + \varphi\left(x, 2x, 0\right)} \end{aligned}$$

for all $x \in \mathcal{X}$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.7) that $d(f, Jf) \leq 1$.

By Theorem 1.1, there exists a mapping $D : \mathcal{X} \to \mathcal{X}$ satisfying the following:

(1) D is a fixed point of J, i.e.,

$$D\left(\frac{x}{3}\right) = \frac{1}{3}D(x) \tag{2.8}$$

for all $x \in \mathcal{X}$. The mapping D is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that D is a unique mapping satisfying (2.8) such that there exists a $\alpha \in (0, \infty)$ satisfying

$$N(f(x) - D(x), \alpha t) \geq \frac{t}{t + \varphi\left(x, 2x, 0\right)}$$

for all $x \in \mathcal{X}$;

(2) $d(J^k f, D) \to 0$ as $k \to \infty$. This implies the equality

$$N\text{-}\lim_{k\to\infty} 3^k f\left(\frac{x}{3^k}\right) = D(x)$$

for all $x \in \mathcal{X}$;

(3) $d(f, D) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, D) \leq \frac{1}{1-L}.$$

This implies that the inequality (2.7) holds.

It follows from (2.3) that

$$\sum_{k=0}^{\infty} 3^k \varphi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right) < \infty$$

for all $x, y, z \in \mathcal{X}$.

By (2.4),

$$\begin{aligned} N\Bigg(3^k \mu f\left(\frac{y-x}{3^{k+1}}\right) + 3^k \mu f\left(\frac{x-3z}{3^{k+1}}\right) + 3^k \mu f\left(\frac{3x - y + 3z}{3^{k+1}}\right) \\ -3^k f\left(\frac{\mu x}{3^k}\right), 3^k t\Bigg) \geq \frac{t}{t + \varphi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right)} \end{aligned}$$

for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. So

$$N\left(3^k \mu f\left(\frac{y-x}{3^{k+1}}\right) + 3^k \mu f\left(\frac{x-3z}{3^{k+1}}\right) + 3^k \mu f\left(\frac{3x-y+3z}{3^{k+1}}\right) \right.$$
$$\left. -3^k f\left(\frac{\mu x}{3^k}\right), t\right) \geq \frac{\frac{t}{3^k}}{\frac{t}{3^k} + \varphi(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k})} = \frac{t}{t + 3^k \varphi(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k})}$$

for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Since $\lim_{k\to\infty} \frac{t}{t+3^k\varphi(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k})} = 1$ for all $x, y, z \in \mathcal{X}$ and all $t > 0$,

$$N\left(\mu D\left(\frac{y-x}{3}\right) + \mu D\left(\frac{x-3z}{3}\right) + \mu D\left(\frac{3x-y+3z}{3}\right) - D(\mu x), t\right) = 1$$

for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Thus

$$\mu D\left(\frac{y-x}{3}\right) + \mu D\left(\frac{x-3z}{3}\right) + \mu D\left(\frac{3x-y+3z}{3}\right) = D(\mu x) \tag{2.9}$$

for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Letting $x = y = z = 0$ in (2.9), we get $D(0) = 0$. Let $\mu = 1$ and $x = 0$ in (2.9). By the same reasoning as in the proof of Lemma 2.1, one can easily show that D is additive. Letting $y = 2x$ and $z = 0$ in (2.9), we get

$$\mu D(x) = 3\mu D\left(\frac{x}{3}\right) = D(\mu x)$$

for all $x \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1$. By [26, Theorem 2.1], the mapping $D : \mathcal{X} \to \mathcal{X}$ is $\mathbb{C}$-linear.

By (2.5) and letting $v = 0$ in (2.5), we have

$$N\left(3^{nk} f\left(\frac{w^n}{3^{nk}}\right) - 3^{nk} f\left(\frac{w}{3^k}\right) w^{n-1} - 3^{nk} w f\left(\frac{w}{3^k}\right) w^{n-2} - \cdots - 3^{nk} w^{n-2} f\left(\frac{w}{3^k}\right) w \right.$$
$$\left. -3^{nk} w^{n-1} f\left(\frac{w}{3^k}\right), 3^{nk} t\right) \geq \frac{t}{t + \varphi(\frac{w}{3^k}, 0, 0)}$$

for all $w \in \mathcal{X}$ and all $t > 0$. So

$$N\left(3^{nk} f\left(\frac{w^n}{3^{nk}}\right) - 3^{nk} f\left(\frac{w}{3^k}\right) w^{n-1} - 3^{nk} w f\left(\frac{w}{3^k}\right) w^{n-2} - \cdots - 3^{nk} w^{n-2} f\left(\frac{w}{3^k}\right) w \right.$$
$$\left. -3^{nk} w^{n-1} f\left(\frac{w}{3^k}\right), t\right) \geq \frac{\frac{t}{3^{nk}}}{\frac{t}{3^{nk}} + \varphi(\frac{w}{3^k}, 0, 0)} = \frac{t}{t + (3^{n-1}L)^k \varphi(w, 0, 0)}$$

for all $w \in \mathcal{X}$ and all $t > 0$. Since $\lim_{k\to\infty} \frac{t}{t+(3^{n-1}L)^k\varphi(w,0,0)} = 1$ for all $w \in \mathcal{X}$ and all $t > 0$,

$$N(D(w^n) - D(w)w^{n-1} - wD(w)w^{n-2} \cdots w^{n-2}D(w)w - w^{n-1}D(w), t) = 1$$

for all $w \in \mathcal{X}$ and all $t > 0$. Thus

$$D(w^n) - D(w)w^{n-1} - wD(w)w^{n-2} \cdots w^{n-2}D(w)w - w^{n-1}D(w) = 0$$

for all $w \in \mathcal{X}$.

By (2.5) and letting $w = 0$ in (2.5), we have

$$N\left(3^k f\left(\frac{v^*}{3^k}\right) - 3^k f\left(\frac{v}{3^k}\right)^*, 3^k t\right) \geq \frac{t}{t + \varphi(0, \frac{v}{3^k}, 0)}$$

for all $v \in \mathcal{X}$ and all $t > 0$. So

$$N\left(3^k f\left(\frac{v^*}{3^k}\right) - 3^k f\left(\frac{v}{3^k}\right)^*, t\right) \geq \frac{\frac{t}{3^k}}{\frac{t}{3^k} + \varphi(0, \frac{v}{3^k}, 0)} = \frac{t}{t + 3^k\varphi(0, \frac{v}{3^k}, 0)}$$

for all $v \in \mathcal{X}$ and all $t > 0$. Since $\lim_{k\to\infty} \frac{t}{t+3^k\varphi(0,\frac{v}{3^k},0)} = 1$ for all $v \in \mathcal{X}$ and all $t > 0$,

$$N(D(v^*) - D(v)^*, t) = 1$$

for all $x \in \mathcal{X}$ and all $t > 0$. Thus $D(v^*) - D(v)^* = 0$ for all $v \in \mathcal{X}$.

Therefore, the mapping $D : \mathcal{X} \to \mathcal{X}$ is a fuzzy n-Jordan $*$-derivation. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > n$. Let $\mathcal{X}$ be a normed vector space with norm $\|\cdot\|$. Let $f : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying*

$$N\left(\mu f\left(\frac{y-x}{3}\right) + \mu f\left(\frac{x-3z}{3}\right) + \mu f\left(\frac{3x-y+3z}{3}\right) - f(\mu x), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}, \tag{2.10}$$

$$N(f(w^n) - f(w)w^{n-1} - wf(w)w^{n-2} - \cdots - w^{n-2}f(w)w - w^{n-1}f(w) + f(v^*) - f(v)^*, t) \geq \frac{t}{t + \theta(\|w\|^p + \|v\|^p)} \tag{2.11}$$

for all $x, y, z, w, v \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Then $D(x) = N - \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$ exists for each $x \in \mathcal{X}$ and defines a fuzzy n-Jordan $*$-derivation $D : \mathcal{X} \to \mathcal{X}$ such that

$$N(f(x) - D(x), t) \geq \frac{(3^p - 3)t}{(3^p - 3)t + 3^p\theta\|x\|^p}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = 3^{1-p}$. □

Theorem 2.4. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \leq 3L\varphi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right)$$

for all $x, y, z \in \mathcal{X}$. Let $f : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying (2.4) and (2.5). Then $D(x) = N - \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ exists for each $x \in \mathcal{X}$ and defines a fuzzy n-Jordan $*$-derivation $D : \mathcal{X} \to \mathcal{X}$ such that

$$N(f(x) - D(x), t) \geq \frac{(1 - L)t}{(1 - L)t + L\varphi(x, 0, 0)} \tag{2.12}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{3} g(3x)$$

for all $x \in \mathcal{X}$.

It follows from (2.7) that

$$N\left(f(x)-\frac{1}{3}f(3x),\frac{1}{3}t\right)\geq\frac{t}{t+\varphi(3x,0,0)}\geq\frac{t}{t+3L\varphi(x,0,0)}$$

for all $x\in\mathcal{X}$ and all $t>0$. So $d(f,Jf)\leq L$. Hence

$$d(f,D)\leq\frac{L}{1-L},$$

which implies that the inequality (2.12) holds.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 2.5. *Let $\theta\geq 0$ and let p be a positive real number with $p<1$. Let $\mathcal{X}$ be a normed vector space with norm $\|\cdot\|$. Let $f:\mathcal{X}\to\mathcal{X}$ be a mapping satisfying* (2.10) *and* (2.11)*. Then $D(x)=N-\lim_{n\to\infty}\frac{1}{3^n}f(3^nx)$ exists for each $x\in\mathcal{X}$ and defines a fuzzy n-Jordan $*$-derivation $D:\mathcal{X}\to\mathcal{X}$ such that*

$$N(f(x)-D(x),t)\geq\frac{(3-3^p)t}{(3-3^p)t+3^p\theta\|x\|^p}$$

for all $x\in\mathcal{X}$ and all $t>0$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x,y,z)=\theta(\|x\|^p+\|y\|^p+\|z\|^p)$$

and $L=3^{p-1}$. $\square$

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Hyers-Ulam stability of a Tribonacci functional equation in 2-normed spaces

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Abstract. In this paper, we investigate the Hyers-Ulam stability of the Tribonacci functional equation

$$f(x) = f(x-1) + f(x-2) + f(x-3)$$

in 2-Banach spaces.

Keywords: Hyers-Ulam stability, 2-Banach space, Fibonacci functional equation, Tribonacci functional equation.

1. Introduction and preliminaries

The concept of 2-normed spaces was first introduced by S. Gähler [8]. Let X be a complex vector space of a dimension greater than one. Suppose that $\|\cdot,\cdot\|$ is a real valued mapping on $X \times X$ satisfying the following conditions

N1: $\|b,a\| = \|a,b\|$
N2: $\|a,b\| = \mathbf{0} \Leftrightarrow$ *a and b are linearly dependent*
N3: $\|\alpha a, b\| = |\alpha|\ \|a,b\|$
N4: $\|a+\tilde{a}, b\| \ \leq\ \|a,b\| + \|\tilde{a}, b\|$

for all $a, b \in X$ and $\alpha \in \mathbb{C}$. Then $\|\cdot,\cdot\|$ is called a 2-norm on X and the pair $(X, \|\cdot,\cdot\|)$ is called a 2-normed space. Some of the basic properties of 2-norms are that they are non-negative and $\|a, b+\alpha a\| = \|a,b\|$ for all $a, b \in X$ and $\alpha \in \mathbb{C}$. As an example of a 2-normed space, we may take an inner product space $(X, < \cdot,\cdot >)$, and define the standard 2-norm on X by

$$\|a,b\| = \begin{vmatrix} <a,a> & <a,b> \\ <b,a> & <b,b> \end{vmatrix}.$$

A sequence $\{x_n\}$ in a 2-normed space $(X; \|\cdot,\cdot\|)$ is said to converge to some $x \in X$ in the 2-norm if $\|x - x_n, u\| \to 0$ as $n \to \infty$ for all $u \in X$. A sequence $\{x_n\}$ in a 2-normed

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space $(X, \|\cdot,\cdot\|)$ is said to be Cauchy with respect to the 2-norm if

$$\lim_{n,m\to\infty} \|x_n - x_m, u\| = 0$$

for all $u \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be 2-Banach space.

Throughout this paper, we denote by T_n the nth Tribonacci number for $n \in \mathbb{N}$. In particular, we define $T_0 = 0$, $T_1 = T_2 = 1$ and $T_n = T_{n-1}+T_{n-2}+T_{n-3}$ for $n \geq 3$. Similar application of Pascal's triangle in the Fibonacci numbers can be applied to calculate the Tribonacci numbers.

											1
					1						1
				1		1					2
			1		3		1				4
		1		5		5		1			7
	1		7		13	†	7		1		13
1		9		25		25		9		1	⋮

(a) Numbers in the n^{th} row are the sum of three neighbours: $25 = 13 + 5 + 7$.

(b) Sums of shallow diagonals giving Tribonacci numbers: 4= 1+3.

Let X be 2-Banach space. A function $f : R \to X$ is called a Tribonacci function if it satisfies

$$f(x) = f(x-1) + f(x-2) + f(x-3). \tag{1.1}$$

The stability of functional equations originated from a question of Ulam [15] in 1940. In the next year, Hyers [9] proved the problem for the Cauchy functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14]).

Recently, Bidkham and et al. [7] investigated the solution and the Hyers-Ulam stability of (1.1) in normed spaces.

In this paper, we establish the Hyers-Ulam stability of (1.1) in 2-normed spaces.

We denote the roots of the equation $x^3 - x^2 - x - 1 = 0$ By α, β and γ. β and γ are complex, $|\beta| = |\gamma|$ and α is greater than one. We have

$$\alpha + \beta + \gamma = 1\ ,\ \alpha\beta + \alpha\gamma + \beta\gamma = -1,\ \alpha\beta\gamma = 1. \tag{1.2}$$

2. **Main result**

As we shall see in the following theorem, the general solution of the Tribonacci functional equation is strongly related to the Tribonacci numbers T_n.

Theorem 2.1. ([7]) *Let X be a real vector space. A function $f : R \to X$ is a Tribonacci function if and only if there exists a function $g : [-2, 2] \to X$ such that*

$$f(x) = T_{[x]+2}\, g(x - [x]) + T'_{[x]}\, g(x - [x] - 1) + T_{[x]+1}\, g(x - [x] - 2),$$

where $T'_{[x]} = T_{[x]+3} - T_{[x]+2}$ for all $x \in \mathbb{R}$.

In the following theorem, we prove the Hyers-Ulam stability of the Tribonacci functional equation (1.1) in 2-Banach spaces. We try to prove this theorem under condition

$$\|f(x)-[f(x-1)+f(x-2)+f(x-3)],z\|\ \leq\epsilon$$

for all $x\in\mathbb{R}$ and $z\in X$, but this condition is very heavy and often inaccessible. In the following we offer a condition to obtain a best result.

Theorem 2.2. Let $(X,\|\cdot,\cdot\|)$ be a real 2-Banach space. If a function $f:\mathbb{R}\to X$ satisfies the inequality

$$\|f(x),f(x-1)+f(x-2)+f(x-3)\|\ \leq\epsilon$$

for all $x\in\mathbb{R}$ and some $\epsilon>0$, then there exists a Tribonacci function $G:\mathbb{R}\to X$ such that

$$\|f(x),G(x)\|\leq\frac{1}{|\alpha^2(\beta-\gamma)+\beta^2(\gamma-\alpha)+\gamma^2(\alpha-\beta)|}[\frac{2(1+|\beta|)+|\beta|^2}{1-|\beta|^2}]\epsilon$$

for all $x\in\mathbb{R}$.

Proof. By (1.2), it follows from (1.1) that

$$\|f(x),(\alpha+\beta+\gamma)f(x-1)-(\alpha\beta+\alpha\gamma+\beta\gamma)f(x-2)+\alpha\beta\gamma\ f(x-3)\|\ \leq\epsilon$$

for all $x\in\mathbb{R}$. If we replace x by $x-r$ and $x+r$ in the last inequality, then we have

$$\begin{aligned}&\|f(x-r),\alpha[f(x-r-1)-\gamma f(x-r-2)]\\&+\beta\ [f(x-r-1)-(\alpha+\gamma)f(x-r-2)+\alpha\gamma f(x-r-3)]+\gamma f(x-r-1)\|\leq\epsilon,\\&\|f(x-r),\alpha[f(x-r-1)-\beta f(x-r-2)]\\&+\gamma\ [f(x-r-1)-(\alpha+\beta)f(x-r-2)+\alpha\beta f(x-r-3)]+\beta f(x-r-1)\|\leq\epsilon,\\&\|f(x+r),\alpha[f(x+r-1)-\gamma f(x+r-2)]\\&+\beta\ [f(x+r-1)-(\alpha+\gamma)f(x+r-2)+\alpha\gamma f(x+r-3)]+\gamma f(x+r-1)\|\leq\epsilon\end{aligned}$$

for all $x\in\mathbb{R}$ and all $r\in\mathbb{Z}$. Hence we have

$$\begin{aligned}&\|f(x-r),\beta^r\alpha[f(x-r-1)-\gamma f(x-r-2)]+\beta^{r+1}\ [f(x-r-1)\\&\quad-(\alpha+\gamma)f(x-r-2)+\alpha\gamma f(x-r-3)]+\beta^r\gamma f(x-r-1)\|\ \leq|\beta^r|\epsilon,\end{aligned}\tag{2.1}$$

$$\begin{aligned}&\|f(x-r),\gamma^r\beta[f(x-r-1)-\alpha f(x-r-2)]+\gamma^{r+1}\ [f(x-r-1)\\&\quad-(\alpha+\beta)f(x-r-2)+\alpha\beta f(x-r-3)]+\gamma^r\alpha f(x-r-1)\|\ \leq|\gamma^r|\epsilon,\end{aligned}\tag{2.2}$$

$$\begin{aligned}&\|f(x+r),\alpha^{-r}\beta[f(x+r-1)-\gamma f(x-r-2)]+\alpha^{-r+1}\ [f(x+r-1)\\&\quad-(\beta+\gamma)f(x+r-2)+\beta\gamma f(x+r-3)]\alpha^{-r}\beta f(x-r-1)\|\ \leq|\alpha^{-r}|\epsilon\end{aligned}\tag{2.3}$$

for all $x\in\mathbb{R}$ and all $r\in\mathbb{Z}$. Then we have

$$\begin{aligned}&\|f(x),\alpha[f(x-1)-\gamma f(x-2)]+\gamma f(x-1)+\beta^n[f(x-n)-(\alpha+\gamma)f(x-n-1)+\alpha\gamma f(x-n-2)]\|\\&\leq\sum_{r=0}^{n=1}\|f(x-r),\beta^r\alpha[f(x-r-1)-\gamma f(x-r-2)]+\beta^{r+1}[f(x-r-1)-(\alpha+\gamma)f(x-r-2)\\&\qquad+\alpha\gamma f(x-r-3)]+\beta^r\gamma f(x-r-1)\|\ \leq\sum_{r=0}^{n-1}|\beta^r|\epsilon,\end{aligned}\tag{2.4}$$

$$\|f(x),\beta[f(x-1)-\alpha f(x-2)]+\alpha f(x-1)+\gamma^n[f(x-n)-(\alpha+\beta)f(x-n-1)+\alpha\beta f(x-n-2)]\|$$
$$\leq \sum_{r=0}^{n=1}\|f(x-r),\gamma^r\beta[f(x-r-1)-\alpha f(x-r-2)]+\gamma^{r+1}[f(x-r-1)-(\alpha+\beta)f(x-r-2)$$
$$+\alpha\beta f(x-r-3)]+\gamma^r\alpha f(x-r-1)\| \leq \sum_{r=0}^{n-1}|\gamma^r|\epsilon, \tag{2.5}$$
$$\|f(x),\gamma[f(x-1)-\beta f(x-2)]+\gamma f(x-1)+\alpha^{-n}[f(x-n)-(\beta+\gamma)f(x-n-1)+\beta\gamma f(x-n-2)]\|$$
$$\leq \sum_{r=0}^{n=1}\|f(x+r),\alpha^{-r}\beta[f(x+r-1)-\gamma f(x-r-2)]+\alpha^{-r+1}\,[f(x+r-1)-(\beta+\gamma)f(x+r-2)$$
$$+\beta\gamma f(x+r-3)]+\alpha^{-r}\beta f(x-r-1)\| \leq \sum_{r=0}^{n-1}|\alpha^{-r}|\epsilon \tag{2.6}$$
for all $x\in\mathbb{R}$ and all $r\in\mathbb{Z}$.

By (2.1), (2.2) and (2.3), we obtain that
$$\{\beta^n[f(x-r-1)-(\alpha+\gamma)f(x-r-2)+\alpha\gamma f(x-r-3)]\},$$
$$\{\gamma^n[f(x-r-1)-(\alpha+\beta)f(x-r-2)+\alpha\beta f(x-r-3)]\},$$
$$\{\alpha^{-n}[f(x+r-1)-(\beta+\gamma)f(x+r-2)+\beta\gamma f(x+r-3)]\}$$
are Cauchy sequences for any fixed $x\in\mathbb{R}$. Hence we can define the functions $G_1:\mathbb{R}\to X$, $G_2:\mathbb{R}\to X$ and $G_3:\mathbb{R}\to X$ by
$$G_1=\lim_{n\to\infty}\beta^n[f(x-r-1)-(\alpha+\gamma)f(x-r-2)+\alpha\gamma f(x-r-3)],$$
$$G_2=\lim_{n\to\infty}\gamma^n[f(x-r-1)-(\alpha+\beta)f(x-r-2)+\alpha\beta f(x-r-3)],$$
$$G_3=\lim_{n\to\infty}\alpha^{-n}\,[f(x+r-1)-(\beta+\gamma)f(x+r-2)+\beta\gamma f(x+r-3)]$$
for all $x\in\mathbb{R}$ and all $r\in\mathbb{Z}$. Using the above definition of G_1, G_2 and G_3, we show that there are Tribonacci functions

$$G_1(x-1)+G_1(x-2)+G_1(x-3)$$
$$=\beta^{-1}\lim_{n\to\infty}\beta^{n+1}[f(x-(n+1))-(\alpha+\gamma)f(x-(n+1)-1)+\alpha\gamma f(x-(n+1)-2)]$$
$$+\beta^{-2}\lim_{n\to\infty}\beta^{n+2}[f(x-(n+2))-(\alpha+\gamma)f(x-(n+2)-1)+\alpha\gamma f(x-(n+2)-2)]$$
$$+\beta^{-3}\lim_{n\to\infty}\beta^{n+3}[f(x-(n+3))-(\alpha+\gamma)f(x-(n+3)-1)+\alpha\gamma f(x-(n+3)-2)]$$
$$=\beta^{-1}G_1(x)+\beta^{-2}G_1(x)+\beta^{-3}G_1(x)=G_1(x),$$
$$G_2(x-1)+G_2(x-2)+G_2(x-3)$$
$$=\gamma^{-1}\lim_{n\to\infty}\gamma^{n+1}[f(x-(n+1))-(\alpha+\beta)f(x-(n+1)-1)+\alpha\beta f(x-(n+1)-2)]$$

$$+\gamma^{-2}\lim_{n\to\infty}\gamma^{n+2}[f(x-(n+2))-(\alpha+\beta)f(x-(n+2)-1)+\alpha\beta f(x-(n+2)-2)]$$

$$+\gamma^{-3}\lim_{n\to\infty}\gamma^{n+3}[f(x-(n+3))-(\alpha+\beta)f(x-(n+3)-1)+\alpha\beta f(x-(n+3)-2)]$$

$$=\gamma^{-1}G_2(x)+\gamma^{-2}G_2(x)+\gamma^{-3}G_2(x)=G_2(x),$$

$$G_3(x-1)+G_3(x-2)+G_3(x-3)$$

$$=\alpha^{-1}\lim_{n\to\infty}\alpha^{-n+1}\ [f(x-(n+1))-(\beta+\gamma)f(x-(n+1)-1)+\beta\gamma f(x-(n+1)-2)]$$

$$+\alpha^{-2}\lim_{n\to\infty}\alpha^{-n+2}\ [f(x-(n+2))-(\beta+\gamma)f(x-(n+2)-1)+\beta\gamma f(x-(n+2)-2)]$$

$$+\alpha^{-3}\lim_{n\to\infty}\alpha^{-n+3}\ [f(x-(n+3))-(\beta+\gamma)f(x-(n+3)-1)+\beta\gamma f(x-(n+3)-2)]$$

$$=\alpha^{-1}G_3(x)+\alpha^{-2}G_3(x)+\alpha^{-3}G_3(x)=G_3(x)$$

for all $x\in\mathbb{R}$. It follows from (2.4), (2.5) and (2.6) that

$$\|f(x),(\alpha+\gamma)f(x-1)-\alpha\gamma f(x-2)+G_2(x)\|\ \leq\ \frac{1}{1-|\beta|}\epsilon, \tag{2.7}$$

$$\|f(x),(\alpha+\beta)f(x-1)-\alpha\beta f(x-2)+G_2(x)\|\ \leq\ \frac{1}{1-|\gamma|}\epsilon=\frac{1}{1-|\beta|}\epsilon, \tag{2.8}$$

$$\|f(x),(\beta+\gamma)f(x-1)-\beta\gamma f(x-2)+G_3(x)\|\ \leq\ \frac{|\alpha^{-1}|}{1-|\alpha^{-1}|}\epsilon=\frac{|\beta^2|}{1-|\beta^2|}\epsilon \tag{2.9}$$

for all $x\in\mathbb{R}$. Now, put $\Delta=\alpha^2(\beta-\gamma)+\beta^2(\gamma-\alpha)+\gamma^2(\alpha-\beta)$, and define

$$G(x):=\frac{\beta^2(\gamma-\alpha)}{\Delta}G_1(x)+\frac{\gamma^2(\alpha-\beta)}{\Delta}G_2(x)+\frac{\alpha^2(\beta-\gamma)}{\Delta}G_3(x)$$

for all $x\in\mathbb{R}$. By (2.7), (2.8) and (2.9), we have

$$\begin{aligned}
&\|f(x),G(x)\|\\
&=\|f(x),\frac{\beta^2(\gamma-\alpha)}{\Delta}G_1(x)+\frac{\gamma^2(\alpha-\beta)}{\Delta}G_2(x)+\frac{\alpha^2(\beta-\gamma)}{\Delta}G_3(x)\|\\
&\leq\frac{1}{|\Delta|}[\|f(x),\beta^2(\gamma^2-\alpha^2)f(x-1)-\beta^2(\gamma-\alpha)\alpha\gamma f(x-2)+\beta^2(\gamma-\alpha)G_1\|\\
&\quad+\|f(x),\gamma^2(\alpha^2-\beta^2)f(x-1)-\gamma^2(\alpha-\beta)\alpha\beta f(x-2)+\gamma^2(\alpha-\beta)G_2\|\\
&\quad+\|f(x),\alpha^2(\beta^2-\gamma^2)f(x-1)-\alpha^2(\beta-\gamma)\beta\gamma f(x-2)+\alpha^2(\beta-\gamma)G_3\|]\\
&\leq\frac{1}{|\Delta|}[\frac{2}{1-|\beta|}+\frac{|\beta|^2}{1-|\beta|^2}]\epsilon\\
&\leq\frac{1}{|\Delta|}[\frac{2(1+|\beta|)+|\beta|^2}{1-|\beta|^2}]\epsilon
\end{aligned}$$

for all $x\in\mathbb{R}$.

On the other hand, it is easy to show that G is a Tribonacci function and this completes the proof. □

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An identity of the twisted q-Euler polynomials with weak weight α associated with the p-adic q-integrals on $\mathbb{Z}_p$

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Abstract : In [7], we studied the twisted q-Euler numbers and polynomials with weak weight α. By using these numbers and polynomials, we investigate the alternating sums of powers of consecutive integers. By applying the symmetry of the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give recurrence identities the twisted q-Euler polynomials with weak weight α.

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1. Introduction

The Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics(see [1-12]). Throughout this paper, we always make use of the following notations: $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \text{ (cf. [1-6])} .$$

Hence, $\lim_{q \to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic q-integral was defined by Kim as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{[2]_q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \text{ see [1-5]} . \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we easily see that

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0). \tag{1.2}$$

Let $T_p = \cup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\zeta | \zeta^{p^N} = 1\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$.

In [7], we defined the twisted q-Euler numbers and polynomials with weak weight α and investigate their properties. For $\alpha \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, and $\zeta \in T_p$, the twisted q-Euler polynomials $\widetilde{E}^{(\alpha)}_{n,q,\zeta}(x)$ with weak weight α are defined by

$$\widetilde{F}^{(\alpha)}_{q,\zeta}(x,t) = \sum_{n=0}^{\infty} \widetilde{E}^{(\alpha)}_{n,q,\zeta}(x) \frac{t^n}{n!} = \frac{[2]_{q^\alpha}}{\zeta q^\alpha e^t + 1} e^{xt}. \tag{1.3}$$

The twisted q-Euler numbers $\widetilde{E}^{(\alpha)}_{n,q,\zeta}$ with weak weight α are defined by the generating function:

$$\widetilde{F}^{(\alpha)}_{q,\zeta}(t) = \sum_{n=0}^{\infty} \widetilde{E}^{(\alpha)}_{n,q,\zeta} \frac{t^n}{n!} = \frac{[2]_{q^\alpha}}{\zeta q^\alpha e^t + 1}. \tag{1.4}$$

The following elementary properties of the q-Euler numbers $\widetilde{E}^{(\alpha)}_{n,q,\zeta}$ and polynomials $\widetilde{E}^{(\alpha)}_{n,q,\zeta}(x)$ with weak weight α are readily derived form (1.1), (1.2), (1.3) and (1.4) (see, for details, [7]). We, therefore, choose to omit details involved.

Theorem 1(Witt formula). For $\alpha \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, and $\zeta \in T_p$, we have

$$\widetilde{E}^{(\alpha)}_{n,q,\zeta} = \int_{\mathbb{Z}_p} \zeta^x x^n d\mu_{-q^\alpha}(x), \quad \widetilde{E}^{(\alpha)}_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \zeta^y (x+y)^n d\mu_{-q^\alpha}(y).$$

Theorem 2. For any positive integer n, we have

$$\widetilde{E}^{(\alpha)}_{n,q,\zeta}(x) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{E}^{(\alpha)}_{k,q,\zeta} x^{n-k}.$$

In this paper, by using the symmetry of p-adic q-integral on $\mathbb{Z}_p$, we obtain the recurrence identities the twisted q-Euler polynomials with weak weight α.

2. The alternating sums of powers of consecutive q-integers

Let q be a complex number with $|q| < 1$ and ζ be the p^N-th root of unity. By using (1.3), we give the alternating sums of powers of consecutive q-integers as follows:

$$\sum_{n=0}^{\infty} \widetilde{E}^{(\alpha)}_{n,q,\zeta} \frac{t^n}{n!} = \frac{[2]_{q^\alpha}}{\zeta q^\alpha e^t + 1} = [2]_{q^\alpha} \sum_{n=0}^{\infty} (-1)^n \zeta^n q^{\alpha n} e^{nt}.$$

From the above, we obtain

$$-\sum_{n=0}^{\infty} (-1)^n \zeta^n q^{\alpha n} e^{(n+k)t} + \sum_{n=0}^{\infty} (-1)^{n-k} \zeta^{n-k} q^{\alpha(n-k)} e^{nt} = \sum_{n=0}^{k-1} (-1)^{n-k} \zeta^{n-k} q^{\alpha(n-k)} e^{nt}.$$

Thus, we have

$$\begin{aligned} &- [2]_{q^\alpha} \sum_{n=0}^{\infty} (-1)^n \zeta^n q^{\alpha n} e^{(n+k)t} + [2]_{q^\alpha} (-1)^{-k} \zeta^{-k} q^{-\alpha k} \sum_{n=0}^{\infty} (-1)^n \zeta^n q^{\alpha n} e^{nt} \\ &= [2]_{q^\alpha} (-1)^{-k} \zeta^{-k} q^{-\alpha k} \sum_{n=0}^{k-1} (-1)^n \zeta^n q^{\alpha n} e^{nt}. \end{aligned} \tag{2.1}$$

By using (1.3)and (1.4), and (2.1), we obtain

$$-\sum_{j=0}^{\infty} \widetilde{E}^{(\alpha)}_{j,q,\zeta}(k) \frac{t^j}{j!} + (-1)^{-k} \zeta^{-k} q^{-\alpha k} \sum_{j=0}^{\infty} \widetilde{E}^{(\alpha)}_{j,q,\zeta} \frac{t^j}{j!} = [2]_{q^\alpha} \sum_{j=0}^{\infty} \left((-1)^{-k} \zeta^{-k} q^{-\alpha k} \sum_{n=0}^{k-1} (-1)^n \zeta^n q^{\alpha n} n^j \right) \frac{t^j}{j!}.$$

By comparing coefficients of $\frac{t^j}{j!}$ in the above equation, we obtain

$$\sum_{n=0}^{k-1} (-1)^n \zeta^n q^{\alpha n} n^j = \frac{(-1)^{k+1} \zeta^k q^{\alpha k} \widetilde{E}^{(\alpha)}_{j,q,\zeta}(k) + \widetilde{E}^{(\alpha)}_{j,q,\zeta}}{[2]_{q^\alpha}}.$$

By using the above equation we arrive at the following theorem:

Theorem 3. Let k be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$. Then we obtain

$$\widetilde{T}^{(\alpha)}_{j,q,\zeta}(k-1) = \sum_{n=0}^{k-1}(-1)^n \zeta^n q^{\alpha n} n^j = \frac{(-1)^{k+1}\zeta^k q^{\alpha k}\widetilde{E}^{(\alpha)}_{j,q,\zeta}(k) + \widetilde{E}^{(\alpha)}_{j,q,\zeta}}{[2]_{q^\alpha}}.$$

Remark 4. For $\zeta = 1$, we have

$$\lim_{q\to 1}\widetilde{T}^{(\alpha)}_{j,q,\zeta}(k-1) = \sum_{n=0}^{k-1}(-1)^n n^j = \frac{(-1)^{k+1}E_j(k) + E_j}{2},$$

where $E_j(x)$ and E_j denote the Euler polynomials and Euler numbers, respectively.

Next, we assume that $q \in \mathbb{C}_p$ and $\zeta \in T_p$. We obtain recurrence identities the q-Euler polynomials and the q-analogue of alternating sums of powers of consecutive integers. By using (1.1), we have

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1}(-1)^{n-1-l} q^l g(l),$$

where $g_n(x) = g(x+n)$. If n is odd from the above, we obtain

$$q^n I_{-q}(g_n) + I_{-q}(g) = [2]_q \sum_{l=0}^{n-1}(-1)^{n-1-l} q^l g(l) \text{ (cf. [1-5]).} \tag{2.2}$$

It will be more convenient to write (2.2) as the equivalent integral form

$$q^{\alpha n}\int_{\mathbb{Z}_p} g(x+n)d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-q^\alpha}(x) = [2]_{q^\alpha}\sum_{k=0}^{n-1}(-1)^k q^{\alpha k} g(k). \tag{2.3}$$

Substituting $g(x) = \zeta^x e^{xt}$ into the above, we obtain

$$\zeta^n q^{\alpha n}\int_{\mathbb{Z}_p} \zeta^x e^{(x+n)t}d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p} \zeta^x e^{xt}d\mu_{-q^\alpha}(x) = [2]_{q^\alpha}\sum_{j=0}^{n-1}(-1)^j \zeta^j q^{\alpha j} e^{jt}. \tag{2.4}$$

After some elementary calculations, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \zeta^x e^{xt}d\mu_{-q^\alpha}(x) = \frac{[2]_{q^\alpha}}{\zeta q^\alpha e^t + 1}, \\
&\int_{\mathbb{Z}_p} \zeta^x e^{(x+n)t}d\mu_{-q^\alpha}(x) = e^{nt}\frac{[2]_{q^\alpha}}{\zeta q^\alpha e^t + 1}.
\end{aligned} \tag{2.5}$$

By using (2.4) and (2.5), we have

$$\zeta^n q^{\alpha n}\int_{\mathbb{Z}_p} \zeta^x e^{(x+n)t}d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p} \zeta^x e^{xt}d\mu_{-q^\alpha}(x) = \frac{[2]_{q^\alpha}(1+\zeta^n q^{\alpha n} e^{nt})}{\zeta q^\alpha e^t + 1}.$$

From the above, we get

$$\frac{[2]_{q^\alpha}(1+\zeta^n q^{\alpha n} e^{nt})}{\zeta q^\alpha e^t + 1} = \frac{[2]_{q^\alpha}\int_{\mathbb{Z}_p}\zeta^x e^{xt}d\mu_{-q^\alpha}(x)}{\int_{\mathbb{Z}_p}\zeta^{nx} q^{\alpha(n-1)x} e^{ntx}d\mu_{-q^\alpha}(x)}. \tag{2.6}$$

By substituting Taylor series of e^{xt} into (2.4), we obtain

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left(\zeta^n q^{\alpha n}\int_{\mathbb{Z}_p}\zeta^x (x+n)^m d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p}\zeta^x x^m d\mu_{-q^\alpha}(x)\right)\frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty}\left([2]_{q^\alpha}\sum_{j=0}^{n-1}(-1)^j\zeta^j q^{\alpha j} j^m\right)\frac{t^m}{m!}.
\end{aligned}$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation, we obtain

$$\zeta^n q^{\alpha n}\sum_{k=0}^{m}\binom{m}{k}n^{m-k}\int_{\mathbb{Z}_p}\zeta^x x^k d\mu_{-q^\alpha}(x)+\int_{\mathbb{Z}_p}\zeta^x x^m d\mu_{-q^\alpha}(x)=[2]_{q^\alpha}\sum_{j=0}^{n-1}(-1)^j\zeta^j q^{\alpha j}j^m.$$

By using Theorem 3, we have

$$\zeta^n q^{\alpha n}\sum_{k=0}^{m}\binom{m}{k}n^{m-k}\int_{\mathbb{Z}_p}\zeta^x x^k d\mu_{-q^\alpha}(x)+\int_{\mathbb{Z}_p}\zeta^x x^m d\mu_{-q^\alpha}(x)=[2]_{q^\alpha}\widetilde{T}^{(\alpha)}_{m,q,\zeta}(n-1). \tag{2.7}$$

By using (2.6) and (2.7), we arrive at the following theorem:

Theorem 5. Let n be odd positive integer. Then we have

$$\frac{\int_{\mathbb{Z}_p}\zeta^x e^{xt}d\mu_{-q^\alpha}(x)}{\int_{\mathbb{Z}_p}\zeta^{nx}q^{\alpha(n-1)x}e^{ntx}d\mu_{-q^\alpha}(x)}=\sum_{m=0}^{\infty}\left(\widetilde{T}^{(\alpha)}_{m,q,\zeta}(n-1)\right)\frac{t^m}{m!}.$$

Let w_1 and w_2 be odd positive integers. By (2.5), Theorem 5, and after some elementary calculations, we obtain the following theorem.

Theorem 6. Let w_1 and w_2 be odd positive integers. Then we have

$$\frac{\int_{\mathbb{Z}_p}\zeta^{w_2x}e^{w_2xt}d\mu_{-q^{w_2\alpha}}(x)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{\alpha(w_1w_2-1)x}e^{w_1w_2tx}d\mu_{-q^\alpha}(x)}=\frac{[2]_{q^{w_2\alpha}}}{[2]_{q^\alpha}}\sum_{m=0}^{\infty}\left(\widetilde{T}^{(\alpha)}_{m,q^{w_2},\zeta^{w_2}}(w-1)w_2^m\right)\frac{t^m}{m!}. \tag{2.8}$$

By (1.1), we obtain

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\zeta^{w_1x_1+w_2x_2}e^{(w_1x_1+w_2x_2+w_1w_2x)t}d\mu_{-q^{w_1\alpha}}(x_1)d\mu_{-q^{w_2\alpha}}(x_2)}{\int_{\mathbb{Z}_p}\zeta^{w_2x_2}q^{\alpha(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q^\alpha}(x)}\\&=\frac{e^{w_1w_2xt}\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{w_1x_1t}d\mu_{-q^{w_1\alpha}}(x_1)\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{w_2x_2t}d\mu_{-q^{w_2\alpha}}(x_2)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{\alpha(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q^\alpha}(x)}.\end{aligned} \tag{2.9}$$

By using (2.8) and (2.9), after elementary calculations, we obtain

$$\begin{aligned}a&=\left(\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{(w_1x_1+w_1w_2x)t}d\mu_{-q^{w_1\alpha}}(x_1)\right)\left(\frac{\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{x_2w_2t}d\mu_{-q^{w_2\alpha}}(x_2)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{\alpha(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q^\alpha}(x)}\right)\\&=\left(\sum_{m=0}^{\infty}\widetilde{E}^{(\alpha)}_{m,q^{w_1},\zeta^{w_1}}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(\frac{[2]_{q^{w_2\alpha}}}{[2]_{q^\alpha}}\sum_{m=0}^{\infty}\widetilde{T}^{(\alpha)}_{m,q^{w_2},\zeta^{w_2}}(w_1-1)w_2^m\frac{t^m}{m!}\right).\end{aligned} \tag{2.10}$$

By using Cauchy product in the above, we have

$$a=\sum_{m=0}^{\infty}\left(\frac{[2]_{q^{w_2\alpha}}}{[2]_{q^\alpha}}\sum_{j=0}^{m}\binom{m}{j}\widetilde{E}^{(\alpha)}_{j,q^{w_1},\zeta^{w_1}}(w_2x)w_1^j\widetilde{T}^{(\alpha)}_{m-j,q^{w_2},\zeta^{w_2}}(w_1-1)w_2^{m-j}\right)\frac{t^m}{m!}. \tag{2.11}$$

By using the symmetry in (2.10), we obtain

$$\begin{aligned}a&=\left(\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{(w_2x_2+w_1w_2x)t}d\mu_{-q^{w_2\alpha}}(x_2)\right)\left(\frac{\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{x_1w_1t}d\mu_{-q^{w_1\alpha}}(x_1)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{\alpha(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q^\alpha}(x)}\right)\\&=\left(\sum_{m=0}^{\infty}\widetilde{E}^{(\alpha)}_{m,q^{w_2},\zeta^{w_2}}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(\frac{[2]_{q^{w_1\alpha}}}{[2]_{q^\alpha}}\sum_{m=0}^{\infty}\widetilde{T}^{(\alpha)}_{m,q^{w_1},\zeta^{w_1}}(w_2-1)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we obtain

$$a=\sum_{m=0}^{\infty}\left(\frac{[2]_{q^{w_1\alpha}}}{[2]_{q^{\alpha}}}\sum_{j=0}^{m}\binom{m}{j}\widetilde{E}^{(\alpha)}_{j,q^{w_2},\zeta^{w_2}}(w_1x)w_2^j\widetilde{T}^{(\alpha)}_{m-j,q^{w_1},\zeta^{w_1}}(w_2-1)w_1^{m-j}\right)\frac{t^m}{m!}. \tag{2.12}$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (2.11) and (2.12), we arrive at the following theorem.

Theorem 7. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&[2]_{q^{w_2\alpha}}\sum_{j=0}^{m}\binom{m}{j}\widetilde{E}^{(\alpha)}_{j,q^{w_1},\zeta^{w_1}}(w_2x)w_1^j\widetilde{T}^{(\alpha)}_{m-j,q^{w_2},\zeta^{w_2}}(w_1-1)w_2^{m-j}\\&=[2]_{q^{w_1\alpha}}\sum_{j=0}^{m}\binom{m}{j}\widetilde{E}^{(\alpha)}_{j,q^{w_2},\zeta^{w_2}}(w_1x)w_2^j\widetilde{T}^{(\alpha)}_{m-j,q^{w_1},\zeta^{w_1}}(w_2-1)w_1^{m-j},\end{aligned}$$

where $\widetilde{E}^{(\alpha)}_{k,q,\zeta}(x)$ and $\widetilde{T}^{(\alpha)}_{m,q,\zeta}(k)$ denote the twisted q-Euler polynomials with weak weight α and the q-analogue of alternating sums of powers of consecutive integers, respectively.

By using Theorem 2, we have the following corollary:

Corollary 8. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&[2]_{q^{w_1\alpha}}\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^jx^{j-k}\widetilde{E}^{(\alpha)}_{k,q,\zeta^{w_2}}\widetilde{T}^{(\alpha)}_{m-j,q^{w_1},\zeta^{w_1}}(w_2-1)\\&=[2]_{q^{w_2\alpha}}\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^jw_2^{m-k}x^{j-k}\widetilde{E}^{(\alpha)}_{k,q,\zeta^{w_1}}\widetilde{T}^{(\alpha)}_{m-j,q^{w_2},\zeta^{w_2}}(w_1-1).\end{aligned}$$

By using (2.9), we have

$$\begin{aligned}a&=\left(e^{w_1w_2xt}\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{x_1w_1t}d\mu_{-q^{w_1\alpha}}(x_1)\right)\left(\frac{\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{x_2w_2t}d\mu_{-q^{w_2\alpha}}(x_2)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{\alpha(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q^{\alpha}}(x)}\right)\\&=\frac{[2]_{q^{w_2\alpha}}}{[2]_{q^{\alpha}}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}q^{w_2\alpha j}\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{\left(x_1+w_2x+j\frac{w_2}{w_1}\right)(w_1t)}d\mu_{-q^{w_1\alpha}}(x_1)\\&=\sum_{n=0}^{\infty}\left(\frac{[2]_{q^{w_2\alpha}}}{[2]_{q^{\alpha}}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}q^{w_2\alpha j}\widetilde{E}^{(\alpha)}_{n,q^{w_1},\zeta^{w_1}}\left(w_2x+j\frac{w_2}{w_1}\right)w_1^n\right)\frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

By using the symmetry property in (2.13), we also have

$$\begin{aligned}a&=\left(e^{w_1w_2xt}\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{x_2w_2t}d\mu_{-q^{w_2\alpha}}(x_2)\right)\left(\frac{\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{x_1w_1t}d\mu_{-q^{w_1\alpha}}(x_1)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{\alpha(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q^{\alpha}}(x)}\right)\\&=\frac{[2]_{q^{w_1\alpha}}}{[2]_{q^{\alpha}}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}q^{w_1\alpha j}\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{\left(x_2+w_1x+j\frac{w_1}{w_2}\right)(w_2t)}d\mu_{-q^{w_2\alpha}}(x_2)\\&=\sum_{n=0}^{\infty}\left(\frac{[2]_{q^{w_1\alpha}}}{[2]_{q^{\alpha}}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}q^{w_1\alpha j}\widetilde{E}^{(\alpha)}_{n,q^{w_2},\zeta^{w_2}}\left(w_1x+j\frac{w_1}{w_2}\right)w_2^n\right)\frac{t^n}{n!}.\end{aligned} \tag{2.14}$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (2.13) and (2.14), we have the following theorem.

Theorem 9. Let w_1 and w_2 be odd positive integers. Then we have

$$\begin{aligned}&[2]_{q^{w_2\alpha}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2 j}q^{w_2\alpha j}\widetilde{E}^{(\alpha)}_{n,q^{w_1},\zeta^{w_1}}\left(w_2x+j\frac{w_2}{w_1}\right)w_1^n\\ =&[2]_{q^{w_1\alpha}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1 j}q^{w_1\alpha j}\widetilde{E}^{(\alpha)}_{n,q^{w_2},\zeta^{w_2}}\left(w_1x+j\frac{w_1}{w_2}\right)w_2^n.\end{aligned}\tag{2.15}$$

Remark 10. Let w_1 and w_2 be odd positive integers. If $q\to 1$ and $\zeta=1$, we have

$$\sum_{j=0}^{w_1-1}(-1)^jE_n\left(w_2x+j\frac{w_2}{w_1}\right)w_1^n=\sum_{j=0}^{w_2-1}(-1)^jE_n\left(w_1x+j\frac{w_1}{w_2}\right)w_2^n.$$

Substituting $w_1=1$ into (2.15), we arrive at the following corollary.

Corollary 11. Let w_2 be odd positive integer. Then we obtain

$$\widetilde{E}^{(\alpha)}_{n,q,\zeta}(x)=\frac{[2]_{q^\alpha}}{[2]_{q^{w_2\alpha}}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^jq^{\alpha j}\widetilde{E}^{(\alpha)}_{n,q^{w_2},\zeta^{w_2}}\left(\frac{x+j}{w_2}\right)w_2^n.$$

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Two-Level Hierarchical Basis Preconditioner for Elliptic Equations with Jump Coefficients*

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Abstract. This paper provides a proof of robustness of the two-level hierarchical basis preconditioner for the linear finite element approximation of second order elliptic problems with strongly discontinuous coefficients. As a result, we prove that the convergence rate of the conjugate gradient method with two-level preconditioner is uniform with respect to large jumps and mesh sizes.

Key words. Jump Coefficients, Conjugate Gradient, Effective Condition Number, Two-Level Hierarchical Basis

AMS subject classifications. 65N30, 65N55, 65F10

1 Introduction

In this paper, we will discuss the two-level hierarchical basis preconditioned conjugate gradient methods for the linear finite element approximation of the second order elliptic boundary value problem

$$\begin{cases} -\nabla \cdot (\omega \nabla u) = f & in \quad \Omega \\ u = g_D & on \quad \Gamma_D \\ -\omega \dfrac{\partial u}{\partial n} = g_N & on \quad \Gamma_N \end{cases} \tag{1.1}$$

where $\Omega \in R^d(d = 1, 2 \text{ or } 3)$ is a polygonal or polyhedral domain with

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Dirichlet boundary Γ_D and Neumann boundary Γ_N. The coefficient $\omega = \omega(x)$ is a positive and piecewise constant function. More precisely, we assume that there are M open disjointed polygonal or polyhedral regions $\Omega_m(m = 1, \cdots, M)$ satisfying $\cup_{m=1}^{M} \overline{\Omega}_m = \overline{\Omega}$ with

$$\omega_m = \omega|_{\Omega_m}, m = 1, \cdots, M$$

where each $\omega_m > 0$ is a constant. The analysis can be carried through to a more general case when $\omega(x)$ varies moderately in each subdomain.

We assume that the subdomain $\Omega_m : m = 1, \cdots, M$ are given and fixed but may possibly have complicated geometry. We are concerned with the robustness of the preconditioned conjugate gradient method in regard to both the meshsize and jump coefficients. This model problem is relevant to many applications, such as groundwater flow [1, 15], fluid pressure prediction [19], electromagnetics [13], semiconductor modeling [9], electrical power network modeling [14] and fuel cell modeling [20, 21], where the coefficients have large discontinuities across interfaces between subdomains with different material properties.

The goal of the current paper is to provide proof of the robustness of the two-level hierarchical basis preconditioner (Two-Level-PCG).

The rest of the paper is organized as follows. To the paper is comprehensive and self-contained, we refer directly to parts of contents in [23] and [25].(Section 2 in the paper). In Section 2, we introduce some basic notation, the PCG algorithm and some theoretical foundations. In Section 3, we introduce the two-level hierarchical basis method and preconditioner. In Section 4, we analyze the eigenvalue distribution of the two-level preconditioned system and prove the convergence rate of the PCG algorithm. Section 5 is the conclusions. Following [22], short notation $x \lesssim y$ means $x \leq Cy$; and $x \eqsim y$ means $cx \leq y \leq Cx$.

2 Preliminaries

2.1 Notation

We introduce the bilinear form

$$a(u, v) = \sum_{m=1}^{M} \omega_m (\nabla u, \nabla v)_{L^2(\Omega_m)}, \quad \forall u, v \in H_D^1(\Omega),$$

where $H_D^1(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$, and introduce the H^1-norm and seminorm with respect to any subregion Ω_m by

$$|u|_{1,\Omega_m} = \|\nabla u\|_{0,\Omega_m}, \quad \|u\|_{1,\Omega_m} = (\|u\|_{0,\Omega_m}^2 + |u|_{1,\Omega_m}^2)^{\frac{1}{2}}.$$

Thus,

$$a(u,u) = \sum_{m=1}^{M} \omega_m |u|_{1,\Omega_m}^2 := |u|_{1,\omega}^2.$$

We also need the weighted L^2-inner product

$$(u,v)_{0,\omega} = \sum_{m=1}^{M} \omega_m (u,v)_{L^2(\Omega_m)}$$

and the weighted L^2- and H^1-norms

$$\|u\|_{0,\omega} = (u,u)_{0,\omega}^{\frac{1}{2}}, \quad \|u\|_{1,\omega} = (\|u\|_{0,\omega}^2 + |u|_{1,\omega}^2)^{\frac{1}{2}}.$$

For any subset $O \subset \Omega$, let $|u|_{1,\omega,O}$ and $\|u\|_{0,\omega,O}$ be the restrictions of $|u|_{1,\omega}$ and $\|u\|_{0,\omega}$ on the subset O, respectively.

For the distribution of the coefficients, we introduce the index set

$$I = \{m : meas(\partial\Omega_m \cap \Gamma_D) = 0\}$$

where meas$(\cdot)$ is the $d-1$ measure. In other words, I is the index set of all subregions which do not touch the Dirichlet boundary. We assume that the cardinality of I is m_0. We shall emphasize that m_0 is a constant which depends only on the distribution of the coefficients.

2.2 The Discrete Systems

Given a quasi-uniform triangulation $\mathcal{T}_h$ with the meshsize h, let

$$\mathcal{V}_h = \{v \in H_D^1(\Omega) : v|_\tau \in \mathcal{P}_1(\tau), \forall \tau \in \mathcal{T}_h\}$$

be the piecewise linear finite element space, where $\mathcal{P}_1$ denotes the set of linear polynomials. The finite element approximation of (1.1) is the function $u \in \mathcal{V}_h$, such that

$$a(u,v) = (f,v) + \int_{\Gamma_N} g_N v, \quad \forall v \in \mathcal{V}_h.$$

We define a linear symmetric positive definite operator $A : \mathcal{V}_h \to \mathcal{V}_h$ by

$$(Au, v)_{0,\omega} = a(u, v).$$

The related inner product and the induced energy norm are denoted by

$$(\cdot, \cdot)_A := a(\cdot, \cdot), \quad \| \cdot \|_A := \sqrt{a(\cdot, \cdot)}.$$

Then we have the following operator equation,

$$Au = F. \tag{2.1}$$

2.3 Preconditioned Conjugate Gradient (PCG) Methods

The well known conjugate gradient method is the basis of all the preconditioning techniques to be studied in this paper. The PCG methods can be viewed as a conjugate gradient method applied to the preconditioned system

$$BAu = BF.$$

Here, B is an SPD operator, known as a preconditioner of A. Note that BA is symmetric with respect to the inner product $(\cdot, \cdot)_{B^{-1}}$ (or $(\cdot, \cdot)_A$). For the implementation of the PCG algorithm, we refer to the monographs [2, 17, 18].

Let $u_k, k = 0, 1, \cdots$, be the solution sequence of the PCG algorithm. It is well known that

$$\|u - u_k\|_A \leq 2 \left(\frac{\sqrt{k(BA)} - 1}{\sqrt{k(BA)} + 1} \right)^k \|u - u_0\|_A, \tag{2.2}$$

which implies that the PCG method generally converges faster with a smaller condition number $k(BA)$.

Even though the estimate given in (2.2) is sufficient for many applications, in general it is not sharp. One way to improve the estimate is to look at the eigenvalue distribution of BA(see [2, 12] for more details). More specifically, suppose that we can divide $\sigma(BA)$, the spectrum of BA, into two sets, $\sigma_0(BA)$ and $\sigma_1(BA)$, where σ_0 consists of all "bad" eigenvalues and the remaining eigenvalues in σ_1 are bounded above and below, then we have the following theorem.

Theorem 2.1 *Suppose that $\sigma(BA) = \sigma_0(BA) \cup \sigma_1(BA)$ such that there are m elements in $\sigma_0(BA)$ and $\lambda \in [a, b]$ for each $\lambda \in \sigma_1(BA)$. Then*

$$\|u - u_k\|_A \leq 2K \left(\frac{\sqrt{b/a} - 1}{\sqrt{b/a} + 1} \right)^{k-m} \|u - u_0\|_A, \tag{2.3}$$

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where

$$K = \max_{\lambda \in \sigma_1(BA)} \prod_{\mu \in \sigma_0(BA)} \left| 1 - \frac{\lambda}{\mu} \right|.$$

If there are only m small eigenvalues in σ_0, say

$$0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_m \ll \lambda_{m+1} \leq \cdots \leq \lambda_n,$$

then

$$K = \prod_{i=1}^{m} \left| 1 - \frac{\lambda_n}{\lambda_i} \right| \leq \left(\frac{\lambda_n}{\lambda_1} - 1 \right)^m = (k(BA) - 1)^m.$$

In this case, the convergence rate estimate (2.3) becomes

$$\frac{\|u - u_k\|_A}{\|u - u_0\|_A} \leq 2(k(BA) - 1)^m \left(\frac{\sqrt{b/a} - 1}{\sqrt{b/a} + 1} \right)^{k-m}. \quad (2.4)$$

Based on (2.4), given a tolerance $0 < \epsilon < 1$, the number of iterations of the PCG algorithm needed for $\frac{\|u-u_k\|_A}{\|u-u_0\|_A} < \epsilon$ is given by

$$k \geq m + \left(log\left(\frac{2}{\epsilon} \right) + mlog(k(BA) - 1) \right) / log\left(\frac{\sqrt{b/a} + 1}{\sqrt{b/a} - 1} \right). \quad (2.5)$$

Observing the convergent estimate (2.4), if there are only a few small eigenvalues of BA in $\sigma_0(BA)$, then the convergent rate of the PCG methods will be dominated by the factor $\frac{\sqrt{b/a}+1}{\sqrt{b/a}-1}$, i.e., by b/a where $b = \lambda_n(BA)$ and $a = \lambda_{m+1}(BA)$. We define this quantity as the "effective condition number".

Definition. ([23]) Let $\mathcal{V}$ be a Hilbert space. The m-th effective condition number of an operator $A : \mathcal{V} \to \mathcal{V}$ is defined by

$$k_{m+1}(A) = \frac{\lambda_{max}(A)}{\lambda_{m+1}(A)}$$

where $\lambda_{m+1}(A)$ is the (m+1)-th minimal eigenvalue of A.

To estimate the effective condition number, we need to estimate $\lambda_{m+1}(A)$. A fundamental tool is the following *Courant-Fisher min-max theorem.*

Theorem 2.2 *The eigenvalues of a SPD operator $A : \mathcal{V} \to \mathcal{V}$ are characterized by the relation*

$$\lambda_{m+1}(A) = \min_{S, dim(S) = n-m.} \max_{x \in S, x \neq 0} \frac{(Ax, x)}{(x, x)}. \quad (2.6)$$

Especially, for any subspace $\mathcal{V}_0 \subset \mathcal{V}$ with $dim(\mathcal{V}_0) = n - m$, the following estimation of $\lambda_{m+1}(A)$ holds:

$$\lambda_{m+1}(A) \geq \min_{0 \neq x \in \mathcal{V}_0} \frac{(Ax, x)}{(x, x)}. \qquad (2.7)$$

3 Two-Level Hierarchical Basis Preconditioner

The classical two-level hierarchical basis method was proposed and developed by Axelsson, Bank, Dupont, and Yserentant [3, 4, 5, 6, 24]. As usual, we assume that V is decomposed as a direct sum

$$V = SV_s \oplus PV_c. \qquad (3.1)$$

for some components V_s and V_c isomorphic to $\mathbb{R}^{n_s}$ and $\mathbb{R}^{n_c}$ respectively, with $n = n_s + n_c$. A typical and simple example to keep in mind is $S = \begin{pmatrix} I \\ 0 \end{pmatrix}$ and $P = \begin{pmatrix} W \\ I \end{pmatrix}$ for some W such that the square matrix (S, P) is unit upper triangular, and hence invertible.

3.1 Some notation

Two ingredients (the space decomposition (3.1) and the smoother M) are important in the two-level hierarchical basis method. Various restrictions of M and A to the subspaces mentioned before will be needed. We first define the exact coarse grid matrix A_c and its hierarchical complement A_s as follows

$$A_c = P^T AP, \quad A_s = S^T AS.$$

Later we will see, in the case of a two level hierarchical basis preconditioner, one needs M to be well-defined only on the first component SV_s. In that case, we refer to M as M_s. Then

$$M_s = S^T MS.$$

In order to define the hierarchical basis preconditioner, we also need two symmetrized version of the smoother M:

$$\widetilde{M} = M^T(M^T + M - A)^{-1}M, \qquad (3.2)$$

$$\overline{M} = M(M^T + M - A)^{-1}M^T. \qquad (3.3)$$

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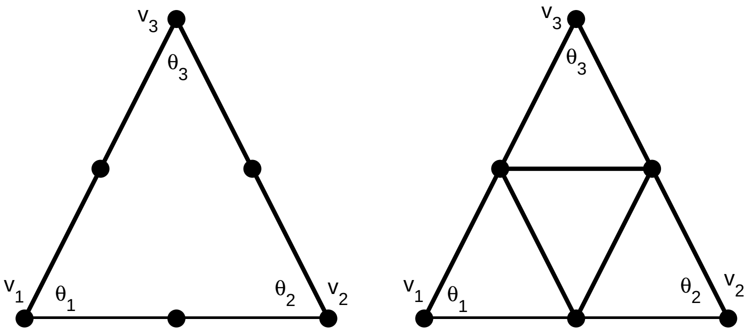

Figure 1: Quadratic element(left) and piecewise linear element(right).

If we assume that $A = D - L - L^T$ where D, L, L^T are the diagonal, lower triangle, and upper triangle part of A, and let $M = D - L^T$, then

$$\widetilde{M_s}^{-1} = (D - L^T)_s^{-1} D_s (D - L)_s^{-1}, \qquad (3.4)$$

where $D_s = S^T DS$, $(D-L^T)_s = S^T(D-L^T)S$, and $(D-L)_s = S^T(D-L)S$.

3.2 The Element Stiffness Matrix for The Hierarchical Basis

In this subsection, we consider the stiffness matrix for the hierarchical basis in each element. Following Braess [7], and Bank [6], simply we let $\omega = 1$ in (1.1) and let t be a triangle with vertices v_i, edges e_i, and angles θ_i, $1 \le i \le 3$. Here, we consider two kinds of different hierarchical basis: the quadratic element and piecewise linear element.

For the space of continuous quadratic finite elements (illustrated on the left in Figure 1), we let $\phi_i, 1 \le i \le 3$ denote the linear basis functions for element t. Then on element t, the subspace PV_c will be the span of $\langle \phi_i \rangle_{i=1}^3$.

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And the subspace SV_s is composed of the quadratic bump functions $\langle\psi_i\rangle_{i=1}^3$, where $\psi_i = 4\phi_j\phi_k$, and (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

For the space of continuous piecewise linear polynomials on a refined mesh (illustrated on the right in Figure 1), let PV_c be defined as the quadratic finite elements above. But the subspace SV_s contains the continuous piecewise polynomials on the fine grid that are zero at the vertices of t. Then on element t, the subspace $SV_s = \langle\widehat{\phi_i}\rangle_{i=1}^3$, where $\widehat{\phi_i}$ is the standard nodal piecewise linear basis functions associated with the midpoint of edge e_i of t.

Following [6] and [7], we can establish the relation

$$L_i = cot\theta_i = -2|t|\nabla\phi_j\cdot\nabla\phi_k, \tag{3.5}$$

where $|t|$ is measure of element t, it is about $h^d, d = 1, 2, 3$.

Then the element stiffness matrix for the quadratic hierarchical basis can be shown to be

$$A_Q^t = \begin{pmatrix} * & * \\ * & A_s^t \end{pmatrix}, \tag{3.6}$$

where A_s^t is the restriction of A_s on the element t, and

$$A_s^t = \frac{4}{3}\begin{pmatrix} L_1+L_2+L_3 & -L_3 & -L_2 \\ -L_3 & L_1+L_2+L_3 & -L_1 \\ -L_2 & -L_1 & L_1+L_2+L_3 \end{pmatrix}. \tag{3.7}$$

The diagonal of A_s^t is

$$D_s^t = \frac{4}{3}\begin{pmatrix} L_1+L_2+L_3 & 0 & 0 \\ 0 & L_1+L_2+L_3 & 0 \\ 0 & 0 & L_1+L_2+L_3 \end{pmatrix}. \tag{3.8}$$

The element stiffness matrix for the piecewise linear hierarchical basis is given by

$$A_L^t = \begin{pmatrix} * & * \\ * & A_s^t \end{pmatrix}. \tag{3.9}$$

In this case,

$$A_s^t = \begin{pmatrix} L_1+L_2+L_3 & -L_3 & -L_2 \\ -L_3 & L_1+L_2+L_3 & -L_1 \\ -L_2 & -L_1 & L_1+L_2+L_3 \end{pmatrix}, \tag{3.10}$$

and

$$D_s^t = \begin{pmatrix} L_1+L_2+L_3 & 0 & 0 \\ 0 & L_1+L_2+L_3 & 0 \\ 0 & 0 & L_1+L_2+L_3 \end{pmatrix}. \tag{3.11}$$

3.3 The Two-Level Hierarchical Basis Preconditioner

In this subsection, we can define the two-level hierarchical basis preconditioner using some notations in above subsections. Let

$$\widehat{B}_{TL} = \begin{pmatrix} I & 0 \\ P^T ASM_s^{-1} & I \end{pmatrix} \begin{pmatrix} \overline{M}_s & 0 \\ 0 & A_c \end{pmatrix} \begin{pmatrix} I & M_s^{-T} S^T AP \\ 0 & I \end{pmatrix}. \quad (3.12)$$

Then, the two-level hierarchical basis preconditioner is defined by

$$B_{TL}^{-1} = \begin{pmatrix} S, P \end{pmatrix} \widehat{B}_{TL}^{-1} \begin{pmatrix} S, P \end{pmatrix}^T. \quad (3.13)$$

4 The Condition Number Analysis of $B_{TL}^{-1}A$

Following [11], for the two-level hierarchical basis preconditioner B_{TL}^{-1}, we have following estimate.

Lemma 4.1 *Assume that $(M_s + M_s^T - A_s)$ is S.P.D, for any $v \in V$, the following bounds hold:*

$$\frac{1}{K} v^T B_{TL} v \le v^T A v \le v^T B_{TL} v, \quad K \lesssim \sup_w \frac{1}{\lambda(\widetilde{M}_s^{-1} A_s)}. \quad (4.1)$$

If $\widetilde{M}_s^{-1}$ is given by (3.4), then we have the following relationship between the symmetric Gauss-Seidel preconditioner and the Jacobi preconditioner.

Lemma 4.2 *For any $v \in V$, we have*

$$\frac{1}{4} v^T D_s v \le v^T \widetilde{M}_s v \le v^T D_s v. \quad (4.2)$$

Proof:

$$\widetilde{M}_s = (D - L)_s D_s^{-1} (D - L^T)_s.$$

Following the Schwarz inequality we can prove the second inequality. Then, we prove the first inequality.

Using the fact that D_s and A_s are S.P.D, then for any $v \in V$ we have

$$((D - L)_s v, v)_A = \frac{1}{2}((A_s + D_s)v, v)_A \ge \frac{1}{2}(D_s v, v)_A.$$

Taking $v = (D - L^T)_s^{-1} w$, we have for all $w \in V$:

$$\frac{1}{2}(D_s(D-L)_s^{-T} w, (D-L)_s^{-T} w)_A \le ((D-L)_s^{-1} w, w)_A = (D_s(D-L)_s^{-T} w, D_s^{-1} w)_A.$$

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On the other hand,

$$\frac{1}{2}(\widetilde{M}_s^{-1}w, w)_A \leq (\widetilde{M}_s^{-1}w, w)_A^{1/2}(D_s^{-1}w, w)_A^{1/2}.$$

Consequently,

$$(\widetilde{M}_s^{-1}w, w)_A \leq 4(D_s^{-1}w, w)_A,$$

and

$$(\widetilde{M}_s v, v)_A \geq \frac{1}{4}(D_s v, v)_A.$$

The proof of Lemma 4.2 can be found in [26].

Following lemma provides the eigenvalue estimate of Jacobi preconditioner.

Lemma 4.3 *For any $v \in V$, we have*

$$v^T D_s v \eqsim h^{-2}\|v\|_{0,\omega}^2.$$

Proof: Note that on each element, we have

$$\sum_{i=1}^{3} L_i = -2h^d(\nabla\phi_1\nabla\phi_2 + \nabla\phi_1\nabla\phi_3 + \nabla\phi_2\nabla\phi_3) \eqsim h^{d-2}.$$

Consequently, following [23] we have

$$v^T D_s v \eqsim h^{d-2}(v, v)_{l^2,\omega} \eqsim h^{-2}\|v\|_{0,\omega}^2.$$

This completes the proof.

In order to research the effective condition number for the Jacobi preconditioner, we need to define the space

$$\widetilde{\mathcal{V}}_h = \left\{ v \in \mathcal{V}_h : \int_{\Omega_m} v = 0, m \in I \right\}.$$

On this space, the following Poincare-Friedrichs inequality holds:

$$\|v\|_{0,\omega} \lesssim \|\nabla v\|_{0,\omega}, \quad \forall v \in \widetilde{\mathcal{V}}_h. \tag{4.3}$$

Then, we have following important lemma.

Lemma 4.4 *Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform, then we have*

$$h^2 J(\omega)^{-1} v^T D_s v \lesssim v^T A_s v, \quad \forall v \in \mathcal{R}^n, \tag{4.4}$$

and

$$h^2 v^T D_s v \lesssim v^T A_s v, \quad \forall v \in \widetilde{\mathcal{V}}_h. \tag{4.5}$$

where $J(\omega) = \frac{\max_m \omega_m}{\min_m \omega_m}$.

Proof: In fact, we have

$$a(v,v) \geq \min_m\{\omega_m\}|v|^2_{1,\Omega} \gtrsim \min_m\{\omega_m\}\|v\|^2_{0,\Omega} \geq \frac{\min_m\{\omega_m\}}{\max_m\{\omega_m\}}h^{-2}(h^2)\|v\|^2_{0,\Omega}.$$

Applying Lemma 4.3 and inequality (4.3), we also have

$$v^T D_s v \lesssim \sum_{\tau\in\mathcal{T}_h} h^{-2}\|v\|^2_{0,\omega,\tau} = h^{-2}\sum_{m=1}^{M}\omega_m\|v\|^2_{0,\Omega_m} \lesssim h^{-2}|v|^2_{1,\omega} = h^{-2}v^T A_s v.$$

This completes the proof. Followed by Lemmas 4.1-4.4, we have the following results regarding the condition number of $B^{-1}_{TL}A$.

Theorem 4.1 *For the hierarchical basis preconditioner B^{-1}_{TL} defined by (3.13), the condition number and m_0-th effective condition number satisfies:*

$$k(B^{-1}_{TL}A) \leq J(\omega)h^{-2}, \quad k_{m_0+1}(B^{-1}_{TL}A) \leq h^{-2}.$$

Theorem 4.2 *For the hierarchical basis preconditioned conjugate gradient methods, we have the following convergence rate*

$$\frac{\|u-u_k\|_A}{\|u-u_0\|_A} \leq 2(C_1J(\omega)h^{-2}-1)^{m_0}\left(1-\frac{2}{1+C_2h^{-1}}\right)^{k-m_0}, k \geq m_0. \quad (4.6)$$

5 Conclusions

In this paper, we provided a proof of robustness of the two-level hierarchical basis preconditioner for the linear finite element approximation of second order elliptic problems with strongly discontinuous coefficients. We discussed the eigenvalue distribution of the Two-Level-preconditioner and found that only a few small eigenvalues infected by the large jump.

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A New Fourth-Order Explicit Finite Difference Method for the Solution of Parabolic Partial Differential Equation with Nonlocal Boundary Conditions

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Abstract

In this paper, a new fourth-order explicit finite difference method is proposed for solving linear and nonhomogeneous parabolic partial differential equation with nonlocal boundary conditions. The advantage of the explicit finite difference methods is easier to implement than the implicit methods. Moreover, the explicit method need lesser CPU time than the implicit schemes. Numerical results show that the proposed method is very accurate and effective.

Key words: Finite difference method, Fourth-order explicit method, Nonlocal boundary conditions, nonhomogeneous parabolic partial differential equation.

1 Introduction

Many physical phenomena can be modelled by parabolic partial differential equations which involve integral terms in the boundary conditions. These boundary conditions are called nonlocal boundary conditions. One-dimensional parabolic equation with nonlocal boundary conditions have important applications in electro-chemistry, porous media flow, thermo-elasticity, heat conduction and several others. The existence, uniqueness and theoretical aspects of these equations have been studied by [17, 20, 35]. Generally, it is difficult to find the analytical solution of parabolic partial differential equations with nonlocal boundary conditions.

Approximate and numerical techniques for obtaining approximate solution of these equations have been developed by many researchers [5, 6, 7, 8, 9, 11, 19, 23, 27, 29, 28]. Some standard numerical methods have been used for the solution of one dimensional diffusion equation with nonlocal boundary conditions such as finite difference method, finite element method, adomian decomposition method (ADM), Chebyshev spectral collocation method, reducing kernel space method and method of lines [1, 13, 21, 24, 25, 26, 30].

In this paper a method based on explicit finite difference method is introduced and applied to obtain the numerical solution of the following parabolic equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t), \qquad 0 \le x \le 1 \quad , \quad 0 \le t \le T, \tag{1.1}$$

with initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le 1, \tag{1.2}$$

and subject to the boundary conditions

$$u(0,t) = \int_0^1 \phi(x,t)u(x,t)dx + g_1(t), \qquad 0 < t \le T, \tag{1.3}$$

$$u(1,t) = \int_0^1 \psi(x,t)u(x,t)dx + g_2(t), \qquad 0 < t \le T, \tag{1.4}$$

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where $q(x,t)$, $f(x)$, $g_1(t)$, $g_2(t)$, $\phi(x,t)$ and $\psi(x,t)$ are known functions.

Many authors applied various type of finite difference methods to obtain the numerical solution of equations (1.1)–(1.4). Dehghan [15, 16] applied the forward Euler, backward Euler, BTCS (backward in time and centered in space) schemes, Crandall's implicit formula, FTCS (forward in time and centered in space) method for the heat equation. The nonlocal boundary conditions have been approximated by Trapezoidal rule and fourth-order Simpson composite formula. Zhou et al. [36], Mu and Du [25] introduced an efficient technique based on reproducing kernel space to solve the partial differential equations with nonlocal boundary conditions. The BTCS and explicit Crandall's formula have been developed by Martin-Vaquero and Vigo-Aguiar [32, 31] to solve the above mentioned equations .

The aim of this paper is to describe an efficient technique based on explicit finite difference method to find out the numerical solution of parabolic equation with nonlocal boundary conditions. The new method is of fourth order and it is compared with BTCS, Crank-Nicolson and Crandall's formula. The basic idea of this approach is to write $q(x,t)$ as a linear combination of q_{i-1}^{n+1}, q_i^{n+1}, q_{i+1}^{n+1}, q_{i-1}^{n}, q_i^n and q_{i+1}^n. The objective of this technique is to improve the results obtained by many researchers in some papers [3, 10, 16, 26, 31, 32, 36]. We considered that coefficient q_{i+1}^{n+1}, q_{i-1}^{n+1} and q_{i-1}^{n}, q_{i+1}^{n} are not equal. The nonlocal boundary conditions are solved by higher order Integration rules.

This paper is organized as follows: In section 2, the new fourth-order explicit technique is presented, the composite Simpson rule and sixth-order formula for the nonlocal boundary conditions are also introduced. Numerical results are presented in section 3. Finally conclusion is given in section 4.

2 Explicit Finite Difference Method

The domain $[0,1]\times[0,T]$ is divided into an $M\times N$ mesh with a spatial size of $h=1/M$ and temporal size $k=T/N$. The grid points (x_i,t_n) are defined by

$$\begin{aligned} x_i &= ih, \qquad & i &= 0,1,...,M,\\ t_n &= nk, \qquad & n &= 0,1,...,N, \end{aligned}$$

where M and N are integers. The notation u_i^n, q_i^n, ϕ_i^n, ψ_i^n, g_1^n and g_2^n represents, respectively, the finite difference approximations of $u(x_i,t_n)$, $q(x_i,t_n)$, $\phi(x_i,t_n)$, $\psi(x_i,t_n)$, $g_1(t_n)$ and $g_2(t_n)$. The FTCS (forward in time and centrad in space) finite difference scheme for the heat equation (1.1) can be written as

$$u_i^{n+1} = ru_{i-1}^n + (1-2r)u_i^n + ru_{i+1}^n + kq_i^n, \tag{2.1}$$

for $i=1,...,M-1$, $n=0,1,...,N-1$ and $r=\frac{k}{h^2}$. The stability condition for this method is proved in [16]:

$$r \le \frac{1}{2}.$$

The local truncation error of this method can be written as [31, 33]:

$$\tau = (u_t - u_{xx} - q) + \frac{6ru_{tt} - u_{xxxx}}{12}h^2 + \frac{60r^2u_{ttt} - u_{xxxxxx}}{360}h^4 + O(h^6). \tag{2.2}$$

Now it can be verified that

$$u_{tt} = u_{xxxx} + q_{xx} + q_t. \tag{2.3}$$

By substituting (2.3) into (2.2) the truncation error can be obtained as

$$\tau = \frac{6rq_t + 6rq_{xx} + (6r-1)u_{xxxx}}{12}h^2 + \frac{60r^2u_{ttt} - u_{xxxxxx}}{360}h^4 + O(h^6). \tag{2.4}$$

It is clear that (2.4) is second order. Now, we write q_i^n using a linear combination of q_{i-1}^{n+1}, q_i^{n+1}, q_{i+1}^{n+1}, q_{i-1}^n, q_i^n and q_{i+1}^n, then we have

$$\frac{u_i^{n+1} - u_i^n}{k} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + a_1q_{i-1}^{n+1} + a_2q_i^{n+1} + a_3q_{i+1}^{n+1} + a_4q_{i-1}^n + a_5q_i^n + a_6q_{i+1}^n. \tag{2.5}$$

By using the Taylor's expansion in (2.5), we can obtain

$$\left(\sum_{i=1}^{6} a_i - 1\right) + h(a_3 - a_1 - a_4 + a_6)q_x + rh^2(a_1 + a_2 + a_3)q_t + \frac{h^2}{2}(a_1 + a_3 + a_4 + a_6)q_{xx} - \frac{h^3}{6}(a_6 - a_4)q_{xxx} + rh^3(a_3 - a_1)q_{xt} + A(u(x,t), q)h^4 + O(h^6) = 0. \tag{2.6}$$

where

$$A(u(x,t), q) = \frac{-15q_{tt} + 10u_{tt} - 15q_{xxxx} - 6u_{xxxxxx} + (60 - 180a_5)q_{xxt}}{2160}. \tag{2.7}$$

By substituting (2.4), (2.6) and (2.7) into (2.5), we get the following system of linear equations

$$\begin{aligned}
&a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1,\\
&a_3 - a_1 - a_4 + a_6 = 0,\\
&a_1 + a_2 + a_3 = \frac{1}{2},\\
&a_6 - a_4 = 0,\\
&a_3 - a_1 = 0,\\
&a_1 + a_3 + a_4 + a_6 = r,
\end{aligned} \tag{2.8}$$

and $r = 1/6$. By selecting $a_5 = m$ thus equation (2.5) can be written as

$$\begin{aligned}
u_i^{n+1} &= ru_{i-1}^n + (1-2r)u_i^n + ru_{i+1}^n + \frac{k}{12}[(6m-2)(q_{i-1}^{n+1} + q_{i+1}^{n+1})\\
&+ (10 - 12m)q_i^{n+1} - (6m-3)(q_{i-1}^n + q_{i+1}^n) + 12mq_i^n].
\end{aligned} \tag{2.9}$$

It should be noted that this technique is fourth order accurate when $r = 1/6$. By considering (2.9), we can consider many values for m such that the solution of these equation become converges to the exact solution. Our goal in this paper is to improve the results obtained in the literature. To find the optimal value of m, we can apply the following algorithm.

1. Step 1: We consider, $\quad m_1 = \dfrac{a_1}{b_1}, \qquad m_2 = \dfrac{a_2}{b_2}$
2. Step 2: We calculate, E_{m_1} and E_{m_2}
3. Step 3: If $E_{m_1} < E_{m_2}$ then, $m_2 = \dfrac{a_1 + a_2}{b_1 + b_2}, \qquad$ else $m_1 = \dfrac{a_1 + a_2}{b_1 + b_2}$
4. Step 4: If $|E_{m_i}| < l$ then $m = m_i$ is optimal $i = 1$ or 2, else we repeat Step 1.

Equation (2.9) has $M-1$ linear equations and $M+1$ unknowns. Thus two more equations are needed. The integral in the boundary conditions can be approximated by composite Simpson rule and sixth order formula.

2.1 Composite Simpson formula

The Simpson composite formula for solving the nonlocal boundary conditions (1.3) and (1.4) can be written as [16, 31]:

$$\begin{aligned}
u_0^{n+1} &= \int_0^1 \phi(x, t^{n+1})u(x, t^{n+1})dx + g_1^{n+1} = \frac{h}{3}\left(\phi_0^{n+1}u_0^{n+1}\right.\\
&+ 4\sum_{i=1}^{M/2} \phi_{2i-1}^{n+1}u_{2i-1}^{n+1} + 2\sum_{i=1}^{M/2-1} \phi_{2i}^{n+1}u_{2i}^{n+1} + \phi_M^{n+1}u_M^{n+1}) + g_1^{n+1} + O(h^4),
\end{aligned} \tag{2.10}$$

and

$$u_M^{n+1} = \int_0^1 \psi(x,t^{n+1})u(x,t^{n+1})dx + g_2^{n+1} = \frac{h}{3}(\psi_0^{n+1}u_0^{n+1} + 4\sum_{i=1}^{M/2}\psi_{2i-1}^{n+1}u_{2i-1}^{n+1} + 2\sum_{i=1}^{M/2-1}\psi_{2i}^{n+1}u_{2i}^{n+1} + \psi_M^{n+1}u_M^{n+1}) + g_2^{n+1} + O(h^4). \tag{2.11}$$

Thus

$$(h\phi_0^{n+1}-3)u_0^{n+1} + 4h\sum_{i=1}^{M/2}\phi_{2i-1}^{n+1}u_{2i-1}^{n+1} + 2h\sum_{i=1}^{M/2-1}\phi_{2i}^{n+1}u_{2i}^{n+1} + h\phi_M^{n+1}u_M^{n+1} = -3g_1^{n+1}, \tag{2.12}$$

$$h\psi_0^{n+1}u_0^{n+1} + 4h\sum_{i=1}^{M/2}\psi_{2i-1}^{n+1}u_{2i-1}^{n+1} + 2h\sum_{i=1}^{M/2-1}\psi_{2i}^{n+1}u_{2i}^{n+1} + (h\psi_M^{n+1}-3)u_M^{n+1} = -3g_1^{n+1}, \tag{2.13}$$

Combining (2.12) and (2.13) with (2.9) gives $(M+1)\times(M+1)$ linear system of equations. We can obtain

$$u_0^{n+1} = \frac{F_1(\Phi,U)(h\psi_M^{n+1}-3) - hF_2(\Psi,U)\phi_M^{n+1}}{J(\Phi,\Psi,U)},$$

$$u_M^{n+1} = \frac{F_2(\Psi,U)(h\phi_0^{n+1}-3) - hF_1(\Phi,U)\psi_0^{n+1}}{J(\Phi,\Psi,U)},$$

where

$$F_1(\Phi,U) = -4h\left(\sum_{i=1}^{M/2}\phi_{2i-1}^{n+1}u_{2i-1}^{n+1}\right) - 2h\left(\sum_{i=1}^{M/2-1}\phi_{2i}^{n+1}u_{2i}^{n+1}\right) - 3g_1^{n+1},$$

$$F_2(\Psi,U) = -4h\left(\sum_{i=1}^{M/2}\psi_{2i-1}^{n+1}u_{2i-1}^{n+1}\right) - 2h\left(\sum_{i=1}^{M/2-1}\psi_{2i}^{n+1}u_{2i}^{n+1}\right) - 3g_2^{n+1},$$

and

$$J(\Phi,\Psi,U) = (h\phi_0^{n+1}-3)(h\psi_M^{n+1}-3) - h^2\phi_M^{n+1}\psi_0^{n+1} \neq 0.$$

2) Sixth-order formula

The sixth-order integration formula can be used to approximate numerically the integral present in the boundary conditions (1.3) and (1.4). We can write the sixth-order formula as [31]:

$$\begin{aligned} u_0^{n+1} = u(0,t_{n+1}) &= \int_0^1 \phi(x,t_{n+1})u(x,t_{n+1})dx + g_1(t_{n+1}) \\ &= \frac{2h}{45}[7\phi_0^{n+1}u_0^{n+1} + 32\sum_{i=1}^{M/2}\phi_{2i-1}^{n+1}u_{2i-1}^{n+1} + 12\sum_{i=0}^{M/4-1}\phi_{4i+2}^{n+1}u_{4i+2}^{n+1} \\ &+ 14\sum_{i=0}^{M/4-2}\phi_{4i+4}^{n+1}u_{4i+4}^{n+1} + 7\phi_M^{n+1}u_M^{n+1}] + g_1^{n+1} + O(h^6), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} u_M^{n+1} = u(1,t_{n+1}) &= \int_0^1 \psi(x,t_{n+1})u(x,t_{n+1})dx + g_1(t_{n+1}) \\ &= \frac{2h}{45}[7\psi_0^{n+1}u_0^{n+1} + 32\sum_{i=1}^{M/2}\phi_{2i-1}^{n+1}u_{2i-1}^{n+1} + 12\sum_{i=0}^{M/4-1}\psi_{4i+2}^{n+1}u_{4i+2}^{n+1} \\ &+ 14\sum_{i=0}^{M/4-2}\psi_{4i+4}^{n+1}u_{4i+4}^{n+1} + 7\psi_M^{n+1}u_M^{n+1}] + g_1^{n+1} + O(h^6), \end{aligned} \tag{2.15}$$

where M should be a multiple of 4. So

$$(14h\phi_0^{n+1} - 45)u_0^{n+1} + 14h\phi_M^{n+1}u_M^{n+1} = F_1(\Phi, U), \tag{2.16}$$

$$14h\psi_0^{n+1}u_0^{n+1} + (14h\psi_M^{n+1} - 45)u_M^{n+1} = F_2(\Psi, U), \tag{2.17}$$

where

$$F_1(\Phi, U) = -64h\sum_{i=1}^{M/2}\phi_{2i-1}^{n+1}u_{2i-1}^{n+1} - 24h\sum_{i=0}^{M/4-1}\phi_{4i+2}^{n+1}u_{4i+2}^{n+1} - 28h\sum_{i=0}^{M/4-2}\phi_{4i+4}^{n+1}u_{4i+4}^{n+1} - 45g_1^{n+1},$$

$$F_2(\Psi, U) = -64h\sum_{i=1}^{M/2}\psi_{2i-1}^{n+1}u_{2i-1}^{n+1} - 24h\sum_{i=0}^{M/4-1}\psi_{4i+2}^{n+1}u_{4i+2}^{n+1} - 28h\sum_{i=0}^{M/4-2}\psi_{4i+4}^{n+1}u_{4i+4}^{n+1} - 45g_1^{n+1},$$

Combining (2.16), (2.17) with (2.9) gives $(M+1)\times(M+1)$ linear system of equations. We can obtain

$$u_0^{n+1} = \frac{F_1(\Phi, U)(14h\psi_M^{n+1} - 45) - 14hF_2(\Psi, U)\phi_M^{n+1}}{J(\Phi, \Psi, U)},$$

$$u_M^{n+1} = \frac{F_2(\Psi, U)(14h\phi_0^{n+1} - 45) - 14hF_1(\Phi, U)\psi_0^{n+1}}{J(\Phi, \Psi, U)},$$

where

$$J(\Phi, \Psi, U) = 196h^2(\phi_0^{n+1}\psi_M^{n+1} - \phi_M^{n+1}\psi_0^{n+1}) - 630h(\phi_0^{n+1} + \psi_M^{n+1}) + 2025.$$

It should be noted that the system (2.8) does not have unique solution. Thus it can help us to obtain the optimal value of m whilst it was not considered in [31]. To check the accuracy of present method, we compared our results with the results obtained in [31]. It should also be noted that the explicit finite difference methods are easier to implement than the implicit schemes or Crank-Nicolson method, because in explicit schemes there is only one unknown is involved in the finite difference formula. Moreover, implicit finite difference schemes require the solution of a large number of simultaneous linear algebraic equations at each steps resulting in an extensive amount of CPU time utilized compared to explicit finite difference methods for the same values of s and h.

3 Illustrative Examples

In this section, the new explicit finite difference method (NFTCS) applied to linear and nonhomogeneous parabolic partial differential equation (1.1) with nonlocal boundary conditions (1.3)–(1.4). The results show that the described method is very accurate, capable and powerful. The numerical results indicate that the approximate solution convergence to the exact solution as h tends to zero. The Simpson formula and sixth-order formula are used to approximate the integral in the examples. The MATHEMATICA software is used to find the approximate solution and CPU time. For describing the error, we define relative error E_R and the absolute error E_A as follows:

$$E_R(u(x,t)) = \frac{|u(ih, jk)_{\text{approx}} - u(ih, jk)_{\text{exact}}|}{|u(ih, jk)_{\text{exact}}|}$$

and

$$E_A(u(x,t)) = |u(ih, jk)_{\text{approx}} - u(ih, jk)_{\text{exact}}|$$

where $u(ih, jk)_{\text{approx}}$ is the approximate solution and $u(ih, jk)_{\text{exact}}$ is the exact solution.

Example 1:

We consider the nonhomogeneous parabolic partial differential equation [16, 31, 32, 33, 36]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t), \qquad 0 \le x \le 1 \quad , \quad 0 \le t \le T, \tag{3.1}$$

with the following initial and boundary conditions

$$\begin{aligned}
&f(x) = x^2, && && 0 < x < 1,\\
&\phi(x,t) = x, && 0 < x < 1, && 0 < t < 1,\\
&\psi(x,t) = x, && 0 < x < 1, && 0 < t < 1,\\
&g_1(t) = \frac{-1}{4(t+1)^2}, && && 0 < t < 1,\\
&g_2(t) = \frac{3}{4(t+1)^2}, && && 0 < t < 1,\\
&q(x,t) = \frac{-2(x^2+t+1)}{(t+1)^3}, && 0 \le t \le 1, && 0 < x < 1.
\end{aligned}$$

It can be verified that the exact solution is

$$u(x,t) = \left(\frac{x}{t+1}\right)^2.$$

By applying the algorithm which is introduced in section 2, we take $m = 0.427487$ in (2.9) and obtained results are shown in tables 1 and 2 for various values of x and h. The Simpson composite rule and sixth-order formula are used to approximate the nonlocal boundary conditions (1.3) and (1.4). It can be seen that the errors are very small. In the last row in each table, we have obtained the CPU time consumed in the implementation of NFTCS for various step size h and x at $t = 1$. As expected, the CPU time increases as the step size h decrease.

Table 1: Absolute error NFTCS at $t = 1$ by using the Simpson formula.

x/h	0.25	0.125	0.0625	0.03125
0	1.97727×10^{-9}	2.73501×10^{-13}	4.75005×10^{-14}	1.46390×10^{-14}
0.25	8.07057×10^{-8}	4.87357×10^{-9}	3.03945×10^{-10}	1.89884×10^{-11}
0.5	3.92653×10^{-8}	2.26709×10^{-9}	1.40977×10^{-10}	8.80097×10^{-12}
0.75	3.27403×10^{-8}	2.21648×10^{-9}	1.39182×10^{-10}	8.70845×10^{-12}
1	1.97727×10^{-9}	2.73392×10^{-13}	4.75009×10^{-14}	1.43112×10^{-14}
CPU	0.078	2.248	76	1006.45

Table 2: Absolute error NFTCS at $t = 1$ by using the sixth-order formula.

x/h	0.25	0.125	0.0625	0.03125
0	2.32131×10^{-10}	7.89537×10^{-13}	5.05059×10^{-14}	3.47797×10^{-14}
0.25	7.77293×10^{-8}	4.86301×10^{-9}	3.03905×10^{-10}	1.89883×10^{-11}
0.5	3.60167×10^{-8}	2.25557×10^{-9}	1.40933×10^{-10}	8.80078×10^{-11}
0.75	3.57167×10^{-8}	2.22705×10^{-9}	1.39223×10^{-10}	8.70862×10^{-12}
1	2.32131×10^{-10}	7.89538×10^{-13}	5.05068×10^{-14}	3.47777×10^{-14}
CPU	0.078	1.391	53.687	807.840

The relative error E_R for $u(0.5,1)$ is obtained for different step size h and compared the results obtain by [31] (scheme FTCS4). We used the algorithm introduced in section 2 with optimal value of m. The Simpson formula (NFTCS4) and sixth-order formula (NFTCS6) have been used for approximating the integrals in the nonlocal boundary conditions.

Table 3: Relative error E_R for $u(0.5,1)$ at various spatial length.

h	FTCS4 [31]	NFTCS4	NFTCS6
0.25	0.000127462	6.28724×10^{-7}	5.76746×10^{-7}
0.125	7.96658×10^{-6}	3.63034×10^{-8}	3.61190×10^{-8}
0.0625	4.97912×10^{-7}	2.25753×10^{-9}	2.25683×10^{-9}
0.03125	3.11195×10^{-8}	1.40816×10^{-10}	1.40812×10^{-10}

In table 4, we compared the relative error E_R at $x = 0.5$ and $t = 1$ by using NFTCS described in this paper and FTCS, explicit Crandall's formula (ECF) and implicit Crandall's formula (ICF) obtained in [31]. It is observed that the our results are better than the obtained results in [31].

Table 4: Comparison between relative error E_R in [31] and our solution for $u(0.5, 1)$.

h	FTCS [31]	ECF [31]	ICF [31]	Our method
1/8	7.96658×10^{-6}	6.19972×10^{-6}	1.32985×10^{-5}	3.63034×10^{-8}
1/16	4.97912×10^{-7}	3.87463×10^{-7}	8.31040×10^{-7}	2.25753×10^{-9}
1/32	3.11195×10^{-8}	2.42163×10^{-8}	5.19395×10^{-8}	4.37617×10^{-11}

In Figure 1, we show the absolute error E_A using the NFTCS at $x = 0.125$ and $x = 1$ with $h = 1/8$, while the Simpson formula is used for approximating the integrals in the boundary conditions. Similarly, the graph of absolute error NFTCS at $x = 0.0625$ and $x = 1$ with $h = 1/16$ is shown in Figure 2.

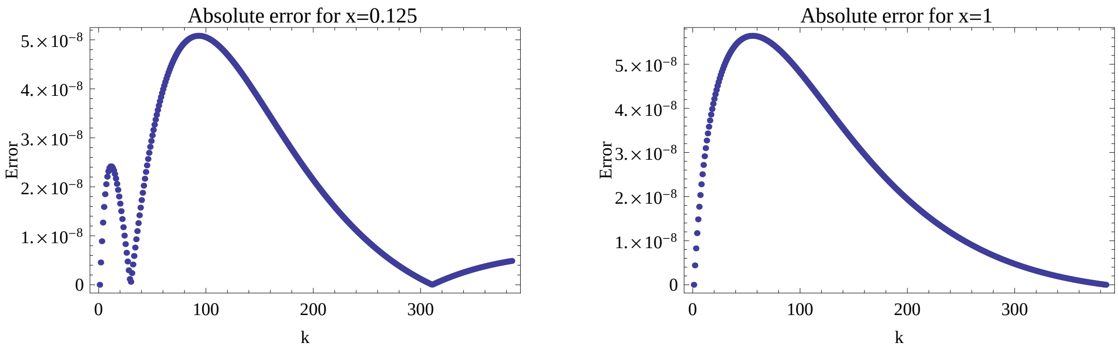

Figure 1: The absolute error between solution obtained by using NFTCS and the exact solution for $x = 0.125$ and $x = 1$ with $h = 1/8$, while the Simpson formula is used for approximating the integrals in boundary conditions.

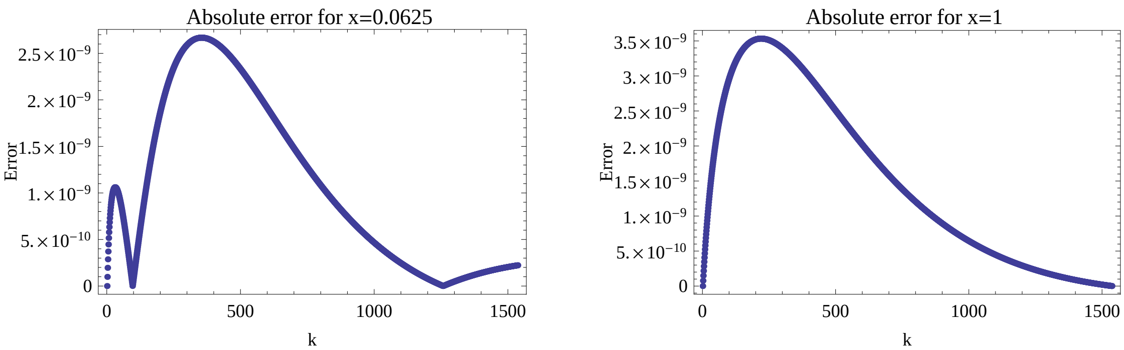

Figure 2: The absolute error between solution obtained by using NFTCS and the exact solution for $x = 0.0625$ and $x = 1$ with $h = 1/16$, while the Simpson formula is used for approximating the integrals in boundary conditions.

The consumed CPU time of three numerical schemes FTCS [31], FTCS4, and FTCS6 is obtained and the graph of CPU time is shown in Figure 3. It is clear from the Figure 3 that with the same step size h, our method consumed less CPU time than other two numerical schemes.

Example 2:

To check the performance of the explicit finite difference scheme described in section 2, we take the

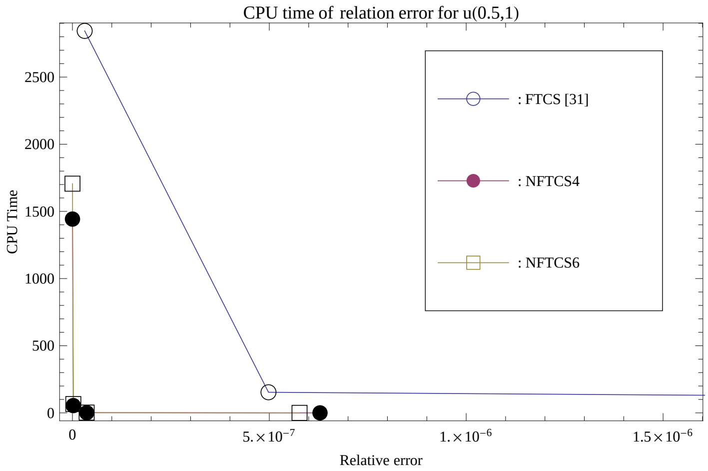

Figure 3: The CPU time spent to find the relative error E_R for $u(0.5, 1)$.

heat equation (1.1) with the initial condition and the boundary conditions as follows [12, 24, 18]:

$$\begin{aligned}
&f(x) = x^2 - x + \frac{\delta}{6(1+\delta)}, && && 0 < x < 1,\\
&\phi(x,t) = -\delta, && 0 < x < 1, && 0 \le t \le 1,\\
&\psi(x,t) = -\delta, && 0 < x < 1, && 0 \le t \le 1,\\
&g_1(t) = 0, && && 0 < x < 1,\\
&g_2(t) = 0, && && 0 < x < 1,\\
&q(x,t) = -\left(x^2 - x + \frac{\delta}{6(1+\delta)} + 2\right) e^{-t}, && 0 < x < 1, && 0 \le t \le 1,
\end{aligned}$$

with $\delta = 0.0144$. It can be verified that the exact solution is

$$u(x,t) = \left(x^2 - x + \frac{\delta}{6(1+\delta)}\right) e^{-t}.$$

We take $m = 0.5166667$, in equation (2.9) and the obtained results are shown in table 5 and table 6. The Simpson composite rule and sixth-order formula are used to approximate the nonlocal boundary conditions (1.3) and (1.4). It can be seen that the errors are very small with different value of x and t. In the last row in each table, we observed that the consumed CPU time increase with the decrease of step size h .

Table 5: Absolute error NFTCS at $t = 1$ by using the Simpson formula.

x/h	0.25	0.125	0.0625	0.03125
0	7.11516×10^{-12}	5.71649×10^{-13}	3.62120×10^{-14}	2.23064×10^{-15}
0.25	1.37993×10^{-9}	8.67588×10^{-11}	5.42373×10^{-12}	3.36300×10^{-13}
0.5	2.54796×10^{-9}	1.58515×10^{-10}	9.90530×10^{-12}	6.22738×10^{-13}
0.75	1.37993×10^{-9}	8.67587×10^{-11}	5.42373×10^{-12}	3.36300×10^{-13}
1	7.11516×10^{-12}	5.71649×10^{-13}	3.62121×10^{-14}	2.23064×10^{-15}
CPU	0.093	1.407	53.578	1670.780

Table 6: Absolute error NFTCS at $t = 1$ by using the Sixth-order formula.

x/h	0.25	0.125	0.0625	0.03125
0	9.18984×10^{-12}	5.79563×10^{-13}	3.62435×10^{-14}	2.23118×10^{-15}
0.25	1.37764×10^{-9}	8.67501×10^{-11}	5.42373×10^{-12}	3.36300×10^{-13}
0.5	2.55032×10^{-9}	1.58524×10^{-10}	9.90531×10^{-12}	6.22738×10^{-13}
0.75	1.37764×10^{-9}	8.67501×10^{-11}	5.42373×10^{-12}	3.36300×10^{-13}
1	9.18984×10^{-12}	5.79563×10^{-13}	3.62435×10^{-14}	2.23117×10^{-15}
CPU	0.079	1.406	52.985	1706.580

In table 7, we present a comparison between the numerical solution of this problem by using new explicit finite difference method and those obtained by the method described in [31]. we observed that the relative error E_R of present method is better than the method described in [31]. Further more the Simpson formula (NFTCS4) and sixth-order formula (NFTCS6) have been used for approximating the integrals in the nonlocal boundary conditions.

Table 7: Relative error E_R for $u(0.5, 1)$ at various spatial length.

h	Method in [31]	NFTCS4	NFTCS6
0.25	3.12655×10^{-5}	2.79690×10^{-8}	2.79949×10^{-8}
0.125	1.95407×10^{-6}	1.74002×10^{-9}	1.74012×10^{-9}
0.0625	1.22130×10^{-7}	1.08731×10^{-10}	1.08731×10^{-10}
0.03125	7.63314×10^{-9}	6.83580×10^{-12}	6.83580×10^{-12}

In Figure 4, we show the absolute error E_A using the NFTCS at $x = 0.125$ and $x = 1$ with step size $h = 1/8$ when the Simpson formula is used for approximating the integrals in the boundary conditions. Similarly, we show the absolute error NFTCS at $x = 0.0625$ and $x = 1$ with step size $h = 1/16$ in Figure 5.

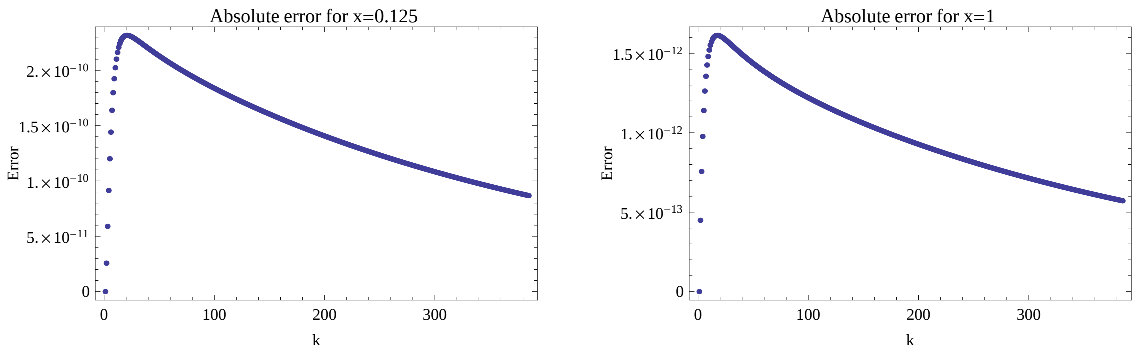

Figure 4: The absolute error between solution obtained by using NFTCS and the exact solution for $x = 0.125$ and $x = 1$ with $h = 1/8$, while the Simpson formula is used for approximating the integrals in boundary conditions.

The consumed CPU time to obtain the numerical solution of the present method with different algorithm to to solve the non local boundary conditions are shown Figure 6. It is clear that for the all small value of step size h, the consumed CPU time in applying the our methods and consumed CPU time by the algorithm proposed in [31] are almost same.

4 Conclusion

Several approaches have been developed for obtaining the numerical solution of heat equation with non-local boundary conditions. A new fourth-order explicit finite difference method has been applied to

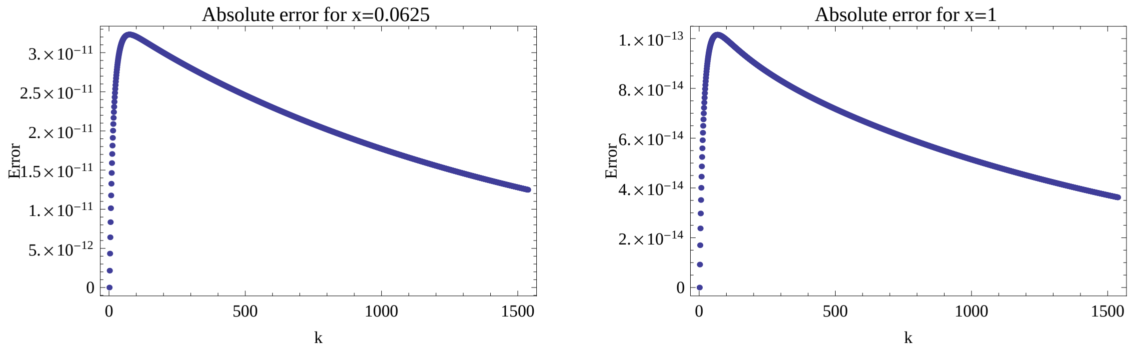

Figure 5: The absolute error between solution obtained by using NFTCS and the exact solution for $x = 0.0625$ and $x = 1$ with $h = 1/16$, while the Simpson formula is used for approximating the integrals in boundary conditions.

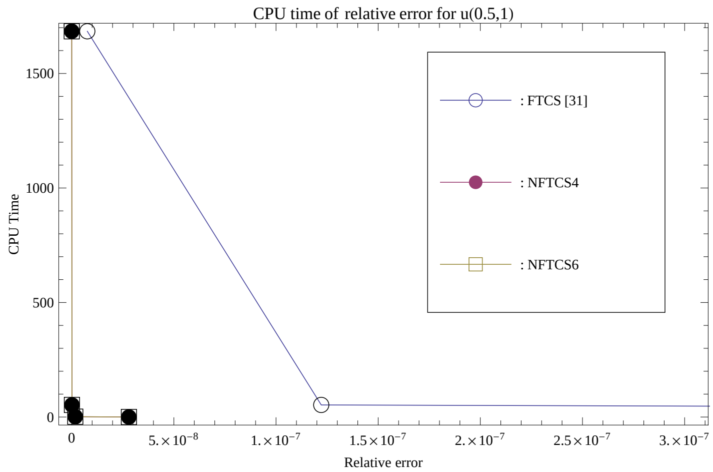

Figure 6: The CPU time spent to find the relation error E_R for $u(0.5, 1)$ in table 7.

obtain numerical solution of one dimensional linear and non-homogeneous parabolic partial differential equation with nonlocal boundary conditions in this paper. The present method is also capable for solving parabolic type partial differential equations with non-local boundary conditions.

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Numerical Solutions of Fourth Order Lidstone Boundary Value Problems Using Discrete Quintic Splines

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Abstract. In this paper, the numerical treatment of a fourth order Lidstone boundary value problem is proposed with the use of a discrete quintic spline based on central differences. It is shown that the method is of order 4 if a parameter takes a specific value, else it is of order 2. A well known numerical example is presented to illustrate our method as well as to compare the performance with other numerical methods proposed in the literature.

Keywords : Discrete quintic spline, central difference, Lidstone boundary value problem, numerical solution, fourth order.

1 Introduction

We consider the fourth order Lidstone boundary value problem

$$\begin{aligned} &y^{(4)}(x) = f(x)y(x) + g(x), \quad a \le x \le b \\ &y(a) = A_1, \quad y(b) = B_1, \quad y''(a) = A_2, \quad y''(b) = B_2 \end{aligned} \tag{1.1}$$

where $f(x)$ and $g(x)$ are continuous on $[a, b]$ and $A_i, B_i, i = 1, 2$ are arbitrary real finite constants.

Lidstone boundary value problems have received a lot of attention in the literature, notably on the existence of positive solutions, see for example [1, 7, 22] and the references cited therein. The fourth order Lidstone boundary value problem (1.1) considered arises from the physical problem of bending a rectangular simply supported beam resting on an elastic foundation [14, 17], here y is the vertical deflection of the plate. The use of polynomial splines in the numerical treatment of (1.1) has gathered substantial interests over the years. Usmani and Warsi [20] have used quintic and sextic splines respectively to develop second and fourth order convergent methods for (1.1). Thereafter, quartic splines

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are employed by Usmani [19] to formulate second order convergent method. Also, during their investigation on fourth order obstacle boundary value problems, Al-Said and Noor [2] and Al-Said et al [3] have respectively used cubic and quartic splines to obtain second order convergent methods for (1.1). Recently, nonpolynomial spline functions have been proposed by Ramadan et al [12] to obtain second and fourth order convergent methods for (1.1), these methods reduced to those of [2, 19, 20] when certain parameters take certain values. A related problem to (1.1) arises from the bending of a long uniformly loaded rectangular plate supported over the entire surface by an elastic foundation and rigidly supported along the edges [14, 17], here the boundary conditions are the conjugate type $y(a) - A_1 = y(b) - B_1 = y'(a) - A_2 = y' - B_2 = 0$. For this problem, second order convergent methods based on quintic splines have been established in [13, 16, 18], while fourth order convergent method based on sextic splines has been discussed in [18]. The general observation from all these research is that spline methods usually give better (or comparable) approximation than finite difference methods and shooting type methods.

Motivated by all the above research especially the use of splines in solving (1.1), in this paper we shall employ a *discrete quintic spline* to get a numerical solution of (1.1). Our proposed method is fourth-order convergent when a parameter takes certain value, else it is second-order convergent. Through a well know numerical example, we illustrate that our method outperforms other spline methods for solving (1.1) in the literature [2, 3, 12, 19, 20].

Discrete splines were first introduced by Mangasarian and Schumaker [11] in 1971 as solutions to constrained minimization problems in real Euclidean space, which are discrete analogs of minimization problems in Banach space whose solutions are generalized splines. Subsequent investigations on discrete splines can be found in the work of Schumaker [15], Astor and Duris [4], Lyche [9, 10] and Wong et al [5, 6, 21]. Following [9, 10], the discrete spline we use will involve central differences.

The plan of the paper is as follows. In section 2, we shall derive our method. The matrix form of the method is presented in section 3 and its convergence analysis is performed. In section 4, we present a well known example and compare the performance of our method with other methods in the literature.

2 Numerical Method for (1.1)

Suppose $P : a = x_0 < x_1 < \cdots < x_n = b$ is a uniform mesh of $[a, b]$ with $x_i - x_{i-1} = p,\ 1 \le i \le n$, i.e., the step size $p = \frac{b-a}{n}$.

Let $h \in (0, p]$ be a given constant. We recall the *central difference operator*

D_h applying to a function $F(x)$ gives

$$D_h^{\{0\}}F(x) = F(x); \qquad D_h^{\{1\}}F(x) = \frac{F(x+h) - F(x-h)}{2h};$$
$$D_h^{\{2\}}F(x) = \frac{F(x+h) - 2F(x) + F(x-h)}{h^2};$$
$$D_h^{\{3\}}F(x) = \frac{F(x+2h) - 2F(x+h) + 2F(x-h) - F(x-2h)}{2h^3};$$
$$D_h^{\{4\}}F(x) = \frac{F(x+2h) - 4F(x+h) + 6F(x) - 4F(x-h) + F(x-2h)}{h^4}.$$

We also use the basic polynomials $x^{\{j\}}$ introduced by [10]

$$x^{\{j\}} = x^j, \quad j = 0, 1, 2; \quad x^{\{3\}} = x(x^2 - h^2),$$
$$x^{\{4\}} = x^2(x^2 - h^2), \quad x^{\{5\}} = x(x^2 - h^2)(x^2 - 4h^2).$$

It is noted that $D_h^{\{1\}}x^{\{j\}} = jx^{\{j-1\}},\ j = 0, 1, 2, 3, 5$ and $D_h^{\{1\}}x^{\{4\}} = 2x(2x^2 + h^2)$.

Definition 1. Let $S(x; h)$ be a piecewise continuous function defined over $[a, b]$ (with mesh P) and $S_i(x)$ be its restriction in $[x_{i-1}, x_i],\ 1 \le i \le n$ passing through the points (x_{i-1}, s_{i-1}) and (x_i, s_i). We say $S(x; h)$ is a *discrete quintic spline* if $S_i(x),\ 1 \le i \le n$ is a polynomial of degree 5 or less and

$$(S_{i+1} - S_i)(x_i + jh) = 0, \quad j = -2, -1, 0, 1, 2, \quad 1 \le i \le n-1. \tag{2.1}$$

The above definition is in the spirit of *discrete cubic spline* studied in [10]. In fact, in terms of *central differences*, the condition (2.1) has the following equivalent form

$$D_h^{\{j\}}S_i(x_i) = D_h^{\{j\}}S_{i+1}(x_i), \quad j = 0, 1, 2, 3, 4, \quad 1 \le i \le n-1. \tag{2.2}$$

Throughout, we shall use the notations

$$y_i^{(k)} = y^{(k)}(x_i), \quad f_i = f(x_i), \quad g_i = g(x_i), \quad s_i = S_i(x_i),$$
$$M_i = D_h^{\{2\}}S_i(x_i), \quad F_i = D_h^{\{4\}}S_i(x_i), \qquad 0 \le i \le n.$$

We propose s_i's to be the numerical solution of (1.1) at the mesh points, i.e.,

$$y_i \cong s_i, \quad 0 \le i \le n. \tag{2.3}$$

Discretizing (1.1) and noting the Lidstone boundary conditions, we set

$$\begin{gathered} s_0 = y_0 = A_1, \quad s_n = y_n = B_1, \quad M_0 = y_0'' = A_2, \\ M_n = y_n'' = B_2, \quad F_i = f_i s_i + g_i, \quad 0 \le i \le n. \end{gathered} \tag{2.4}$$

We shall now obtain an explicit expression of $S_i(x)$ in terms of its central differences. To begin, let the functions $h_i(x)$, $\bar{h}_i(x)$ and $\bar{\bar{h}}_i(x)$ satisfy the following for $0 \le i, j \le n$:

$$\begin{aligned}
h_i(x_j) = \delta_{ij}, &\quad D_h^{\{2\}} h_i(x_j) = D_h^{\{4\}} h_i(x_j) = 0, \\
D_h^{\{2\}} \bar{h}_i(x_j) = \delta_{ij}, &\quad \bar{h}_i(x_j) = D_h^{\{4\}} \bar{h}_i(x_j) = 0, \\
D_h^{\{4\}} \bar{\bar{h}}_i(x_j) = \delta_{ij}, &\quad \bar{\bar{h}}_i(x_j) = D_h^{\{2\}} \bar{\bar{h}}_i(x_j) = 0.
\end{aligned}$$

By direct computation, we obtain the explicit expressions:

$$\begin{aligned}
h_i(x) &= \frac{x - x_{i-1}}{p}, \quad x \in [x_{i-1}, x_i], \quad 1 \le i \le n \\
&= \frac{x_{i+1} - x}{p}, \quad x \in [x_i, x_{i+1}], \quad 0 \le i \le n-1 \\
&= 0, \qquad \text{otherwise}; \\
\bar{h}_i(x) &= \frac{(x - x_{i-1})^{\{3\}}}{6p} - \frac{(p^2 - h^2)(x - x_{i-1})}{6p}, \\
&\qquad x \in [x_{i-1}, x_i], \quad 1 \le i \le n \\
&= \frac{(x_{i+1} - x)^{\{3\}}}{6p} - \frac{(p^2 - h^2)(x_{i+1} - x)}{6p}, \\
&\qquad x \in [x_i, x_{i+1}], \quad 0 \le i \le n-1 \\
&= 0, \qquad \text{otherwise}; \\
\bar{\bar{h}}_i(x) &= \frac{(x - x_{i-1})^{\{5\}}}{120p} - \frac{(p^2 - h^2)(x - x_{i-1})^{\{3\}}}{36p} \\
&\quad + \frac{(x - x_{i-1})(p^2 - h^2)(7p^2 + 2h^2)}{360p}, \\
&\qquad x \in [x_{i-1}, x_i], \quad 1 \le i \le n \\
&= \frac{(x_{i+1} - x)^{\{5\}}}{120p} - \frac{(p^2 - h^2)(x_{i+1} - x)^{\{3\}}}{36p} \\
&\quad + \frac{(x_{i+1} - x)(p^2 - h^2)(7p^2 + 2h^2)}{360p}, \\
&\qquad x \in [x_i, x_{i+1}], \quad 0 \le i \le n-1 \\
&= 0, \qquad \text{otherwise}.
\end{aligned}$$

Clearly, $S_i(x)$, the restriction of $S(x; h)$ in $[x_{i-1}, x_i]$, can be expressed as

$$\begin{aligned}
S_i(x) = s_{i-1} h_{i-1}(x) + s_i h_i(x) + M_{i-1} \bar{h}_{i-1}(x) + M_i \bar{h}_i(x) + F_{i-1} \bar{\bar{h}}_{i-1}(x) \\
+ F_i \bar{\bar{h}}_i(x), \qquad x \in [x_{i-1}, x_i], \ 1 \le i \le n.
\end{aligned} \tag{2.5}$$

Using (2.5), the 'continuity' requirement $D_h^{\{1\}} S_i(x_i) = D_h^{\{1\}} S_{i+1}(x_i),\ 1 \le i \le n-1$ leads to the equation

$$\begin{aligned}&(p^2-h^2)M_{i-1} + 2(h^2+2p^2)M_i + (p^2-h^2)M_{i+1}\\ &\quad = 6(s_{i-1}-2s_i+s_{i+1}) + \frac{(p^2-h^2)}{60}\left[(2h^2+7p^2)F_{i-1} + 4(4p^2-h^2)F_i\right.\\ &\qquad \left. +(2h^2+7p^2)F_{i+1}\right].\end{aligned} \tag{2.6}$$

Further, the 'continuity' requirement $D_h^{\{3\}} S_i(x_i) = D_h^{\{3\}} S_{i+1}(x_i),\ 1 \le i \le n-1$ yields

$$M_{i-1} - 2M_i + M_{i+1} = \frac{1}{6}\left[(p^2-h^2)F_{i-1} + 2(h^2+2p^2)F_i + (p^2-h^2)F_{i+1}\right]. \tag{2.7}$$

Using (2.6) and (2.7) in a lengthy algebraic procedure, we are able to eliminate M's and get the 'F-equation' as

$$\begin{aligned}&a_1F_{i-2} + a_2F_{i-1} + a_3F_i + a_2F_{i+1} + a_1F_{i+2}\\ &\quad = s_{i-2} - 4s_{i-1} + 6s_i - 4s_{i+1} + s_{i+2}, \qquad 2 \le i \le n-2\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}a_1 &= \frac{(p^2-h^2)(p^2-4h^2)}{120}, \quad a_2 = \frac{2(p^2-h^2)(8h^2+13p^2)}{120},\\ a_3 &= \frac{6(4h^4+5h^2p^2+11p^4)}{120}.\end{aligned} \tag{2.9}$$

Upon substituting $F_j = f_j s_j + g_j$ into (2.8), we see that (2.8) gives $(n-3)$ equations with $(n-1)$ unknowns $s_i,\ 1 \le i \le n-1$.

In order to solve for the unknown s_i's, we need two more equations which we write as

$$b_1F_0 + b_2F_1 + b_3F_2 + b_4F_3 = p^2M_0 + b_5s_0 + b_6s_1 + b_7s_2 + b_8s_3 \tag{2.10}$$

and

$$c_1F_{n-3} + c_2F_{n-2} + c_3F_{n-1} + c_4F_n = p^2M_n + c_5s_{n-3} + c_6s_{n-2} + c_7s_{n-1} + c_8s_n \tag{2.11}$$

where b_i and $c_i,\ 1 \le i \le 8$ are real numbers. We require the local truncation errors in both (2.10) and (2.11) to be $O(p^8)$ (the reason will be clear when we perform the convergence analysis in section 3). To fulfill this, we carry out Taylor series expansion in (2.10) about x_0 and set the coefficients of $s_0^{(k)},\ 0 \le k \le 7$ to zeros. This yields 8 equations which we can solve to get $b_i,\ 1 \le i \le 8$. Similarly, in (2.11) we expand about x_n and set the coefficients of $s_n^{(k)},\ 0 \le k \le 7$ to zeros, then we solve 8 equations to get $c_i,\ 1 \le i \le 8$. The resulting (2.10) and (2.11) are given as follows

$$\frac{p^4}{360}(28F_0 + 245F_1 + 56F_2 + F_3) - p^2M_0 = -2s_0 + 5s_1 - 4s_2 + s_3, \tag{2.12}$$

$$\frac{p^4}{360}(F_{n-3}+56F_{n-2}+245F_{n-1}+28F_n)-p^2M_n = s_{n-3}-4s_{n-2}+5s_{n-1}-2s_n. \tag{2.13}$$

Once again, we substitute $F_j = f_j s_j + g_j$ into (2.12) and (2.13) to give two equations in $s_i,\ i = 1, 2, 3, n-3, n-2, n-1$.

Noting (2.4) the values of $s_0,\ s_n,\ M_0$ and M_n are already known, hence we can now solve (2.8), (2.12), (2.13) to obtain the values of $s_i,\ 1 \le i \le n-1$. The solvability of the linear system will be discussed in section 3.

3 Convergence Analysis

In this section, we shall establish the existence of a unique solution for (2.8), (2.12), (2.13) and also conduct a convergence analysis for the method presented in section 2. To begin, we define the norms of a column vector $T = [t_i]$ and a matrix $A = [a_{ij}]$ as follows:

$$\|T\| = \max_i |t_i| \qquad \text{and} \qquad \|A\| = \max_i \sum_j |a_{ij}|.$$

Let $e_i = y_i - s_i,\ 1 \le i \le n-1$ be the errors. Let $Y = [y_i],\ S = [s_i],\ W = [w_i],\ T = [t_i]$ and $E = [e_i]$ be $(n-1)$-dimensional column vectors. The system (2.8), (2.12), (2.13) can be written as

$$AS = W \tag{3.1}$$

where

$$A = A_0 + Q, \qquad Q = BF, \qquad F = \mathrm{diag}(f_i),\ i = 1, 2, \ldots, n-1, \tag{3.2}$$

A_0 and B are $(n-1)\times(n-1)$ five-band symmetric matrices given by

$$A_0 = \begin{pmatrix} 5 & -4 & 1 & & & & & \\ -4 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & & & \ddots & & & & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{pmatrix}, \tag{3.3}$$

$$B = \begin{pmatrix} -\frac{245p^4}{360} & -\frac{56p^4}{360} & -\frac{p^4}{360} & & & & \\ -a_2 & -a_3 & -a_2 & -a_1 & & & \\ -a_1 & -a_2 & -a_3 & -a_2 & -a_1 & & \\ & & & \ddots & & & \\ & & -a_1 & -a_2 & -a_3 & -a_2 & -a_1 \\ & & & -a_1 & -a_2 & -a_3 & -a_2 \\ & & & & -\frac{p^4}{360} & -\frac{56p^4}{360} & -\frac{245p^4}{360} \end{pmatrix} \tag{3.4}$$

and for the vector $W = [w_i]$, we have

$$w_i = \begin{cases} 2s_0 - p^2M_0 + \dfrac{p^4}{360}(28f_0s_0 + 28g_0 + 245g_1 + 56g_2 + g_3), & i = 1 \\ -s_0 + a_1f_0s_0 + a_1g_0 + a_2g_1 + a_3g_2 + a_2g_3 + a_1g_4, & i = 2 \\ a_1g_{i-2} + a_2g_{i-1} + a_3g_i + a_2g_{i+1} + a_1g_{i+2}, & 3 \le i \le n-3 \\ -s_n + a_1g_{n-4} + a_2g_{n-3} + a_3g_{n-2} + a_2g_{n-1} + a_1g_n + a_1f_ns_n, & i = n-2 \\ 2s_n - p^2M_n + \dfrac{p^4}{360}(g_{n-3} + 56g_{n-2} + 245g_{n-1} + 28g_n + 28f_ns_n), & i = n-1. \end{cases} \tag{3.5}$$

From (3.1) we have $A(Y - E) = W$ or

$$AY = W + T \tag{3.6}$$

where

$$T = AE. \tag{3.7}$$

For $2 \le i \le n-2$, the i-th equation of the linear system (3.7) is

$$y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} = a_1y_{i-2}^{(4)} + a_2y_{i-1}^{(4)} + a_3y_i^{(4)} + a_2y_{i+1}^{(4)} + a_1y_{i+2}^{(4)} + t_i$$

where t_i's are the local truncation errors given by

$$t_i = \frac{p^4(p^2 - 3h^2)}{12}y_i^{(6)} + \frac{p^4(4p^4 - 15p^2h^2 + 8h^4)}{240}y_i^{(8)} + O(p^9). \tag{3.8}$$

For $i = 1, n-1$, the i-th equations of the linear system (3.7) are respectively

$$-2y_0 + 5y_1 - 4y_2 + y_3 = \frac{p^4}{360}\left(28y_0^{(4)} + 245y_1^{(4)} + 56y_2^{(4)} + y_3^{(4)}\right) - p^2y_0'' + t_1$$

and

$$\begin{aligned} &y_{n-3} - 4y_{n-2} + 5y_{n-1} - 2y_n \\ &\quad = \frac{p^4}{360}\left(y_{n-3}^{(4)} + 56y_{n-2}^{(4)} + 245y_{n-1}^{(4)} + 28y_n^{(4)}\right) - p^2y_n'' + t_{n-1} \end{aligned}$$

where t_1 and t_{n-1} are the local truncation errors given by

$$t_1 = t_{n-1} = -\frac{241}{60480}p^8 y_i^{(8)} + O(p^9). \tag{3.9}$$

Remark 1. For the special case $h = \frac{p}{\sqrt{3}}$, it is clear from (3.8) that

$$t_i = -\frac{p^8}{2160}y_i^{(8)} + O(p^9), \quad 2 \le i \le n-2.$$

Thus, taking (3.9) into consideration, we have

$$\|T\| = \frac{241}{60480}p^8 L \tag{3.10}$$

where $L = \max_x |y^{(8)}(x)|$.

Lemma 1. [2] The matrix A_0 is invertible and

$$\|A_0^{-1}\| \le \frac{5n^4 + 4n^2}{384} = \frac{5(b-a)^4 + 4(b-a)^2 p^2}{384 p^4}. \tag{3.11}$$

Lemma 2. [8] Let D be a square matrix such that $\|D\| < 1$. Then, $(I + D)$ is nonsingular and

$$\|(I + D)^{-1}\| \le \frac{1}{1 - \|D\|}.$$

Our first result guarantees the existence of a unique solution for (2.8), (2.12), (2.13).

Theorem 1. The system (3.1) has a unique solution if

$$\frac{489}{480}K\hat{f} < 1 \tag{3.12}$$

where $K = \frac{5(b-a)^4+4(b-a)^2p^2}{384}$ and $\hat{f} = \max_{1\le i\le n-1} |f_i|$.

Proof. If (3.1) has a unique solution, then it can be written as

$$S = A^{-1}W = (A_0 + Q)^{-1}W = [A_0(I + A_0^{-1}Q)]^{-1}W = (I + A_0^{-1}Q)^{-1}A_0^{-1}W. \tag{3.13}$$

From Lemma 1 the inverse A_0^{-1} exists, hence it remains to show that $(I+A_0^{-1}Q)$ is nonsingular.

From (3.4), a direct computation gives $\|B\| \le \frac{489}{480}p^4$. Since $Q = BF$, we find

$$\|Q\| \le \|B\| \; \|F\| \le \frac{489}{480}p^4 \hat{f}. \tag{3.14}$$

It follows from (3.11) and (3.14) that

$$\|A_0^{-1}Q\| \le \|A_0^{-1}\|\ \|Q\| \le \frac{5(b-a)^4 + 4(b-a)^2p^2}{384p^4}\left(\frac{489}{480}p^4\hat{f}\right) = \frac{489}{480}K\hat{f} < 1 \tag{3.15}$$

where we have used (3.12) in the last inequality. Since $\|A_0^{-1}Q\| < 1$, we conclude from Lemma 2 that $(I + A_0^{-1}Q)$ is nonsingular. Hence, (3.1) has a unique solution given by (3.13). □

The next result gives the order of convergence of our method.

Theorem 2. Suppose $\frac{489}{480}K\hat{f} < 1$. Then,

$$\|E\| \cong O(p^4) \quad \text{if} \quad h = \frac{p}{\sqrt{3}}$$

and $\|E\| \cong O(p^2)$ for other values of $h \in (0, p]$, i.e., the method (3.1) is fourth order convergent if $h = \frac{p}{\sqrt{3}}$ and is second order convergent otherwise.

Proof. First, we consider the special case when $h = \frac{p}{\sqrt{3}}$. From (3.7) we have

$$E = A^{-1}T = (A_0 + Q)^{-1}T = (I + A_0^{-1}Q)^{-1}A_0^{-1}T$$

Noting (3.15) we apply Lemma 2, and together with (3.11) and (3.10), we find

$$\begin{aligned}
\|E\| &\le \|(I + A_0^{-1}Q)^{-1}\|\ \|A_0^{-1}\|\ \|T\| \\
&\le \frac{\|A_0^{-1}\|\ \|T\|}{1 - \|A_0^{-1}Q\|} \\
&\le \frac{5(b-a)^4 + 4(b-a)^2p^2}{384p^4}\left(\frac{241}{60480}p^8L\right)\left(\frac{1}{1 - \frac{489}{480}K\hat{f}}\right) \\
&= \frac{241KLp^4}{60480\left(1 - \frac{489}{480}K\hat{f}\right)} \cong O(p^4).
\end{aligned}$$

This inequality shows that (3.1) is a fourth order convergence method when $h = \frac{p}{\sqrt{3}}$.

For other values of $h \in (0, p]$, from (3.8) and (3.9) we have $\|T\| \cong O(p^6)$. Using a similar argument as above, we see that (3.1) is second order convergent. □

4 Numerical Example

In this section, we present a numerical example to demonstrate our proposed method as well as to illustrate the comparative performance with some well known numerical methods for solving (1.1).

Consider the Lidstone boundary value problem

$$\begin{aligned} &y^{(4)} + xy = -(8+7x+x^3)e^x, \quad 0 \le x \le 1 \\ &y(0) = y(1) = 0, \quad y''(0) = 0, \quad y''(1) = -4e. \end{aligned} \tag{4.1}$$

The analytical solution of (4.1) is

$$y(x) = x(1-x)e^x.$$

In this example, we have $a = 0,\ b = 1,\ f(x) = -x$ and $g(x) = -(8+7x+x^3)e^x$. So $K = \frac{5+4p^2}{384}$ and $\hat{f} < 1$. For any $p \in (0,1)$, we have $\frac{489}{480}K\hat{f} < 1$ and hence it follows from Theorem 1 that our method gives a unique numerical solution for (4.1).

To compute the numerical solution of (4.1), first we fix the mesh P (and hence the step size p) and choose $h = \frac{p}{\sqrt{3}}$. Then, we solve the system (2.8), (2.12), (2.13) to get s_i, $1 \le i \le n-1$, which approximates y_i.

The maximum absolute errors ($\max_i |y_i - s_i|$) obtained by our method as well as by other methods in the literature are presented in Table 1. From the table we can see that our method is fourth-order convergent when $h = \frac{p}{\sqrt{3}}$. Moreover, a clear comparison shows that our method *outperforms* continuous polynomial spline (cubic, quartic, quintic, sextic) and nonpolynomial spline (quintic) methods.

Table 1: Maximum absolute errors $\max_i |y_i - s_i|$

Methods	$p = 1/8$	$p = 1/16$	$p = 1/32$
Our method	$7.48e-08$	$5.30e-09$	$4.91e-10$
Quintic nonpolynomial spline (4th order)	$2.09e-07$	$7.92e-09$	$1.27e-09$
Sextic spline [20]	$1.26e-06$	$7.87e-08$	$4.91e-09$
Quintic nonpolynomial spline (2nd order) [12]	$9.42e-05$	$6.17e-06$	$3.95e-07$
Quartic spline [19]	$4.24e-04$	$1.08e-04$	$2.70e-05$
Cubic spline [3]	$5.69e-04$	$1.47e-04$	$3.71e-05$
Quintic spline [20]	$8.67e-04$	$2.16e-04$	$5.40e-05$
Quartic spline [2]	$1.62e-03$	$6.39e-04$	$5.88e-05$

A brief description of the methods listed in Table 1:

(i) In [12], second and fourth order convergent methods are derived using a nonpolynomial spline function that has a polynomial part and a trigonometric part. The methods of [2, 19, 20] are special cases of nonpolynomial spline methods when certain parameters take certain values.

(ii) In [20], quintic and sextic splines are employed respectively to establish second and fourth order convergent methods.

(iii) In [19], a second order convergent method is formulated using quartic splines. Here, the consistency relations are obtained at the *midknots*, this approach is different from other spline methods where consistency relations are usually obtained at the uniformly spaced knots.

(iv) In [3], cubic splines are used to develop a second order convergent method.

(v) In [2], a second order convergent method is proposed based on quartic splines.

5 Conclusion

We have developed a numerical method for fourth order Lidstone boundary value problems using discrete quintic splines. The method is shown to be fourth order convergent when the parameter $h = \frac{p}{\sqrt{3}}$, and second order convergent for other values of $h \in (0, p]$. A well known numerical example is presented to demonstrate the outperformance of our method over other continuous spline methods in the literature.

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Periodicity and Global Attractivity of Difference Equation of Higher Order

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In this paper we investigate the dynamics and behavior of the recursive sequence

$$x_{n+1} = ax_{n-k} + \frac{bx_{n-l} + cx_{n-s} + dx_{n-r}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}}, \quad n = 0, 1, ...$$

where the parameters a ; b; c; d;α ; β and γ are positive real numbers and the initial conditions are positive real numbers.

1 Introduction

Recently there has been a lot of interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations see for example [[1]-[14]].

The study of the nonlinear rational difference equations of a higher order is quite challenging and rewarding, and the results about these equations offer prototypes towards the development of the basic theory of the global behavior of nonlinear difference equations of a big order, recently, many researchers have investigated the behavior of the solution of difference equations for example: In [4] Elabbasy et al. investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$$

Yalnkaya et al. [15], [16] considered the dynamics of the difference equations

$$x_{n+1} = \frac{ax_{n-k}}{b + cx_n^p}, \qquad x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$$

For some related work see [[15]-[19]].
Our goal in this paper is to investigate the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_{n-k} + \frac{bx_{n-l} + cx_{n-s} + dx_{n-r}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}}, \quad n = 0, 1, ... \tag{1}$$

where the parameters a ; b; c; d;α ; β and γ are positive real numbers and the initial conditions $x_{-t}, x_{-t+1}, ..., x_{-1}$ and x_0 are positive real numbers where $t = \max\{k, l, s, r\}$.
Here, we recall some basic definitions and some theorems that we need in the sequel.
Let I be some interval of real numbers and let

$$F : I^{k+1} \longrightarrow I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), n = 0, 1, ... \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$

Theorem 1.1. *[9] Assume that* $p_i \in R$,$i = 1, 2, ..., k$ *and* $k \in \{0, 1, 2, ...\}$ *.Then* $\sum_{i=1}^{k} |p_i| \leq 1$ *, is a sufficient condition for the asymptotic stability of the difference equation*

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0 \qquad , n = 0, 1, ...$$

Theorem 1.2. *[10] Let* $g : [a, b]^{k+1} \to [a, b]$ *be a continuous function, where* k *is a positive integer, and where* $[a, b]$ *is an interval of real numbers. Consider the difference equation*

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), n = 0, 1, ... \tag{3}$$

Suppose that g *satisfies the following conditions:*

1. *For each integer* i *with* $1 \leq i \leq k + 1$ *,the function* $g(z_1, z_2, ..., z_{k+1})$ *is weakly monotonic in* z_i *for fixed* $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.
2. *If If* (m, M) *is a solution of the system*

$$m = g(m_1, m_2, ..., m_{k+1}) \quad , \text{and} \quad M = g(M_1, M_2, ..., M_{k+1})$$

then $m = M$*, where for each* $i = 1, 2, ..., k + 1$*, we set*

$$m_i = \begin{cases} m & \text{if } g \text{ is non-decreasing in } z_i \\ M & \text{if } g \text{ is non-increasing in } z_i \end{cases}$$

$$M_i = \begin{cases} M & \text{if } g \text{ is non-decreasing in } z_i \\ m & \text{if } g \text{ is non-increasing in } z_i \end{cases}$$

Then there exists exactly one equilibrium $\overline{x}$ *of equation* (3)*, and every solution of equation* (3) *converges to* $\overline{x}$*.*

2 Local Stability of the Equilibrium Point of equation (1)

This section deals with study the local stability character of the equilibrium point of Equation (1) .Equation (1) has equilibrium point and is given by $\overline{x} = a\overline{x} + \frac{b+c+d}{\alpha+\beta+\gamma}$. If $a \leq 1$, then the only positive equilibrium point of equation (1) is given by $\overline{x} = \frac{b+c+d}{(\alpha+\beta+\gamma)(1-a)}$.
$f : (0,\infty)^4 \to (0,\infty)$ be a continuously differentiable function defined by

$$f(u,v,w,t) = au + \frac{bv + cw + dt}{\alpha v + \beta w + \gamma t} \tag{4}$$

Therefore it follows that

$$\frac{\partial f(u,v,w,t)}{\partial u} = a, \quad \frac{\partial f(u,v,w,t)}{\partial v} = \frac{(b\beta - c\alpha)w + (b\gamma - d\alpha)t}{(\alpha v + \beta w + \gamma t)^2}$$

$$\frac{\partial f(u,v,w,t)}{\partial w} = \frac{-(b\beta - c\alpha)v + (c\gamma - d\beta)t}{(\alpha v + \beta w + \gamma t)^2}$$

$$\frac{\partial f(u,v,w,t)}{\partial t} = \frac{-(b\gamma - d\alpha)v - (c\gamma - d\beta)w}{(\alpha v + \beta w + \gamma t)^2}$$

Then we see that

$$\frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial u} = a = -a_3$$

$$\frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial v} = \frac{(b\beta - c\alpha) + (b\gamma - d\alpha)}{(\alpha+\beta+\gamma)^2\overline{x}} = \frac{(b\beta - c\alpha) + (b\gamma - d\alpha)}{(\alpha+\beta+\gamma)^2 \frac{b+c+d}{(\alpha+\beta+\gamma)(1-a)}}$$

$$= \frac{[(b\beta - c\alpha) + (b\gamma - d\alpha)](1-a)}{(\alpha+\beta+\gamma)(b+c+d)} = -a_2$$

$$\frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial w} = \frac{[-(b\beta - c\alpha) + (c\gamma - d\beta)](1-a)}{(\alpha+\beta+\gamma)(b+c+d)} = -a_1$$

$$\frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial t} = \frac{[-(b\gamma - d\alpha) - (c\gamma - d\beta)](1-a)}{(\alpha+\beta+\gamma)(b+c+d)} = -a_0$$

Then the linearized equation of Equation (1) about $\overline{x}$ is

$$y_{n+1} + a_3 y_n + a_2 y_{n-1} + a_1 y_{n-2} + a_0 y_{n-3} = 0 \tag{5}$$

whose characteristic equation is

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \tag{6}$$

Theorem 2.1. *Assume that*

$$(\alpha+\beta+\gamma)(b+c+d) > \max\left\{|2\alpha(c+d)-2b(\beta+\gamma)|, |2\gamma(b+c)-2d(\alpha+\beta)|\right.$$
$$\left., |2\beta(b+d)-2c(\alpha+\gamma)|\right\} \quad (7)$$

Then the positive equilibrium point of Equation (1) *is locally asymptotically stable.*

Proof. It is follows by theorem (1.1) that equation (5) is asymptotically stable if all roots of equation (6) lie in the open disc, $|\lambda| < 1$ that is if

$$|a_3| + |a_2| + |a_1| + |a_0| < 1$$

$$|a| + \left|\frac{[(b\beta - c\alpha) + (b\gamma - d\alpha)](1-a)}{(\alpha+\beta+\gamma)(b+c+d)}\right|$$
$$+\left|\frac{[-(b\beta - c\alpha) + (c\gamma - d\beta)](1-a)}{(\alpha+\beta+\gamma)(b+c+d)}\right| + \left|\frac{[-(b\gamma - d\alpha) - (c\gamma - d\beta)](1-a)}{(\alpha+\beta+\gamma)(b+c+d)}\right| < 1$$

and so (after dividing the denominator and numerator by $(1-a)$ gives)

$$|(b\beta - c\alpha) + (b\gamma - d\alpha)| + |-(b\beta - c\alpha) + (c\gamma - d\beta)|$$
$$+|-(b\gamma - d\alpha) - (c\gamma - d\beta)| < (\alpha+\beta+\gamma)(b+c+d) \quad (8)$$

Suppose that
$B_1 = (b\beta - c\alpha) + (b\gamma - d\alpha)$, $\quad B_2 = -(b\beta - c\alpha) + (c\gamma - d\beta)$,
$B_3 = -(b\gamma - d\alpha) - (c\gamma - d\beta)$
We consider the following cases

1. $B_1 > 0$, $B_2 > 0$, and $B_3 > 0$. In this case we see from equation (8) that

$$(b\beta - c\alpha) + (b\gamma - d\alpha) - (b\beta - c\alpha) + (c\gamma - d\beta) - (b\gamma - d\alpha) - (c\gamma - d\beta)$$
$$< (\alpha+\beta+\gamma)(b+c+d)$$

   if and only if $(\alpha+\beta+\gamma)(b+c+d) > 0$ which is always true.

2. $B_1 > 0$, $B_2 > 0$, and $B_3 < 0$. It follows from equation (8) that

$$(b\beta - c\alpha) + (b\gamma - d\alpha) - (b\beta - c\alpha) + (c\gamma - d\beta) + (b\gamma - d\alpha) + (c\gamma - d\beta)$$
$$< (\alpha+\beta+\gamma)(b+c+d)$$

   if and only if $(\alpha+\beta+\gamma)(b+c+d) > 2\gamma(b+c) - 2d(\alpha+\beta)$ which is satisfied by Condition (7).

Also, we can prove the other cases. The proof is complete. □

3 Boundedness of Solutions of Equation (1)

Here we study the boundedness nature and persistence of solutions of Equation (1).

Theorem 3.1. *Every solution of Equation* (1) *is bounded and persists if* $a < 1$.

Proof. Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$x_{n+1} = ax_{n-k} + \frac{bx_{n-l} + cx_{n-s} + dx_{n-r}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}}$$

$$= ax_{n-k} + \frac{bx_{n-l}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}} + \frac{cx_{n-s}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}} + \frac{dx_{n-r}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}}$$

Then

$$x_{n+1} \leq ax_{n-k} + \frac{bx_{n-l}}{\alpha x_{n-l}} + \frac{cx_{n-s}}{\beta x_{n-s}} + \frac{dx_{n-r}}{\gamma x_{n-r}} = x_{n-k} + \frac{b}{\alpha} + \frac{c}{\beta} + \frac{d}{\gamma} \quad \text{for all} \quad n \geq 1$$

By using a comparison, we see that

$$\lim_{n\to\infty} \sup x_n \leq \frac{b\beta\gamma + c\alpha\gamma + d\alpha\beta}{\alpha\beta\gamma(1-a)} = M \tag{9}$$

Thus the solution is bounded. Now we wish to show that there exists $m > 0$ such that $x_n \geq m$ for all $n \geq 1$. The transformation $x_n = \frac{1}{y_n}$ will reduce Equation (1) to the equivalent form

$$y_{n+1} = \frac{y_{n-k}(\alpha y_{n-s}y_{n-r} + \beta y_{n-l}y_{n-r} + \gamma y_{n-l}y_{n-s})}{a(\alpha y_{n-s}y_{n-r} + \beta y_{n-l}y_{n-r} + \gamma y_{n-l}y_{n-s}) + y_{n-k}(by_{n-s}y_{n-r} + cy_{n-l}y_{n-r} + dy_{n-l}y_{n-s})}$$

It follows that

$$y_{n+1} \leq \frac{y_{n-k}(\alpha y_{n-s}y_{n-r} + \beta y_{n-l}y_{n-r} + \gamma y_{n-l}y_{n-s})}{y_{n-k}(by_{n-s}y_{n-r} + cy_{n-l}y_{n-r} + dy_{n-l}y_{n-s})}$$

$$\leq \frac{\alpha y_{n-s}y_{n-r}}{by_{n-s}y_{n-r}} + \frac{\beta y_{n-l}y_{n-r}}{cy_{n-l}y_{n-r}} + \frac{\gamma y_{n-l}y_{n-s}}{dy_{n-l}y_{n-s}} = \frac{\alpha}{b} + \frac{\beta}{c} + \frac{\gamma}{d}$$

$$= \frac{\alpha cd + \beta bd + \gamma bc}{bcd} = H \quad \text{for all} \quad n \geq 1$$

Thus we obtain

$$x_n = \frac{1}{y_n} \geq \frac{1}{H} = \frac{bcd}{\alpha cd + \beta bd + \gamma bc} = m \quad \text{for all} \quad n \geq 1 \tag{10}$$

From Equations (9) and (10) we see that $m \leq x_n \leq M$ for all $n \geq 1$. Therefore every solution of Equation (1) is bounded and persists. □

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Theorem 3.2. *Every solution of Equation* (1) *is unbounded if* $a > 1$.

Proof. Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of Equation (1). Then from Equation (1) we see that

$$x_{n+1} = ax_{n-k} + \frac{bx_{n-l} + cx_{n-s} + dx_{n-r}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}} > ax_{n-k} \quad \text{for all} \quad n \geq 1$$

We see that the right hand side can write as follows

$$y_{n+1} = ay_{n-k} \Rightarrow y_{kn+i} = a^n y_{k+i}, \qquad i = 0, 1, ..., k,$$

and this equation is unstable because $a > 1$, and $\lim\limits_{n\to\infty} y_n = \infty$ Then by using ratio test $\{x_n\}_{n=-t}^{\infty}$ is unbounded from above. □

4 Existence of Periodic Solutions

In this section we study the existence of periodic solutions of equation (1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two and there is clear that there exists a sixteen cases of the indexes s, l, k, r as we see in the following theorem and lemmas.

Theorem 4.1. *Equation* (1) *has positive prime period two solutions if and only if one of the following statements holds*

1. $(b+d-c)(\alpha+\gamma-\beta)(1+a)+4(a\beta(b+d)+c(\alpha+\gamma)) > 0$, $\alpha+\gamma > \beta$, $b+d > c$ *and* l, r*-odd,* k, s*-even.*
2. $(c + d - b)(\beta + \gamma - \alpha)(1 + a) + 4(a\alpha(c + d) + b(\beta + \gamma)) > 0$, $\beta + \gamma > \alpha$, $c + d > b$ *and* k, r*-odd,* l, s*-even.*
3. $(b + c - d)(\alpha + \beta - \gamma)(1 + a) + 4(a\gamma(b + c) + d(\alpha + \beta)) > 0$, $\alpha + \beta > \gamma$, $b + c > d$ *and* k, l*-odd,* r, s*-even.*
4. $(b - c - d)(\alpha - \beta - \gamma)(1 + a) + 4(ab(\beta + \gamma) + \alpha(c + d)) > 0$, $\alpha > \beta + \gamma$, $b > c + d$ *and* l*-odd,* k, s, r*-even.*
5. $(c - b - d)(\beta - \alpha - \gamma)(1 + a) + 4(ac(\alpha + \gamma) + \beta(b + d)) > 0$, $\beta > \alpha + \gamma$, $c > b + d$ *and* k*-odd,* l, s, r*-even.*
6. $(d - b - c)(\gamma - \alpha - \beta)(1 + a) + 4(ad(\beta + \alpha) + \gamma(b + c)) > 0$, $\gamma > \alpha + \beta$, $d > b + c$ *and* r*-odd,* l, k, s*-even.*
7. $(c + d - b)(\alpha - \beta - \gamma) - 4b(\beta + \gamma) > 0$, $a < 1$, $\alpha > \beta + \gamma$, $c + d > b$ *and* k, s, r*-odd,* l*-even*
8. $(b + d - c)(\beta - \alpha - \gamma) - 4c(\alpha + \gamma) > 0$, $a < 1$, $\beta > \alpha + \gamma$, $b + d > c$ *and* l, s, r*-odd,* k*-even.*

9. $(b+c-d)(\gamma-\alpha-\beta)-4d(\alpha+\beta)>0$, $a<1$, $\gamma>\alpha+\beta$, $b+c>d$ *and* k,s,l-*odd*, r-*even.*

10. $(d-b-c)(\alpha+\beta-\gamma)-4\gamma(b+c)>0$, $a<1$, $\alpha+\beta>\gamma$, $d>b+c$ *and* s,r-*odd*, l,k-*even*

11. $(c-b-d)(\alpha+\gamma-\beta)-4\beta(b+d)>0$, $a<1$, $\alpha+\gamma>\beta$, $c>b+d$ *and* s,k-*odd*, l,r-*even.*

12. $(b-c-d)(\beta+\gamma-\alpha)-4\alpha(c+d)>0$, $a<1$, $\beta+\gamma>\alpha$, $b>c+d$ *and* s,l-*odd*, r,k-*even.*

Proof. We will prove the theorem when Condition (1) is true and the proof of the other cases are similar and so we will be omitted. First suppose that there exists a prime period two solution $...,p,q,p,q,...$, of equation (1). We will prove that Condition (1) holds.We see from equation (1) that

$$p=aq+\frac{bp+cq+dp}{\alpha p+\beta q+\gamma p}=aq+\frac{ep+cq}{fp+\beta q}$$

where $e=b+d$, $f=\alpha+\gamma$, and

$$q=ap+\frac{bq+cp+dq}{\alpha q+\beta p+\gamma q}=ap+\frac{eq+cp}{fq+\beta p}$$

Then

$$fp^2+\beta pq=afpq+a\beta q^2+ep+cq, \tag{11}$$

$$fq^2+\beta pq=afpq+a\beta p^2+eq+cp, \tag{12}$$

Subtracting (11) from (12) gives $f(p^2-q^2)=-a\beta(p^2-q^2)+(e-c)(p-q)$. Since $p\neq q$, it follows that

$$p+q=\frac{e-c}{f+a\beta} \tag{13}$$

Again, adding (11) and (12) yields

$$(f-a\beta)(p^2+q^2)+2(\beta-af)pq=(e+c)(p+q) \tag{14}$$

It follows by (13),and (14) that

$$pq=\frac{(ea\beta+cf)(e-c)}{(f+a\beta)^2(\beta-f)(1+a)} \tag{15}$$

Now it is clear from equation (13) and equation (15) that p and q are the two distinct roots of the quadratic equation

$$(f+a\beta)t^2-(e-c)t+\frac{(ea\beta+cf)(e-c)}{(f+a\beta)(\beta-f)(1+a)}=0, \tag{16}$$

and so $(e-c)^2 - \frac{4(ea\beta+cf)(e-c)}{(\beta-f)(1+a)} > 0$. Therefore Inequality (1) holds.
Conversely , suppose that Inequality (1) is true. We will show that equation (1) has a prime period two solution. Assume that

$$p = \frac{e-c+\zeta}{2(f+a\beta)}, \quad and \quad q = \frac{e-c-\zeta}{2(f+a\beta)}$$

where $\zeta = \sqrt{(e-c)^2 - \frac{4(ea\beta+cf)(e-c)}{(\beta-f)(1+a)}}$. We see from Inequality (1) that

$$(e-c)(f-\beta)(1+a) + 4(ea\beta+cf) > 0, e > c, f > \beta,$$

which equivalents to $(e-c)^2 > \frac{4(ea\beta+cf)(e-c)}{(\beta-f)(1+a)}$. Therefore p and q are distinct real numbers. Set $x_{-3} = p, x_{-2} = q, x_{-1} = p$ and $x_0 = q$. We wish to show that $x_1 = x_{-1} = p$ and $x_2 = x_0 = q$ .It follows from equation (1) that

$$x_1 = aq + \frac{ep+cq}{fp+\beta q} = a\big(\frac{e-c-\zeta}{2(f+a\beta)}\big) + \frac{e\big(\frac{e-c-\zeta}{2(f+a\beta)}\big) + c\big(\frac{e-c-\zeta}{2(f+a\beta)}\big)}{f\big(\frac{e-c-\zeta}{2(f+a\beta)}\big) + \beta\big(\frac{e-c-\zeta}{2(f+a\beta)}\big)}$$

Dividing the denominator and numerator by $2(f+a\beta)$ gives

$$x_1 = a\big(\frac{e-c-\zeta}{2(f+a\beta)}\big) + \frac{(e-c)[(e+c)+\zeta]}{(f+\beta)(e-c)+(f-\beta)\zeta}$$

Multiplying the denominator and numerator of the right side by
$(f+\beta)(e-c) - (f-\beta)\zeta$ gives

$$x_1 = a\big(\frac{e-c-\zeta}{2(f+a\beta)}\big)$$

$$+\frac{(e-c)\big\{2(e-c)[fc+\beta e - \frac{2(ea\beta+cf)}{1+a}] + 2\zeta(\beta e - cf)\big\}}{4(e-c)\big[\beta f(e-c) + \frac{(\beta-f)(ea\beta+cf)}{(1+a)}\big]}$$

Multiplying the denominator and numerator of the right side by $(1+a)$ we obtain

$$x_1 = \frac{ae - ac - a\zeta + (e-c)(1-a) + \zeta(1+a)}{2(f+a\beta)} = \frac{e-c+\zeta}{2(f+a\beta)} = p.$$

Similarly as before one can easily show that $x_2 = q$. Then it follows by induction that $x_{2n} = q$ and $x_{2n+1} = p$ for all $n \geq -1$.Thus equation (1) has the prime period two solution $..., p, q, p, q, ...$, where p and q are the distinct roots of the quadratic equation (16) and the proof is complete. □

Lemma 4.2. *If l, k, s, r-even. Then there exists a prime period two solutions if and only if $a = -1$.*

Proof. First suppose that there exists a prime period two solution $..., p, q, p, q, ...$, then we see from equation (1) that when l, k, s, r-even

$$p = aq + \frac{b+c+d}{\alpha+\beta+\gamma}, \tag{17}$$

$$q = ap + \frac{b+c+d}{\alpha+\beta+\gamma} \tag{18}$$

Subtracting equation (17) from equation (18) gives $p - q = -a(p - q)$. Since $p \neq q$, it follows that $a = -1$. Again, adding equation (17) and equation (18) yields $p + q = \frac{b+c+d}{\alpha+\beta+\gamma}$. If we take

$$p = \frac{b+c}{\alpha+\beta+\gamma}, q = \frac{d}{\alpha+\beta+\gamma}, \text{if} \qquad b+c \neq d.$$

Set $x_{-s} = q$, $x_{-l} = p$, $x_{-k} = q$,...,$x_{-2} = q$, $x_{-1} = p$ and $x_0 = q$.We wish to show that $x_1 = x_{-1} = p$ and $x_2 = x_0 = q$. It follows from equation (1) that $x_1 = aq + \frac{bq+cq+dq}{\alpha q+\beta q+\gamma q} = p$. Similarly as before one can easily show that $x_2 = q$. Then it follows by induction that $x_{2n} = q$ and $x_{2n+1} = p$ for all $n \geq -1$. Thus equation (1) has the prime period two solution and the proof is complete. □

Lemma 4.3. *If l, k, r-odd, s-even. Then there exists a positive prime period two solutions if and only if $a = -1$.*

Lemma 4.4. *If l, k, s, r-odd (or l, k, r-even, s-odd). Then there no prime period two solution.*

5 Global Attractor of the Equilibrium Point of Equation (1)

In this section we investigate the global asymptotic stability of equation (1).

Lemma 5.1. *For any values of the quotient $\frac{b}{\alpha}, \frac{c}{\beta}$ and $\frac{d}{\gamma}$ the function $f(u, v, w, t)$ defined by equation (4) has the monotonicity behavior in its three arguments.*

Proof. The proof follows by some computations and it will be omitted. □

Remark 5.2. It follows from equation (1), when $\frac{b}{\alpha} = \frac{c}{\beta} = \frac{d}{\gamma}$, that $x_{n+1} = ax_{n-k} + \lambda$ for all $n \geq -t$ and for some constant λ.Whenever the quotients $\frac{\alpha}{A}$, $\frac{\beta}{B}$ and $\frac{\gamma}{C}$ are not equal, we get the following result.

Theorem 5.3. *The equilibrium point $\overline{x}$ is a global attractor of equation (1) if one of the following statements holds*

$$(1)\frac{b}{\alpha} \geq \frac{c}{\beta} \geq \frac{d}{\gamma} \quad \text{and} \quad d \geq b + c \tag{19}$$

$$(2)\frac{b}{\alpha} \geq \frac{d}{\gamma} \geq \frac{c}{\beta} \quad \text{and} \quad c \geq b + d \tag{20}$$

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$$(3)\frac{c}{\beta} \geq \frac{b}{\alpha} \geq \frac{d}{\gamma} \quad and \quad d \geq b+c \tag{21}$$

$$(4)\frac{c}{\beta} \geq \frac{d}{\gamma} \geq \frac{b}{\alpha} \quad and \quad b \geq c+d \tag{22}$$

$$(5)\frac{d}{\gamma} \geq \frac{c}{\beta} \geq \frac{b}{\alpha} \quad and \quad b \geq c+d \tag{23}$$

$$(6)\frac{d}{\gamma} \geq \frac{b}{\alpha} \geq \frac{c}{\beta} \quad and \quad c \geq b+d \tag{24}$$

Proof. Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of equation (1) and again let f be a function defined by equation (4). We will prove the theorem when case (1) is true and the proof of the other cases are similar and so we will be omitted. Assume that (19) is true, then it is easy from the equations after equation (4) to see that the function $f(u,v,w,t)$ is non-decreasing in u,v and non-increasing in t and it is not clear what is going on with w. So we consider the following two cases:-
Case(1) Assume that the function $f(u,v,w,t)$is non-decreasing in w. Suppose that (m,M) is a solution of the system $M = f(M,M,M,m)$ and $m = g(m,m,m,M)$. Then from equation (1), we see that

$$(\alpha+\beta)(1-a)M^2 + \gamma(1-a)Mm = (b+c)M + dm,$$

$$(\alpha+\beta)(1-a)m^2 + \gamma(1-a)Mm = (b+c)m + dM$$

Subtracting this two equations we obtain

$$(M-m)\big\{(\alpha+\beta)(1-a)(M+m) + (d-b-c)\big\} = 0,$$

under the conditions $d \geq b+c$, $a < 1$, we see that $M = m$. It follows by theorem (1.2) that $\overline{x}$ is a global attractor of equation (1) and then the proof is complete.
Case(2) Assume that the function $f(u,v,w,t)$is non-increasing in w. Suppose that (m,M) is a solution of the system $M = f(M,M,m,m)$ and $m = g(m,m,M,M)$.Then from equation (1), we see that

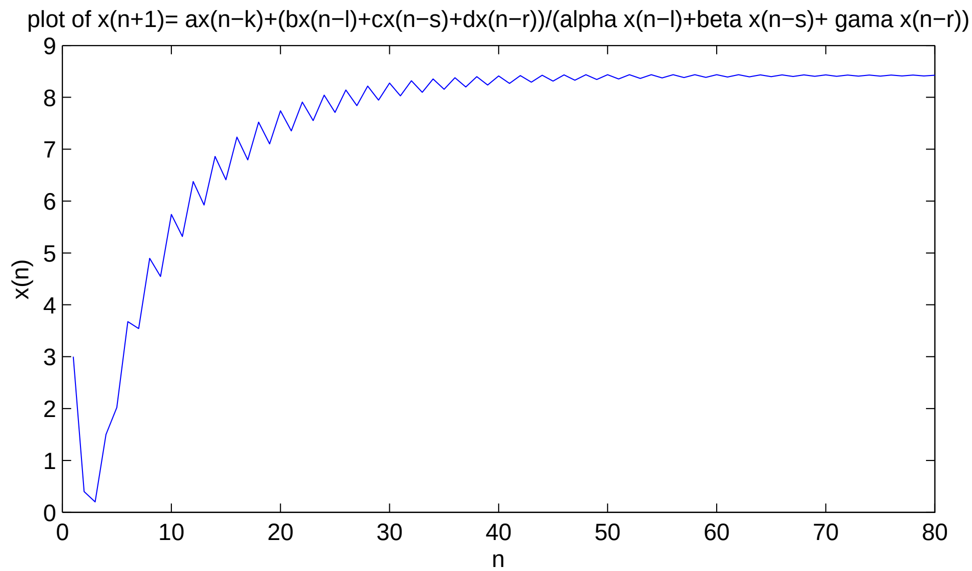
$$M(1-a) = \frac{bM+cm+dm}{\alpha M+\beta m+\gamma m}, \quad m(1-a) = \frac{bm+cM+dM}{\alpha m+\beta M+\gamma M}$$

then under the conditions $d \geq b+c$, $a < 1$, we see that $M = m$. It follows by theorem (1.2) that $\overline{x}$ is a global attractor of equation (1) and then the proof is complete. □

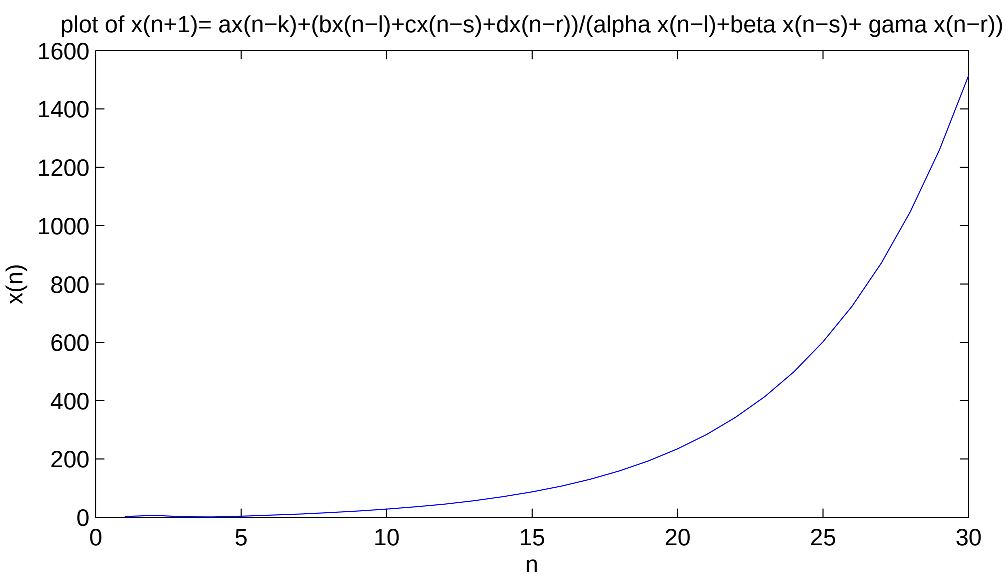
6 Numerical examples

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to equation (1).

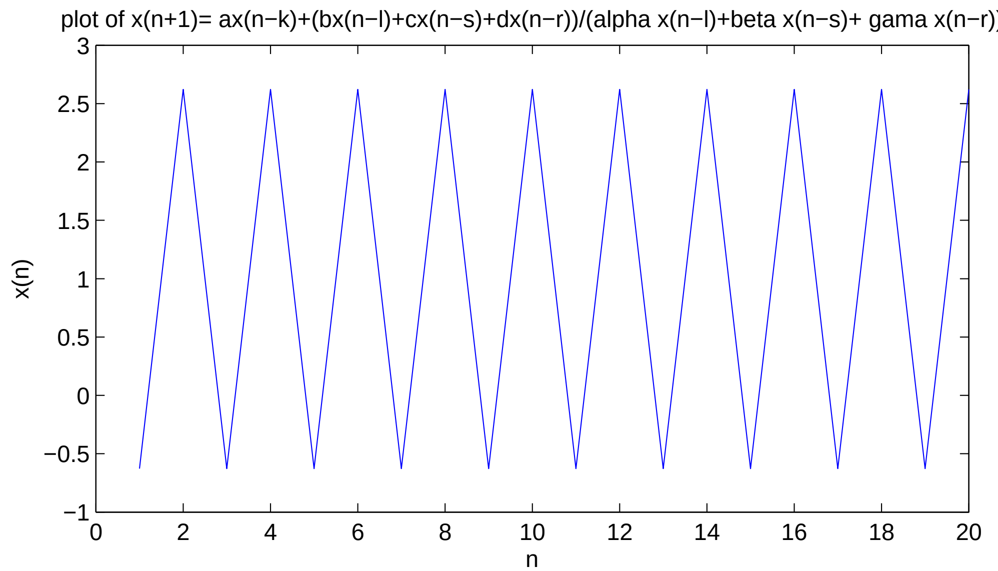
Example 6.1. See Fig.1, since $l = 0, k = 1, s = 2, r = 1, x_{-2} = 3, x_{-1} = 0.4, x_0 = 0.2, a = 0.8, b = 0.4, c = 0.8, d = 2, \alpha = 0.1, \beta = 1, \gamma = 0.8$.



Example 6.2. See Fig.2, since $l = 3, k = 0, s = 1, r = 2, x_{-3} = 3, x_{-2} = 7, x_{-1} = 2, x_0 = 1.5, a = 1.2, b = 3, c = 5, d = 2, \alpha = 1, \beta = 2.1, \gamma = 1.1.$



Example 6.3. See Fig.3, since $l = 1, k = 0, s = 2, r = 3, x_{-3} = x_{-1} = p, x_{-2} = x_0 = q, a = 0.6, b = 7, c = 3, d = 9, \alpha = 3.8, \beta = 0.2, \gamma = 1.2.$



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Quadratic derivations on non-Archimedean Banach algebras

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Abstract. Let A be an algebra and X be an A-module. A quadratic mapping $D : A \to X$ is called a quadratic derivation if

$$D(ab) = D(a)b^2 + a^2D(b)$$

for all $a_1, a_2 \in A$. We investigate the Hyers-Ulam stability of quadratic derivations from a non-Archimedean Banach algebra A into a non-Archimedean Banach A-module.

1. **Introduction**

A definition of stability in the case of homomorphisms between metric groups was proposed by a problem by Ulam [32] in 1940. In 1941, Hyers [17] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Rassias [27] for linear mappings by considering an unbounded Cauchy difference (see [3, 4, 8, 10, 18, 19, 22, 25, 29]).

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y). \tag{1.1}$$

is related to symmetric bi-additive function. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic mapping. It is well known that a mapping f between real vector spaces is quadratic if and only if there exits a unique symmetric bi-additive mapping B such that $f(x) = B(x, x)$ for all x (see [1, 20]). The bi-additive mapping B is given by

$$B(x, y) = \frac{1}{4}(f(x + y) - f(x - y)).$$

The Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for mappings $f : A \to B$, where A is a normed space and B is a Banach space (see [31]). Cholewa [6], Czerwik [7] and Grabiec [16] have generalized the results of stability of quadratic mappings. Borelli and Forti [5] generalized the stability result as follows (cf. [23, 24]): Let G be an Abelian group, and X a Banach space. Assume that a mapping $f : G \to X$ satisfies the functional inequality

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$, where $\varphi : G \times G \to [0, \infty)$ is a function such that

$$\Phi(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $Q : G \to X$ with the property

$$\|f(x) - Q(x)\| \le \Phi(x, x)$$

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for all $x \in G$.

Let $\mathbb{K}$ be a field.

A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have

(i) $|a| \geq 0$ and equality holds if and only if $a = 0$,

(ii) $|ab| = |a||b|$,

(iii) $|a + b| \leq \max\{|a|, |b|\}$.

The condition (iii) is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii) that $|n| \leq 1$ for each integer n. We always assume, in addition, that $|\cdot|$ is nontrivial, i.e., that there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$.

Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

$(NA1)$ $\|x\| = 0$ if and only if $x = 0$;

$(NA2)$ $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

$(NA3)$ the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \qquad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean space. It follows from $(NA3)$ that

$$\|x_m - x_\ell\| \leq \max\{\|x_{\jmath+1} - x_\jmath\| : \ell \leq \jmath \leq m - 1\} \qquad (m > \ell).$$

Therefore, a sequence $\{x_m\}$ is Cauchy in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. A non-Archimedean Banach algebra is a complete non-Archimedean algebra A which satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$. A non-Archimedean Banach space X is a non-Archimedean Banach A-bimodule if X is an A-bimodule which satisfies $\max\{\|xa\|, \|ax\|\} \leq \|a\|\|x\|$ for all $a \in A$, $x \in X$. For more detailed definitions of non-Archimedean Banach algebras, we can refer to [30].

Let A be a normed algebra and let X be a Banach A-module. We say that a mapping $D : A \to X$ is a quadratic derivation if D is a quadratic mapping satisfying

$$D(x_1x_2) = D(x_1)x_2^2 + x_1^2D(x_2) \tag{1.2}$$

for all $x_1, x_2 \in A$.

Recently, the stability of derivations has been investigated by a number of mathematicians including [2, 11, 12, 13, 14, 15, 21, 26, 28] and references therein. More recently, Eshaghi Gordji [9] established the stability of ring derivations on non-Archimedean Banach algebras.

In this paper, we investigate the approximately quadratic derivations on non-Archimedean Banach algebras.

2. Main results

In the following we suppose that A is a non-Archimedean Banach algebra and X is a non-Archimedean Banach A-bimodule. Assume that $|2| \neq 1$.

Theorem 2.1. *Let $f : A \to X$ be a given mapping with $f(0) = 0$ and let $\varphi_1 : A \times A \to \mathbb{R}^+$ and $\varphi_2 : A \times A \to \mathbb{R}^+$ be functions such that*

$$\|f(x_1x_2) - f(x_1)x_2^2 - x_1^2f(x_2)\| \leq \varphi_1(x_1, x_2), \tag{2.1}$$

$$||f(x + y) + f(x - y) - 2f(x) - 2f(y)|| \leq \varphi_2(x, y) \tag{2.2}$$

for all $x_1, x_2, x, y \in A$. Assume that for each $x \in A$

$$\lim_{n\to\infty} \max\left\{\frac{1}{|2|^{2k}}\frac{\varphi_2(2^kx, 2^kx)}{|2|^2} : 0 \leq k \leq n - 1\right\}$$

denoted by $\Psi(x, x)$, exists. Suppose

$$\lim_{n\to\infty} \frac{\varphi_1(2^nx_1, 2^nx_2)}{|2|^{4n}} = \lim_{n\to\infty} \frac{\varphi_2(2^nx, 2^ny)}{|2|^{2n}} = 0$$

for all $x_1, x_2, x, y \in A$. Then there exists a unique quadratic derivation $D : A \to X$ such that

$$\|D(x) - f(x)\| \leq \Psi(x, x) \tag{2.3}$$

for all $x \in A$.

Proof. Setting $y = x$ in (2.2), we get

$$\|f(2x) - 4f(x)\| \leq \varphi_2(x, x) \tag{2.4}$$

for all $x \in A$, and then dividing by $|2|^2$ in (2.4), we obtain

$$\left\|\frac{f(2x)}{2^2} - f(x)\right\| \leq \frac{\varphi_2(x, x)}{|2|^2} \tag{2.5}$$

for all $x \in A$. Replacing x by $2x$ and then dividing by $|2|^2$ in (2.5), we obtain

$$\left\|\frac{f(2^2x)}{2^4} - \frac{f(2x)}{2^2}\right\| \leq \frac{\varphi_2(2x, 2x)}{|2|^4}. \tag{2.6}$$

Combining (2.5), (2.6) and the strong triangle inequality (NA3) yields

$$\left\|\frac{f(2^2x)}{2^4} - f(x)\right\| \leq \max\left\{\frac{\varphi_2(2x, 2x)}{|2|^4}, \frac{\varphi_2(x, x)}{|2|^2}\right\}. \tag{2.7}$$

Following the same argument, one can prove by induction that

$$\left\|\frac{f(2^nx)}{2^{2n}} - f(x)\right\| \leq \max\left\{\frac{1}{|2|^2}\frac{\varphi_2(2^kx, 2^kx)}{|2|^{2k}} : 0 \leq k \leq n-1\right\}. \tag{2.8}$$

Replacing x by $2^{n-1}x$ and dividing by $|2|^{2(n-1)}$ in (2.5), we find that

$$\left\|\frac{f(2^nx)}{2^{2n}} - \frac{f(2^{n-1}x)}{2^{2(n-1)}}\right\| \leq \frac{\varphi_2(2^{n-1}x, 2^{n-1}x)}{|2|^{2n}}$$

for all positive integers n and all $x \in A$. Hence $\{\frac{f(2^nx)}{2^{2n}}\}$ is a Cauchy sequence. Since X is complete, it follows that $\{\frac{f(2^nx)}{2^{2n}}\}$ is convergent. Set $D(x) = \lim_{n\to\infty}\frac{f(2^nx)}{2^{2n}}$. By taking the limit as $n \to \infty$ in (2.8), we see that $\|D(x) - f(x)\| \leq \Psi(x, x)$ and (2.3) holds for all $x \in A$.

In order to show that D satisfies (1.2), replacing x_1, x_2 by $2^nx_1, 2^nx_2$ in (2.1), and dividing both sides of (2.1) by $|2|^{4n}$, we get

$$\left\|\frac{f(2^nx_1 \cdot 2^nx_2)}{2^{4n}} - \frac{f(2^nx_1)}{2^{4n}} \cdot (2^nx_2)^2 - (2^nx_1)^2.\frac{f(2^nx_2)}{2^{4n}}\right\| \leq \frac{\varphi_1(2^nx_1, 2^nx_n)}{|2|^{4n}}.$$

Taking the limit as $n \to \infty$, we find that D satisfies (1.2).

Replacing x by 2^nx and y by 2^ny in (2.2) and dividing by $|2|^{2n}$, we get

$$\left\|\frac{f(2^nx + 2^ny)}{2^{2n}} + \frac{f(2^nx - 2^ny)}{2^{2n}} - 2\frac{f(2^nx)}{2^{2n}} - 2\frac{f(2^ny)}{2^{2n}}\right\| \leq \frac{\varphi_2(2^nx, 2^ny)}{|2|^{2n}}.$$

Taking the limit as $n \to \infty$, we find that D satisfies (1.1).

Now, suppose that there is another such mapping $D' : A \to X$ satisfying $D'(x+y)+D'(x-y) = 2D'(x)+2D'(y)$ and $\|D'(x) - f(x)\| \leq \Psi(x, x)$. Then for all $x \in A$, we have

$$\begin{aligned}
\|D(x) - D'(x)\| &= \lim_{n\to\infty}\frac{1}{|2|^{2n}}\|D(2^nx) - D'(2^nx)\| \\
&\leq \lim_{n\to\infty}\frac{1}{|2|^{2n}}\max\{\|D(2^nx) - f(2^nx)\|, \|D'(2^nx) - f(2^nx)\|\} \\
&\leq \lim_{n\to\infty}\lim_{k\to\infty}\frac{1}{|2|^2}\max\{\frac{\varphi_2(2^jx, 2^jx)}{|2|^{2j}} : n \leq j \leq k+n-1\} = 0.
\end{aligned}$$

It follows that $D(x) = D'(x)$. □

Corollary 2.2. *Let θ_1 and θ_2 be nonnegative real numbers, and let p be a real number such that $p > 4$. Suppose that a mapping $f : A \to X$ satisfies*

$$\|f(x_1x_2) - f(x_1)x_2^2 - x_1^2f(x_2)\| \leq \theta_1(\|x_1\|^p + \|x_2\|^p),$$

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \theta_2(\|x\|^p + \|y\|^p)$$

for all $x_1, x_2, x, y \in A$. *Then there exists a unique quadratic derivation* $D : A \to X$ *such that*

$$\|D(x) - f(x)\| \leq \lim_{n\to\infty} \max\left\{\frac{\theta_2\|x\|^p}{|2|.|2|^{k(2-p)}}\ 0 \leq k \leq n-1\right\}$$

for all $x \in A$.

Proof. Let $\varphi_1 : A \times A \to \mathbb{R}^+$ and $\varphi_2 : A \times A \to \mathbb{R}^+$ be functions such that $\varphi_1(x_1,, x_2) = \theta_1(\|x_1\|^p + \|x_2\|^p)$ and $\varphi_2(x, y) = \theta_2(\|x\|^p + \|y\|^p)$ for all $x_1, x_2, x, y \in A$. Then we have

$$\lim_{n\to\infty} \frac{\varphi_2(2^n x, 2^n y)}{|2|^{2n}} = \lim_{n\to\infty} \theta_2 \cdot |2|^{n(p-2)} \cdot (\|x\|^p + \|y\|^p) = 0 \quad (x, y \in A),$$

$$\lim_{n\to\infty} \frac{\varphi_1(2^n x_1, 2^n x_2)}{|2|^{4n}} = \lim_{n\to\infty} \frac{\theta_1|2|^{pn}}{|2|^{4n}}(\|x_1\|^p + \|x_2\|^p) = 0 \quad (x_1, x_2 \in A).$$

Applying Theorem 2.1, we conclude the required result. □

Theorem 2.3. *Let* $f : A \to X$ *be a mapping and let* $\varphi_1 : A \times A \to \mathbb{R}^+$, $\varphi_2 : A \times A \to \mathbb{R}^+$ *be functions such that*

$$\|f(x_1x_2) - f(x_1)x_2^2 - x_1^2 f(x_2)\| \leq \varphi_1(x_1, x_2), \tag{2.9}$$

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varphi_2(x, y) \tag{2.10}$$

for all $x_1, x_2, x, y \in A$. *Assume that for each* $x \in A$

$$\lim_{n\to\infty} \max\left\{|2|^{2k}\varphi_2\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) : 0 \leq k \leq n-1\right\}$$

denoted by $\Psi(x, x)$, *exists. Suppose*

$$\lim_{n\to\infty} |2|^{4n}\varphi_1\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}\right) = \lim_{n\to\infty} |2|^{2n}\varphi_2\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

for all $x_1, x_2, x, y \in A$. *Then there exists a unique quadratic derivation* $D : A \to X$ *such that*

$$\|D(x) - f(x)\| \leq \Psi(x, x) \tag{2.11}$$

for all $x \in A$.

Proof. Setting $y = x$ in (2.10), we obtain

$$\|f(2x) - 4f(x)\| \leq \varphi_2(x, x). \tag{2.12}$$

Replacing x by $\frac{x}{2}$ in (2.12), one obtains

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \varphi_2\left(\frac{x}{2}, \frac{x}{2}\right). \tag{2.13}$$

Again replacing x by $\frac{x}{2}$ in (2.13) and multiplying by $|2|^2$, we obtain that

$$\left\|2^2 f\left(\frac{x}{2}\right) - 2^4 f\left(\frac{x}{2^2}\right)\right\| \leq |2|^2\varphi_2\left(\frac{x}{2^2}, \frac{x}{2^2}\right). \tag{2.14}$$

By using (2.13), (2.14) and strong triangle inequality (NA3), we get

$$\left\|f(x) - 2^4 f\left(\frac{x}{2^2}\right)\right\| \leq \max\left\{\varphi_2\left(\frac{x}{2}, \frac{x}{2}\right), |2|^2\varphi_2\left(\frac{x}{2^2}, \frac{x}{2^2}\right)\right\} \tag{2.15}$$

for $x \in A$.

Next we prove by induction that

$$\left\|f(x) - 2^{2n} f\left(\frac{x}{2^n}\right)\right\| \leq \max\left\{|2|^{2k}\varphi_2\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) : 0 \leq k \leq n-1\right\}. \tag{2.16}$$

Replacing x by $\frac{x}{2^{n-1}}$ and multiplying by $|2|^{2(n-1)}$ in (2.13), we obtain

$$\left\|2^{2(n-1)} f\left(\frac{x}{2^{n-1}}\right) - 2^{2n} f\left(\frac{x}{2^n}\right)\right\| \leq |2|^{2(n-1)}\varphi_2\left(\frac{x}{2^n}, \frac{x}{2^n}\right) \tag{2.17}$$

for all $x \in A$. Hence $\{2^{2n} f(\frac{x}{2^n})\}$ is a Cauchy sequence. Since X is complete, it follows that $\{2^{2n} f(\frac{x}{2^n})\}$ is convergent. Set $D(x) = \lim_{n\longrightarrow\infty}\{2^{2n} f(\frac{x}{2^n})\}$. By taking the limit as $n \to \infty$ in (2.16), we see that $\|f(x) - D(x)\| \leq \Psi(x, x)$ and (2.11) holds for all $x \in A$.

Replacing x_1, x_2 by $\frac{x_1}{2^n}, \frac{x_2}{2^n}$ in (2.9) and multiplying by $|2|^{4n}$, we get

$$\left\|2^{4n} f\left(\frac{x_1}{2^n} \cdot \frac{x_2}{2^n}\right) - 2^{4n} f\left(\frac{x_1}{2^n}\right)\left(\frac{x_2}{2^n}\right)^2 - 2^{4n}\left(\frac{x_1}{2^n}\right)^2 f\left(\frac{x_2}{2^n}\right)\right\| \leq 2^{4n}\varphi_1\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}\right).$$

Taking the limit as $n \to \infty$, we find that D satisfies (1.2).

Replacing x by $\frac{x}{2^n}$ and y by $\frac{y}{2^n}$ in (2.10) and multiplying by $|2|^{2n}$, we have

$$\left\|2^{2n} f\left(\frac{x}{2^n} + \frac{y}{2^n}\right) + 2^{2n} f\left(\frac{x}{2^n} - \frac{y}{2^n}\right) - 2^{2n} \cdot 2f\left(\frac{x}{2^n}\right) - 2^{2n} \cdot 2f\left(\frac{y}{2^n}\right)\right\| \leq |2|^{2n}\varphi_2\left(\frac{x}{2^n}, \frac{y}{2^n}\right).$$

Taking the limit as $n \to \infty$, we find that D satisfies (1.1).
Now, suppose that there is another such mapping $D' : A \to X$ satisfying $D'(x+y) + D'(x-y) = 2D'(x) + 2D'(y)$ and $\|D'(x) - f(x)\| \leq \Psi(x, x)$. Then for all $x \in A$, we have

$$\begin{aligned}
\|D(x) - D'(x)\| &= \lim_{n\to\infty} |2|^{2n} \left\|D\left(\frac{x}{2^n}\right) - D'\left(\frac{y}{2^n}\right)\right\| \\
&\leq \lim_{n\to\infty} |2|^{2n} \max\left\{\left\|D\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|, \left\|D'\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right\} \\
&\leq \lim_{n\to\infty} \lim_{k\to\infty} \max\left\{\varphi_2\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) : n \leq j \leq k+n-1\right\} = 0
\end{aligned}$$

and so $D(x) = D'(x)$ for all $x \in A$. □

Corollary 2.4. *Let θ_1 and θ_2 be nonnegative real numbers, and let p be a positive real number such that $p < 2$. Suppose that a mapping $f : A \to X$ satisfies*

$$\|f(x_1x_2) - f(x_1)x_2^2 - x_1^2 f(x_2)\| \leq \theta_1(\|x_1\|^p + \|x_2\|^p),$$
$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \theta_2(\|x\|^p + \|y\|^p)$$

for all $x_1, x_2, x, y \in A$. Then there exists a unique quadratic derivation $D : A \to X$ such that

$$\|D(x) - f(x)\| \leq \lim_{n\to\infty} \max\{\theta_2\|x\|^p.|2|^{(k+1)(1-p)}\ 0 \leq k \leq n-1\}$$

for all $x \in A$.

Proof. Let $\varphi_1 : A \times A \to \mathbb{R}^+$ and $\varphi_2 : A \times A \to \mathbb{R}^+$ be functions such that $\varphi_1(x_1, , x_2) = \theta_1(\|x_1\|^p + \|x_2\|^p)$ and $\varphi_2(x, y) = \theta_2(\|x\|^p + \|y\|^p)$ for all $x_1, x_2, x, y \in A$. We have

$$\lim_{n\to\infty} |2|^{2n}\varphi_2\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = \lim_{n\to\infty} (|2|^{n(2-p)})\theta_2(\|x\|^p + \|y\|^p) = 0 \quad (x, y \in A),$$
$$\lim_{n\to\infty} |2|^{4n}\varphi_1\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}\right) = \lim_{n\to\infty} |2|^{n(4-p)}\theta_1(\|x_1\|^p + \|x_2\|^p) = 0 \quad (x_1, x_2 \in A).$$

Applying Theorem 2.4, we conclude the required result. □

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Soft q-ideals of soft BCI-algebras

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Abstract. The notion of soft q-ideals and q-idealistic soft BCI-algebras are introduced, and several properties of them are investigated. Characterizations of a (fuzzy) q-ideals in BCI-algebras are considered. Relations between fuzzy q-ideals and p-idealistic soft BCI-algebras are discussed.

1. Introduction

D. Molodtsov ([2]) introduced introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical applications. Y. B. Jun ([6]) applied first the notion of soft sets by Moldtsov to the theory of BCK-algebras. Y. B. Jun and C. H. Park ([8]) dealt with the algebraic structure of BCK/BCI-algebras by applying soft set theory. They discussed the algebraic properties of soft sets in BCK/BCI-algebras. In [7], Y. B. Jun, K. J. Lee and J. Zhan introduced the notion of soft p-ideals and p-idealistic soft BCI-algebras, and investigated their properties. Y. S. Hwang and S. S. Ahn ([5]) defined the notion of vague q-ideal of a BCI-algebra and studied several properties of them.

In this paper, we introduced the notion of soft q-ideals and q-idealistic soft BCI-algebras, and investigate several properties of them. We also consider characterizations of a (fuzzy) q-ideals in BCI-algebras and study relations between fuzzy q-ideals and p-idealistic soft BCI-algebras.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(a1) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(a2) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(a3) $(\forall x \in X)\ (x * x = 0)$,
(a4) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \Rightarrow\ x = y)$.

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In any BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X has the following properties:

(b1) $(\forall x \in X)\ (x * 0 = x)$.
(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.
(b3) $(\forall x, y \in X)\ (0 * (x * y) = (0 * x) * (0 * y))$.
(b4) $(\forall x, y \in X)\ (x * (x * (x * y)) = x * y)$.
(b5) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$.
(b6) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$.

A non-empty subset S of a BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset A of a BCI-algebra X is called an *ideal* of X if it satisfies:

(c1) $0 \in A$,
(c2) $(\forall x \in X)(\forall y \in A)(x * y \in A \Rightarrow x \in A)$.

Note that every ideal A of a BCI-algebra X satisfies:

$$(\forall x \in X)\,(\forall y \in A)\,(x \leq y \Rightarrow x \in A).$$

A non-empty subset A of a BCI-algebra X is called a *q-ideal* of X if it satisfies (c1) and

(c3) $(\forall x, y, z \in X)(x * (y * z) \in A, y \in A \Rightarrow x * z \in A)$.

Note that any q-ideal is an ideal, but the converse is not true in general.

We refer the reader to the book [4] for further information regarding BCI-algebras.

Molodtsov ([2]) defined the soft set in the following way: Let U be an initial set and E be a set of parameters. Let $\mathscr{P}(U)$ denote the power set of U and $A \subset E$.

Definition 2.1.([2]) A pair $(\mathscr{F}, A)$ is called a *soft set* over U, where $\mathscr{F}$ is a mapping given by

$$\mathscr{F} : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\epsilon \in A$, $\mathscr{F}(\epsilon)$ may be considered as the set of ϵ-approximate elements of the soft set $(\mathscr{F}, A)$. Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [2].

Definition 2.2.([3]) Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be two soft sets over a common universe U. The *intersection* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is defined to be the soft set $(\mathscr{H}, C)$ satisfying the following conditions:

(i) $C = A \cap B$,
(ii) $(\forall e \in C)(\mathscr{H}(e) = \mathscr{F}(e)$ or $\mathscr{G}(e)$, (as both are same sets)).

In this case, we write $(\mathscr{F}, A)\tilde{\cap}(\mathscr{G}, B) = (\mathscr{H}, C)$.

Definition 2.3.([3]) Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be two soft sets over a common universe U. The *union* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is defined to be the soft set $(\mathscr{H}, C)$ satisfying the following conditions:

(i) $C = A \cup B$,

(ii) $\forall e \in C$

$$\mathscr{H}(e) = \begin{cases} \mathscr{F}(e) & \text{if } e \in A \backslash B \\ \mathscr{G}(e) & \text{if } e \in B \backslash A \\ \mathscr{F}(e) \cup \mathscr{G}(e) & \text{if } e \in A \cap B. \end{cases}$$

In this case, we write $(\mathscr{F}, A)\tilde{\cup}(\mathscr{G}, B) = (\mathscr{H}, C)$.

Definition 2.4.([3]) If $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are two soft sets over a common universe U, then "$(\mathscr{F}, A)AND(\mathscr{G}, B)$" denoted by $(\mathscr{F}, A)\tilde{\wedge}(\mathscr{G}, B)$ is defined by $(\mathscr{F}, A)\tilde{\wedge}(\mathscr{G}, B) = (\mathscr{H}, A \times B)$, where $\mathscr{H}(\alpha, \beta) = \mathscr{F}(\alpha) \cap \mathscr{G}(\beta)$ for all $(\alpha, \beta) \in A \times B$.

Definition 2.5.([3]) If $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are two soft sets over a common universe U, then "$(\mathscr{F}, A)OR(\mathscr{G}, B)$" denoted by $(\mathscr{F}, A)\tilde{\vee}(\mathscr{G}, B)$ is defined by $(\mathscr{F}, A)\tilde{\vee}(\mathscr{G}, B) = (\mathscr{H}, A \times B)$, where $\mathscr{H}(\alpha, \beta) = \mathscr{F}(\alpha) \cup \mathscr{G}(\beta)$ for all $(\alpha, \beta) \in A \times B$.

Definition 2.6.([3]) For two soft sets $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ over a common universe U, we say that $(\mathscr{F}, A)$ is a *soft subset* of $(\mathscr{G}, B)$, denoted by $(\mathscr{F}, A)\tilde{\subset}(\mathscr{G}, B)$, if it satisfies:

(i) $A \subset B$,

(ii) For every $\epsilon \in A$, $\mathscr{F}(\epsilon)$ and $\mathscr{G}(\epsilon)$ are identical approximations.

The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh ([11]).

3. Soft q-ideals

In what follows let X and A be a BCI-algebra and a nonempty set, respectively, and R will refer to an arbitrary binary relation between an element of A and an element of X, that is, R is a subset of $A \times X$ without otherwise specified. A set-valued function $\mathscr{F} : A \to \mathscr{P}(X)$ can be defined as $\mathscr{F}(x) = \{y \in X | (x, y) \in R\}$ for all $x \in A$. The pair $(\mathscr{F}, A)$ is then a soft set over X.

Definition 3.1.([8]) Let S be a subalgebra of X. A subset I of X is called an *ideal* of X related to S (briefly, S-ideal of X), denoted by $I \lhd S$, if it satisfies:

(i) $0 \in I$,

(ii) $(\forall x \in S)(\forall y \in I)(x * y \in I \Rightarrow x \in I)$.

Definition 3.2. Let S be a subalgebra of X. A subset I of X is called a *q-ideal* of X related to S (briefly, S-q-ideal of X), denoted by $I \lhd_q S$, if it satisfies:

(i) $0 \in I$,

(ii) $(\forall x, z \in S)(\forall y \in I)(x * (y * z) \in I \Rightarrow x * z \in I)$.

Example 3.3. Let $X := \{0, 1, a, b\}$ be a BCI-algebra ([4]) in which the $*$-operation is given by the following table:

$*$	0	1	a	b
0	0	0	a	a
1	1	0	a	a
a	a	a	0	0
b	b	a	1	0

Then $S := \{0, 1, a\}$ is a subalgebra of X and $I := \{0, 1\}$ is an S-q-ideal of X.

Note that every S-q-ideal of X is an S-ideal of X ($\because$ Put $z := 0$ in Definition 3.2(ii)). But the converse is not true in general as seen in the following example.

Example 3.4. Let $X := \{0, 1, 2, a, b\}$ be a BCI-algebra ([4]) in which the $*$-operation is given by the following table:

$*$	0	1	2	a	b
0	0	0	0	b	a
1	1	0	0	b	a
2	2	1	0	b	a
a	a	a	a	0	b
b	b	b	b	a	0

Then $S := \{0, a, b\}$ is a subalgebra of X and $\{0\}$ is an S-ideal of X, but not an S-q-ideal of X, since $a * (0 * b) = a * a = 0 \in \{0\}$, $0 \in \{0\}$, and $a * b = b \notin \{0\}$.

Definition 3.5.([6]) Let $(\mathscr{F}, A)$ be a soft set over X. Then $(\mathscr{F}, A)$ is called a *soft BCI-algebra* over X if $\mathscr{F}(x)$ is a subalgebra of X for all $x \in A$.

Definition 3.6.([8]) Let $(\mathscr{F}, A)$ be a soft set over X. A soft set $(\mathscr{G}, I)$ over X is called a *soft ideal* of $(\mathscr{F}, A)$, denoted by $(\mathscr{G}, I)\tilde{\lhd}(\mathscr{F}, A)$ if it satisfies:

(i) $I \subset A$,
(ii) $(\forall x \in I)(\mathscr{G}(x) \lhd \mathscr{F}(x))$.

Definition 3.7. Let $(\mathscr{F}, A)$ be a soft set over X. A soft set $(\mathscr{G}, I)$ over X is called a *soft q-ideal* of $(\mathscr{F}, A)$, denoted by $(\mathscr{G}, I)\tilde{\lhd}_q(\mathscr{F}, A)$ if it satisfies:

(i) $I \subset A$,
(ii) $(\forall x \in I)(\mathscr{G}(x) \lhd_q \mathscr{F}(x))$.

Example 3.8. Consider a BCI-algebra $X = \{0, 1, a, b\}$ which is given in Example 3.3. Let $(\mathscr{F}, A)$ be a soft set over X, where $A := \{0, 1, a\} \subset X$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathscr{F}(x) = \{0\} \cup \{y \in X | y * (y * x) \in \{0, 1, a\}\}$$

for all $x \in A$. Then $\mathscr{F}(0) = \mathscr{F}(1) = \mathscr{F}(a) = X$, which are subalgebras of X. Hence $(\mathscr{F}, A)$ is a soft BCI-algebra over X. Let $I := \{0, 1\}$ and $\mathscr{G} : I \to (X)$ be a set-valued function defined by

$$\mathscr{G}(x) = \{0\} \cup \{y \in X | x \leq y\}$$

for all $x \in I$. Then $\mathscr{G}(0) = \{0, 1\} \lhd_q \mathscr{F}(0)$ and $\mathscr{G}(1) = \{0, 1\} \lhd_q \mathscr{F}(1)$. Hence $(\mathscr{G}, I)$ is a soft q-ideal of $(\mathscr{F}, A)$.

Note that every soft q-ideal is a soft ideal. But the converse is not true in general as seen in the following example.

Example 3.9. Let $X := \{0, 1, a, b\}$ be a BCI-algebra ([4]) in which the $*$-operation is given by the following table:

$*$	0	1	a	b
0	0	0	b	a
1	1	0	b	a
a	a	a	0	b
b	b	b	a	0

For $A = \{0, 1\}$, define a set-valued function $\mathscr{F}(x) : A \to \mathscr{P}(X)$ by

$$\mathscr{F}(x) = \{0\} \cup \{y \in X | y * (y * x) = 0\}$$

for all $x \in A$. Then $\mathscr{F}(0) = X$ and $\mathscr{F}(1) = \{0, a, b\}$ are subalgebras of X. Hence $(\mathscr{F}, A)$ is a soft BCI-algebra over X. For $I := \{0\}$, let $\mathscr{G} : I \to \mathscr{P}(X)$ be a set-valued function defined by

$$\mathscr{G}(x) = \{0\} \cup \{y \in X | x \leq y\}$$

for all $x \in I$. Then $\mathscr{G}(0) = \{0, 1\}$. Hence $\mathscr{G}(0) \lhd \mathscr{F}(0)$, but $\mathscr{G}(0) \not\lhd_q \mathscr{F}(0)$ since $a * (0 * b) = 0, 0 \in \{0, 1\}$ and $a * b = b \notin \{0, 1\}$.

Theorem 3.10. *Let $(\mathscr{F}, A)$ be a soft BCI-algebra over X. For any soft sets, $(\mathscr{G}_1, I_1)$ and $(\mathscr{G} I_2)$ over X where $I_1 \cap I_2 \neq \emptyset$, we have*

$$(\mathscr{G}_1, I_1) \tilde{\lhd}_q (\mathscr{F}, A), (\mathscr{G}_2, I_2) \tilde{\lhd}_q (\mathscr{F}, A) \Rightarrow (\mathscr{G}_1, I_1) \tilde{\cap} (\mathscr{G}_2, I_2) \tilde{\lhd}_q (\mathscr{F}, A).$$

Proof. By Definition 2.2, we can write

$$(\mathscr{G}_1, I_1) \tilde{\cap} (\mathscr{G}_2, I_2) = (\mathscr{G}, I),$$

where $I = I_1 \cap I_2$ and $\mathscr{G}(x) = \mathscr{G}_1(x)$ or $\mathscr{G}_2(x)$ for all $x \in I$. Obviously, $I \subset A$ and $\mathscr{G} : I \to \mathscr{P}(X)$ is a mapping. Hence $(\mathscr{G}, I)$ is a soft set over X. Since $(\mathscr{G}_1, I_1) \tilde{\lhd}_q (\mathscr{F}, A)$ and $(\mathscr{G}_2, I_2) \tilde{\lhd}_q (\mathscr{F}, A)$, we know that $\mathscr{G}(x) = \mathscr{G}_1(x) \tilde{\lhd}_q \mathscr{F}(x)$ or $\mathscr{G}(x) = \mathscr{G}_2(x) \tilde{\lhd}_q \mathscr{F}(x)$ for all $x \in I$. Hence

$$(\mathscr{G}_1, I_1) \tilde{\cap} (\mathscr{G}_2, I_2) = (\mathscr{G}, I) \tilde{\lhd}_q (\mathscr{F}, A).$$

This completes the proof. □

Corollary 3.11. *Let $(\mathscr{F}, A)$ be a soft BCI-algebra over X. For any soft sets, $(\mathscr{G}_1, I)$ and $(\mathscr{G}_2, I)$ over X, we have*

$$(\mathscr{G}_1, I) \tilde{\lhd}_q (\mathscr{F}, A), (\mathscr{G}_2, I) \tilde{\lhd}_q (\mathscr{F}, A) \Rightarrow (\mathscr{G}_1, I) \tilde{\cap} (\mathscr{G}_2, I) \tilde{\lhd}_q (\mathscr{F}, A).$$

Proof. Straightforward. □

Theorem 3.12. *Let $(\mathscr{F}, A)$ be a soft BCI-algebra over X. For any soft sets, $(\mathscr{G}, I)$ and $(\mathscr{H}, J)$ over X in where $I \cap J = \emptyset$, we have*

$$(\mathscr{G}, I)\tilde{\lhd}_q(\mathscr{F}, A), (\mathscr{H}, J)\tilde{\lhd}_q(\mathscr{F}, A) \Rightarrow (\mathscr{G}, I)\tilde{\cup}(\mathscr{H}, J)\tilde{\lhd}_q(\mathscr{F}, A).$$

Proof. Assume that $(\mathscr{G}, I)\tilde{\lhd}_q(\mathscr{F}, A)$ and $(\mathscr{H}, J)\tilde{\lhd}_q(\mathscr{F}, A)$. By Definition 2.3, we can write $(\mathscr{G}, I)\tilde{\cup}(\mathscr{H}, J) = (\mathscr{K}, U)$ where $U = I \cup J$ and for every $x \in U$,

$$\mathscr{K}(x) = \begin{cases} \mathscr{G}(x) & \text{if } x \in I\backslash J \\ \mathscr{H}(x) & \text{if } x \in J\backslash I \\ \mathscr{G}(x) \cup \mathscr{H}(x) & \text{if } x \in I \cap J. \end{cases}$$

Since $I \cap J = \emptyset$, either $x \in I\backslash J$ or $x \in J\backslash I$ for all $x \in U$. If $x \in I\backslash J$, then $\mathscr{K}(x) = \mathscr{G}(x) \lhd_q \mathscr{F}(x)$ since $(\mathscr{G}, I)\tilde{\lhd}_q(\mathscr{F}, A)$. If $x \in J\backslash I$, then $\mathscr{K}(x) = \mathscr{H}(x) \lhd_q \mathscr{F}(x)$ since $(\mathscr{H}, J)\tilde{\lhd}_q(\mathscr{F}, A)$. Thus $\mathscr{K}(x) \lhd_q \mathscr{F}(x)$ for all $x \in U$, and so $(\mathscr{G}, I)(\mathscr{H}, J) = (\mathscr{K}, \tilde{\lhd}_q(\mathscr{F}, A)$. □

If I and J are not disjoint in Theorem 3.12, then Theorem 3.12 is not true in general as seen in the following example.

Example 3.13. Consider a BCI-algebra $X = \{0, 1, a, b\}$ which is given in Example 3.3. Let $(\mathscr{F}, A)$ be a soft set over X, where $A := \{0, 1, a\} \subset X$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathscr{F}(x) = \{0\} \cup \{y \in X | y * (y * x) \in \{0, 1, a\}\}$$

for all $x \in A$. Then $(\mathscr{F}, A)$ is a soft BCI-algebra over X (see Example 3.8). Let $I := \{0, 1\}$ and $\mathscr{G} : I \to (X)$ be a set-valued function defined by

$$\mathscr{G}(x) = \{0\} \cup \{y \in X | x \leq y\}$$

for all $x \in I$. Then $(\mathscr{G}, I)$ is a soft q-ideal of $(\mathscr{F}, A)$ (see Example 3.8). Let $J := \{0\}$ and $\mathscr{H} : J \to \mathscr{P}(X)$ be defined by

$$\mathscr{H}(x) = \{x, a\}.$$

Then $\mathscr{H}(0) = \{0, a\} \lhd_q \mathscr{F}(0)$. But $\mathscr{G}(0) \cup \mathscr{H}(0) = \{0, 1, a\} \not\lhd_q \mathscr{F}(0)$ since $b * (1 * 0) = a, 1 \in \{0, 1, a\}$ and $b * 0 = b \notin \{0, 1, a\}$.

4. q-idealistic soft BCI-algebras

Definition 4.1.([8]) Let $(\mathscr{F}, A)$ be a soft set over X. Then $(\mathscr{F}, A)$ is called an *idealistic soft BCI-algebra* over X if $\mathscr{F}(x)$ is an ideal of X for all $x \in A$.

Definition 4.2. Let $(\mathscr{F}, A)$ be a soft set over X. Then $(\mathscr{F}, A)$ is called a *q-idealistic soft BCI-algebra* over X if $\mathscr{F}(x)$ is a q-ideal of X for all $x \in A$.

Example 4.3. Let $X := \{0, a, b, c\}$ be a BCI-algebra ([9]) in which the $*$-operation is given by the following table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $A = X$ and $\mathscr{G} : A \to \mathscr{P}(X)$ be a set-valued function defined by

$$\mathscr{G}(x) = \{0, x\}$$

for all $x \in A$. Then $\mathscr{G}(0) = \{0\}, \mathscr{G}(a) = \{0, a\}$ and $\mathscr{G}(c) = \{0, c\}$, which are ideals of X. Hence $(\mathscr{G}, A)$ is an idealistic soft BCI-algebra over X ([8]). Note that $\mathscr{G}(x)$ is a q-ideal of X for all $x \in A$. Hence $(\mathscr{G}, A)$ is a q-idealistic soft BCI-algebra over X.

For any element x of a BCI-algebra X, we define the order of X, denoted by $o(x)$, as

$$o(x) = \min\{n \in \mathbb{N} | 0 * x^n = 0\}$$

where $0 * x^n = (\cdots((0 * x) * x) * \cdots) * x$ in which x appears n-times.

Example 4.4. Let $X := \{0, a, b, c, de, f, g\}$ be a BCI-algebra ([1]) in which the $*$-operation is given by the following table:

$*$	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Let $(\mathscr{F}, A)$ be a soft set over X, where $A = \{a, b, c\} \subset X$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is a set-valued function defined as follows:

$$\mathscr{F}(x) = \{y \in X | o(x) = o(y)\}$$

for all $x \in A$. Then $\mathscr{F}(a) = \mathscr{F}(b) = \mathscr{F}(c) = \{0, a, b, c\}$ is an ideal of X. Hence $(\mathscr{F}, A)$ is an idealistic soft BCI-algebra over X ([6]). If we take $B := \{a, b, d, f\} \subset X$ and define a set-valued function $\mathscr{G} : B \to \mathscr{P}(X)$ by

$$\mathscr{G}(x) = \{0\} \cup \{y \in X | o(x) = o(y)\}$$

for all $x \in B$, then $(\mathscr{G}, B)$ is not a q-idealistic soft BCI-algebra over X. In fact, since $f * (g * e) = d, g \in \{0, d, e, f, g\}$ and $f * e = b \notin \{0, d, e, f, g\}$, $\mathscr{G}(d) = \{0, d, e, f, g\}$ is not a q-ideal of X.

Obviously, every q-idealistic soft BCI-algebra over X is an idealistic soft BCI-algebra over X, but the converse is not true in general as seen in the following example.

Example 4.5. Consider a BCI-algebra $X := Y \times \mathbb{Z}$, where $(Y, *, 0)$ is a BCI-algebra over X and $(\mathbb{Z}, -, 0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ integers. Let $\mathscr{F} : X \to \mathscr{P}(X)$ be a set-valued function defined as follows:

$$\mathscr{F}(y, n) = \begin{cases} Y \times \mathbb{N}_0 & \text{if } x \in \mathbb{N}_0 \\ \{(0, 0)\} & \text{otherwise} \end{cases}$$

for all $(y, n) \in X$, where $\mathbb{N}_0$ is the set of all non-negative integers. Then $(\mathscr{F}, X)$ is an idealistic soft BCI-algebra over X([8]). But it is not a q-idealistic soft BCI-algebra over X since $\{(0, 0)\}$ is not a q-ideal of X. In fact, $(0, 3) * ((0, 0) * (0, -3)) = (0, 0) \in \{(0, 0)\}$ and $(0, 3) * (0, -3) = (0, 6) \notin \{(0, 0)\}$.

Proposition 4.6. *Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be soft sets over X where $B \subseteq A \subseteq X$. If $(\mathscr{F}, A)$ is a q-idealistic soft BCI-algebra over X, then so is $(\mathscr{G}, B)$.*

Proof. Straightforward. □

The converse of Proposition 4.6 is not true in general as seen in the following example.

Example 4.7. Consider a q-idealistic soft BCI-algebra $(\mathscr{F}, A)$ over X which is described in Example 4.4. If we take $B := \{a, b, c, d\} \supseteq A = \{a, b, c\}$, then $(\mathscr{F}, B)$ is not a q-idealistic soft BCI-algebra over X since $\mathscr{F}(d) = \{d, e, f, g\}$ is not a q-ideal of X.

Theorem 4.8. *Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be two q-idealistic soft BCI-algebra over X. If $A \cap B \neq \emptyset$, then the intersection $(\mathscr{F}, A)\tilde{\cap}(\mathscr{G}, B)$ is a q-idealistic soft BCI-algebra over X.*

Proof. Using Definition 2.2, we can write $(\mathscr{F}, A)\tilde{\cap}(\mathscr{G}, B) = (\mathscr{H}, C)$, where $C = A \cap B$ and $\mathscr{H}(x) = \mathscr{F}(x)$ or $\mathscr{G}(x)$ for all $x \in C$. Note that $\mathscr{H} : C \to \mathscr{P}(X)$ is a mapping, and therefore $(\mathscr{H}, C)$ is a soft set over X. Since $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are q-idealistic soft BCI-algebras over X, it follows that $\mathscr{H}(x) = \mathscr{F}(x)$ is a q-ideal of X, or $\mathscr{H}(x) = \mathscr{G}(x)$ is a q-ideal of X for all $x \in C$. Hence $(\mathscr{H}, C) = (\mathscr{F}, A)\tilde{\cap}(\mathscr{G}, B)$ is a q-idealistic soft BCI-algebra over X. □

Corollary 4.9. *Let $(\mathscr{F}, A)$ and $(\mathscr{G}, A)$ be two q-idealistic soft BCI-algebra over X. Then the intersection $(\mathscr{F}, A)\tilde{\cap}(\mathscr{G}, A)$ is a q-idealistic soft BCI-algebra over X.*

Proof. Straightforward. □

Theorem 4.10. *Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be two q-idealistic soft BCI-algebra over X. If $A \cap B = \emptyset$, then the union $(\mathscr{F}, A)\tilde{\cup}(\mathscr{G}, B)$ is a q-idealistic soft BCI-algebra over X.*

Proof. Using Definition 2.3, we write $(\mathscr{F}, A)\tilde{\cup}(\mathscr{G}, B) = (\mathscr{H}, C)$, where $C = A \cup B$ and for every $x \in C$,

$$\mathscr{H}(x) = \begin{cases} \mathscr{F}(x) & \text{if } x \in A \backslash B \\ \mathscr{G}(c) & \text{if } x \in B \backslash A \\ \mathscr{F}(x) \cup \mathscr{G}(x) & \text{if } x \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, either $x \in A \backslash B$ or $x \in B \backslash A$ for all $x \in C$. If $x \in A \backslash B$, then $\mathscr{H}(x) = \mathscr{F}(x)$ is a q-ideal of X since $(\mathscr{F}, A)$ is a q-idealistic soft BCI-algebra over X. If $x \in B \backslash A$, then

$\mathscr{H}(x) = \mathscr{G}(x)$ is a q-ideal of X since $(\mathscr{G}, B)$ is a q-idealistic soft BCI-algebra over X. Hence $(\mathscr{F}, A)\tilde{\cup}(\mathscr{G}, A)$ is a q-idealistic soft BCI-algebra over X. □

Theorem 4.11. *If $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are q-idealistic soft BCI-algebra over X, then $(\mathscr{F}, A)\tilde{\wedge}(\mathscr{G}, B)$ is a q-idealistic soft BCI-algebra over X.*

Proof. By Definition 2.4,

$$(\mathscr{F}, A)\tilde{\wedge}(\mathscr{G}, B) = (\mathscr{H}, A \times B),$$

where $\mathscr{H}(x, y) = \mathscr{F}(x) \cap \mathscr{G}(y)$ for all $(x, y) \in A \times B$. Since $\mathscr{F}(x)$ and $\mathscr{G}(y)$ are q-ideals of X, the intersection $\mathscr{F}(x) \cap \mathscr{G}(y)$ is also a q-ideal of X. Hence $\mathscr{H}(x, y)$ is a q-ideal of X for all $(x, y) \in A \times B$, and therefore $(\mathscr{F}, A)\tilde{\wedge}(\mathscr{G}, B)$ is a q-idealistic soft BCI-algebra over X. □

Definition 4.12. A q-idealistic BCI-algebra $(\mathscr{F}, A)$ over X is said to be *trivial* (resp., *whole*) if $\mathscr{F}(x) = \{0\}$ (resp., $\mathscr{F}(x) = X$) for all $x \in A$.

Example 4.13. Let $X = \{0, a, b, c\}$ be a BCI-algebra which is given Example 4.3. Let $(\mathscr{F}, A)$ be a soft set over X, where $A := \{a, b, c\} \subset X$, and let $\mathscr{F} : A \to \mathscr{P}(X)$ be a set-valued function defined by

$$\mathscr{F}(x) = \{y \in X | o(x) = o(y)\}$$

for all $x \in X$. Then $\mathscr{F}(a) = \mathscr{F}(b) = \mathscr{F}(c) = X$. It is check that $X \lhd_q X$. Hence $(\mathscr{F}, X \setminus \{0\})$ is a whole q-idealistic soft BCI-algebra over X. Let $\mathscr{G} : \{0\} \to \mathscr{P}(X)$ be a set-valued function defined by $\mathscr{G}(x) = x$ for all $x \in \{0\}$. Then $\mathscr{G}(0) = \{0\}$. It is check that $\{0\} \lhd_q X$. Hence $(\mathscr{G}, \{0\})$ is a trivial q-idealistic soft BCI-algebra over X.

Definition 4.14.([10]) A fuzzy set μ in X is a fuzzy q-ideal of X if it satisfies the following assertions:

(i) $(\forall x \in X)(\mu(0) \geq \mu(x))$,
(ii) $(\forall x, y, z \in X)(\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\})$.

Lemma 4.15. *A fuzzy set μ in X is a fuzzy q-ideal of X if and only if it satisfies:*

$$(\forall t \in [0, 1])(U(\mu; t) \neq \emptyset \Rightarrow U(\mu; t) \text{ is a } q\text{-ideal of } X).$$

Proof. Straightforward. □

Theorem 4.16. *For every fuzzy q-ideal of X, there exists a q-idealistic soft BCI-algebra $(\mathscr{F}, A)$ over X.*

Proof. Let μ be a fuzzy q-ideal of X. Then $U(\mu; t) := \{x \in X | \mu(x) \geq t\}$ is a q-ideal of X for all $t \in Im(\mu)$. If we take $A = Im(\mu)$ and consider a set-valued function $\mathscr{F} : A \to \mathscr{P}(X)$ given by $\mathscr{F}(t) = U(\mu; t)$ for all $t \in A$, then $(\mathscr{F}, A)$ is a q-idealistic soft BCI-algebra over X. □

Conversely, the following theorem is straightforward.

Theorem 4.17. *For any fuzzy set μ in X, if a q-idealistic soft BCI-algebra $(\mathscr{F}, A)$ over X is given by $A = Im(\mu)$ and $\mathscr{F}(t) = U(\mu; t)$ for all $t \in A$, then μ is a fuzzy q-ideal of X.*

Let μ be a fuzzy set in X and let $(\mathscr{F}, A)$ be a soft set over X in which $A = Im(\mu)$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is a set-valued function defined by

(4.1) $(t \in A)(\mathscr{F}(t) = \{x \in X | \mu(x) + t > 1\})$.

Then there exists $t \in A$ such that $\mathscr{F}(t)$ is not a q-ideal of X as seen in the following example.

Example 4.18. For any BCI-algebra X, define a fuzzy set μ in X by $\mu(0) = t_0 < 0.5$ and $\mu(x) = 1 - t_0$ for all $x \neq 0$. Let $A = Im(\mu)$ and $\mathscr{F} : A \to \mathscr{P}(X)$ be a set-valued function given by (4.1). Then $\mathscr{F}(1 - t_0) = X \backslash \{0\}$, which is not a q-ideal of X.

Theorem 4.19. *Let μ be a fuzzy set in X and let $(\mathscr{F}, A)$ be a soft over X in which $A = [0, 1]$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is given by (4.1). Then the following assertions are equivalent:*

(1) *μ is a fuzzy q-ideal of X,*
(2) *for every $t \in A$ with $\mathscr{F}(t) \neq \emptyset$, $\mathscr{F}(t)$ is a q-ideal of X.*

Proof. Assume that μ is a fuzzy q-ideal of X. Let $t \in A$ be such that $\mathscr{F}(t) \neq \emptyset$. If we select $x \in \mathscr{F}(t)$, then $\mu(0) + t \geq \mu(x) + t > 1$, and so $0 \in \mathscr{F}(t)$. Let $t \in A$ and $x, y, z \in X$ be such that $y \in \mathscr{F}(t)$ and $x * (y * z) \in \mathscr{F}(t)$. Then $\mu(y) + t > 1$ and $\mu(x * (y * z)) + t > 1$. Since μ is a fuzzy q-ideal of X, it follows that

$$\begin{aligned} \mu(x * z) + t &\geq \min\{\mu(x * (y * z)), \mu(y)\} + t \\ &= \min\{\mu(x * (y * z)) + t, \mu(y) + t\} \\ &> 1, \end{aligned}$$

so that $x * z \in \mathscr{F}$. Hence $\mathscr{F}(t)$ is a q-ideal of X with $\mathscr{F}(t) \neq \emptyset$.

Conversely, suppose that (2) is valid. If there exists $a \in X$ such that $\mu(0) < \mu(a)$, then we can select $t_a \in A$ such that $\mu(0) + t_a \leq 1 < \mu(a) + t_a$. It follows that $a \in \mathscr{F}(t_a)$ and $0 \notin \mathscr{F}(t_a)$, which is a contradiction. Hence $\mu(0) \geq \mu(x)$ for all $x \in X$. Now, assume that

$$\mu(a * c) < \min\{\mu(a * (b * c)), \mu(b)\}$$

for some $a, b, c \in X$. Then

$$\mu(a * c) + s_0 \leq 1 < \min\{\mu(a * (b * c)), \mu(b)\} + s_0,$$

for some s_0, which implies $a * (b * c) \in \mathscr{F}(s_0)$ and $b \in \mathscr{F}(s_0)$, but $a * c \in \mathscr{F}(s_0)$. This is a contradiction. Therefore

$$\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\},$$

for all $x, y, z \in X$, and thus μ is a fuzzy q-ideal of X. □

Corollary 4.20. *Let μ be a fuzzy set in X such that $\mu(x) > 0.5$ for some $x \in X$, and let $(\mathscr{F}, A)$ be a soft set over X in which*

$$A := \{t \in Im(\mu) | t > 0.5\}$$

and $\mathscr{F} : A \to \mathscr{P}(X)$ is given by (4.1). If μ is a fuzzy q-ideal of X, then $(\mathscr{F}, A)$ is a q-idealistic soft BCI-algebra over X.

Proof. Straightforward. □

Theorem 4.21. Let μ be a fuzzy set in X and let $(\mathscr{F}, A)$ be a soft set over X in which $A = (0.5, 1]$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is defined by

$$(\forall t \in A)(\mathscr{F}(t) = U(\mu; t)).$$

Then $\mathscr{F}(t)$ is a q-ideal of X for all $t \in A$ with $\mathscr{F}(t) \neq \emptyset$ if and only if the following assertions are valid:

(1) $(\forall x \in X)(\max\{\mu(0), 0.5\} \geq \mu(x))$,
(2) $(\forall x, y, z \in X)(\max\{\mu(x * z), 0.5\} \geq \min\{\mu((x * (y * z)), \mu(y)\})$.

Proof. Assume that $\mathscr{F}(t)$ is a q-ideal of X for all $t \in A$ with $\mathscr{F}(t) \neq \emptyset$. If there exists $x_0 \in X$ such that $\max\{\mu(0), 0.5\} < \mu(x_0)$, then we can select $t_0 \in A$ such that $\max\{\mu(0), 0.5\} < t_0 \leq \mu(x_0)$. It follows that $\mu(0) < t_0$, so that $x_0 \in \mathscr{F}(t_0)$ and $0 \notin \mathscr{F}(t_0)$. This is a contradiction, and so (1) is valid. Suppose that there exist $a, b, c \in X$ such that

$$\max\{\mu(a * c), 0.5\} < \min\{\mu(a * (b * c)), \mu(b)\}.$$

Then

$$\max\{\mu(a * c), 0.5\} < u_0 \leq \min\{\mu(a * (b * c)), \mu(b)\}.$$

for some $u_0 \in A$. Thus $a * (b * c) \in \mathscr{F}(u_0)$ and $b \in \mathscr{F}(u_0)$, but $a * c \notin \mathscr{F}(u_0)$. This is a contradiction, and so (2) is valid.

Conversely, suppose that (1) and (2) are valid. Let $t \in A$ with $\mathscr{F}(t) \neq \emptyset$. For any $x \in \mathscr{F}(t)$, we have

$$\max\{\mu(0), 0.5\} \geq \mu(x) \geq t > 0.5$$

and so $\mu(0) \geq t$, i.e., $0 \in \mathscr{F}(t)$. Let $x, y, z \in X$ be such that $y \in \mathscr{F}(t)$ and $x * (y * z) \in \mathscr{F}(t)$. Then $\mu(y) \geq t$ and $\mu(x * (y * z)) \geq t$. It follows from the second condition that

$$\max\{\mu(x * z), 0.5\} \geq \min\{\mu(x * (y * z)), \mu(y)\} \geq t > 0.5,$$

so that $\mu(x * z) \geq t$, i.e., $x * z \in \mathscr{F}(t)$. Therefore $\mathscr{F}(t)$ is a q-ideal of X for all $t \in A$ with $\mathscr{F}(t) \neq \emptyset$. $\square$

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Convergence of parallel multisplitting USAOR methods for block $H-$matrices linear systems

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Abstract: In this paper, We present parallel multisplitting blockwise relaxation methods for solving the large sparse blocked linear systems, which come from the discretizations of many discrential equations, and study the convergence of our methods associated with USAOR multisplitting when the coefficient matrices of the blocked linear systems are block H-matrices. A lot of numerical experiments show that our methods are applicable and efficient.

Key words: Block matrix multisplitting; Blockwise relaxation parallel multisplitting method; Convergence; Block H-matrix.,
2000 MR Subject Classification: 65F10, 65F50

1. Introduction

For the linear system

$$Ax = b, \tag{1.1}$$

where A is an $n \times n$ square matrix, and x and b are n-dimensional vectors. O'Leary and White [6]invented the matrix multisplitting method in 1985 for solving parallely the large sparse linear systems on the multiprocessor systems and was further studied by many authors. For example, Neumann and Plemmons [5] developed some more refined convergence results for one of the cases considered in [6], Elsner [7] established the comparison theorems about the asymptotic convergence rate of this case, Frommer and Mayer [8] discussed the successive overrelaxation (SOR) method in the sense of multisplitting, White [9,10] studied the convergence properties of the above matrix multisplitting methods for the symmetric positive definite matrix class, as well as matrix multisplitting methods as preconditioners, respectively, Bai [4] established the convergence theory of a class of asynchronous multisplitting blockwise relaxation methods, Zhang, Huang, et, al. [3] present local relaxed parallel multisplitting method and global relaxed parallel multisplitting method for H-matrices and so on. On the other hand, Since the finite element or the finite difference discretizations of many partial differential equations usually result in the large sparse systems of linear equations of regularly blocked structures, recently, [1,4] further generalized the matrix multisplitting concept of O'Leary and White [6] to a blocked form and proposed a class of parallel matrix multisplitting blockwise relaxation methods. This class of methods, besides enjoying all the advantages of the existing pointwise parallel matrix multisplitting methods discussed in [6,12], possesses better convergence properties and robuster numerical behaviours. Therefore, the parallel matrix multisplitting blockwise relaxation methods for the solution of large and sparse nonsingular blocked linear system have become more and more obvious.

In the following, we recall the mathematical descriptions of the blocked linear system and the BMM introduced in [1,4].

Let $N(\leq n)$ and $n_i(\leq n)(i = 1,2,\ldots,N)$ be given positive integers satisfying $\sum_{i=1}^{N} n_i = n$, and denote

$$V_n(n_1,\ldots,n_N) = \{x \in \mathbb{R}^n | x = (x_1^T,\ldots,x_N^T)^T, x_i \in \mathbb{R}^{n_i}\},$$

$$\mathbb{L}_n(n_1,\ldots,n_N) = \{A \in \mathbb{R}^{n\times n} | A = (A_{ij})_{N\times N}, A_{ij} \in \mathbb{R}^{n_i \times n_j}\},$$

When the context is clear we will simply use $\mathbb{L}_n$ for $\mathbb{L}_n(n_1,\ldots,n_N)$ and V_n for $V_n(n_1,\ldots,n_N)$. Then, the blocked linear system to be solved can be expressed as the form

$$Ax = b, \quad A \in \mathbb{L}_n, \quad x, b \in V_n \tag{1.2}$$

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where $A \in \mathbb{L}_n$ is nonsingular and $b \in V_n$ are general known coefficient matrix and right-hand vector, respectively, and $x \in V_n$ is the unknown vector.

If blocked matrices $M_k, N_k, E_k \in \mathbb{L}_n (k = 1, 2, \ldots, \alpha)$ satisfy

1. $A = M_k - N_k$, M_k nonsingular, $k = 1, 2, \ldots, \alpha$,
2. $E_k = diag(E_{11}^{(k)}, \ldots, E_{NN}^{(k)})$, $k = 1, 2, \ldots, \alpha$,
3. $\sum_{k=1}^{\alpha} \|E_{ii}^{(k)}\| = 1$, $i = 1, 2, \ldots, N$,

then we call the collection of triples $(M_k, N_k, E_k)(k = 1, 2, \ldots, \alpha)$ is a BMM of the blocked matrix $A \in \mathbb{L}_n$, where $\|\cdot\|$ denotes the consistent matrix norm.

Suppose that we have a multiprocessor with α processors connected to a host processor, that is, the same number of processors as splittings, and that all processors have the last update vector x^k, then the kth processor only computes those entries of the vector

$$M_k^{-1} N_k x^k + M_k^{-1} b,$$

which correspond to the block diagonal entries $E_{ii}^{(k)}$ of the blocked matrix E_k. The processor then scales these entries so as to be able to deliver the vector

$$E_K (M_k^{-1} N_k x^k + M_k^{-1} b)$$

to the host processor, performing the parallel multisplitting scheme

$$x^{m+1} = \sum_{k=1}^{\alpha} E_K M_k^{-1} N_k x^m + \sum_{k=1}^{\alpha} E_K M_k^{-1} b = H x^m + G b, \ m = 0, 1, 2, \ldots$$

Under reasonable restrictions on the relaxation parameters and the multiple splittings, we establish local parallel multisplitting blockwise relaxation method, global parallel multisplitting blockwise relaxation method and global nonstationary parallel multisplitting blockwise relaxation method for solving the large sparse blocked linear systems and study the convergence of our methods associated with USAOR multisplitting when the coefficient matrices of the blocked linear systems are block H-matrices.

2. Establishments of the methods

Given a positive integer $\alpha (\alpha \leq N)$, we separate the number set $\{1, 2, \ldots N\}$ into a nonempty subsets $J_k (k = 1, 2, \ldots, \alpha)$ such that $J_k \subseteq \{1, 2, \ldots, N\}$ and $\bigcup_{k=1}^{\alpha} J_k = \{1, 2, \ldots, N\}$.

Note that there may be overlappings among the subsets $J_1, J_2, \ldots, J_\alpha$. Corresponding to this separation, we introduce matrices

$D = \text{diag}(A_{11}, \ldots, A_{NN}) \in \mathbb{L}_n$,

$L_k = (\mathscr{L}_{ij}^{(k)}) \in \mathbb{L}_n, \quad \mathscr{L}_{ij}^{(k)} = \begin{cases} L_{ij}^{(k)} & \text{for } i, j \in J_k \text{ and } i > j, \\ 0 & \text{otherwise,} \end{cases}$

$U_k = (\mathscr{U}_{ij}^{(k)}) \in \mathbb{L}_n, \quad \mathscr{U}_{ij}^{(k)} = \begin{cases} U_{ij}^{(k)} & \text{for } i \neq j, \\ 0 & \text{otherwise,} \end{cases}$

$E_k = \text{diag}(E_{11}^{(k)}, \ldots, E_{NN}^{(k)}) \in \mathbb{L}_n, \quad E_{ii}^{(k)} = \begin{cases} E_{ii}^{(k)} & \text{for } i \in J_k, \\ 0 & \text{otherwise,} \end{cases} \quad i, j = 1, 2, \ldots, N; \quad k = 1, 2, \ldots, \alpha.$

Obviously, D is a blocked diagonal matrix, $L_k (k = 1, 2, \ldots, \alpha)$ are blocked strictly lower triangular matrices, $U_k (k = 1, 2, \ldots, \alpha)$ are general blocked matrices, and $E_k (k = 1, 2, \ldots, \alpha)$ are blocked diagonal matrices. If they satisfy

1. D is nonsingular;
2. $A = D - L_k - U_k$, $k = 1, 2, \ldots, \alpha$;
3. $\sum_{k=1}^{\alpha} E_k = I$,

then the collection of triples $(D - L_k, U_k, E_k)$ and $(D - U_k, L_k, E_k)$ $(k = 1, 2, \ldots, \alpha)$ are BMM of the blocked matrix $A \in \mathbb{L}_n$. Here, I denotes the identity matrix of order $n \times n$.

We will present *local parallel multisplitting blockwise relaxation USAOR method (LBUSAOR)* and *global parallel multisplitting blockwise relaxation USAOR method (GBUSAOR)*.

Algorithm 2.1. (local parallel multisplitting blockwise relaxation method)
Given the initial vector
For $m = 0, 1, 2, \ldots$ repeat (I) and (II), until convergence.
(I) For $k = 1, 2, \ldots, \alpha$, (parallel) solving y_k:

$$M_k y_k = N_k x^m + b.$$

(II) Computing

$$x^{m+1} = \sum_{k=1}^{\alpha} E_k y_k.$$

Algorithm 2.1 associated with LBUSAOR method can be written as

$$x^{m+1} = H_{LBUSAOR} x^m + G_{LBUSAOR} b, \quad m = 0, 1, \cdots, \tag{2.1}$$

where

$$\begin{array}{rcl} H_{LBUSAOR} & = & \sum_{k=1}^{\alpha} E_k U_{\omega_2 r_2}(k) L_{\omega_1 r_1}(k), \\ U_{\omega_2 r_2}(k) & = & (D - r_2 U_k)^{-1}\{(1-\omega_2)D + (\omega_2 - r_2)U_k + \omega_2 L_k\}, \\ L_{\omega_1 r_1}(k) & = & (D - r_1 L_k)^{-1}\{(1-\omega_1)D + (\omega_1 - r_1)L_k + \omega_1 U_k\}, \\ G_{LBUSAOR} & = & \sum_{k=1}^{\alpha} E_k (D - r_2 U_k)^{-1}\{(\omega_1 + \omega_2 - \omega_1\omega_2)D + \omega_2(\omega_1 - r_1)L_k \\ & & +\omega_1(\omega_2 - r_2)U_k\}(D - r_1 L_k)^{-1} \end{array} \tag{2.2}$$

By using a suitable positive relaxation parameter β, we will establish global parallel multisplitting blockwise relaxation USAOR method which is based on Algorithm 2.1.

Algorithm 2.2. (global parallel multisplitting blockwise relaxation method)
Given the initial vector
For $m = 0, 1, 2, \ldots$ repeat (I) and (II), until convergence.
(I) For $k = 1, 2, \ldots, \alpha$, (parallel) solving y_k:

$$M_k y_k = N_k x^m + b.$$

(II) Computing

$$x^{m+1} = \beta \sum_{k=1}^{\alpha} E_k y_k + (1-\beta) x^m.$$

Algorithm 2.2 associated with GBUSAOR method can be written as

$$x^{m+1} = H_{GBUSAOR} x^m + \beta G_{LBUSAOR} b, \quad m = 0, 1, \cdots, \tag{2.3}$$

where $H_{GBUSAOR} = \beta H_{LBUSAOR} + (1-\beta) I$.

In the standard multisplitting method each local approximation is updated exactly once using the same previous iterate x^m. On the other hand, it is possible to update the local approximations more than once, using different iterates computed earlier. In this case, we call this method a nonstationary multisplitting method [15,16,17]. The main idea of the nonstationary method is that at the mth iteration each processor k solves the system $q(m,k)$ times, using in each time the new calculated vector to update the right-hand side; i.e., we have the following algorithm:

Algorithm 2.3. (global nonstationary parallel multisplitting blockwise relaxation method)
Given the initial vector
For $m = 0, 1, 2, \ldots$ repeat (I) and (II), until convergence.
(I) For $i = 1, 2, \ldots, q(m,k)$, (parallel) solving $y_k^{(i)}$:

$$M_k y_k^{(i)} = N_k y_k^{(i-1)} + b.$$

(II) Computing

$$x^{m+1} = \beta \sum_{k=1}^{\alpha} E_k y_k^{q(m,k)} + (1-\beta)x^m.$$

Algorithm 2.3 associated with GNBUSAOR method can be written as

$$x^{m+1} = H_{GNBUSAOR} x^m + \beta G_{GNBUSAOR} b, \ \ m = 0, 1, \cdots, \tag{2.4}$$

where

$$\begin{array}{rcl} H_{GNBUSAOR} & = & \beta \sum_{k=1}^{\alpha} E_k (P_{\omega r} Q_{\xi\eta})^{q(m,k)} + (1-\beta)I \\ P_{\omega r} & = & (D - r_k U_k)^{-1}\{(1-\omega_k)D + (\omega_k - r_k)U_k + \omega_k L_k\} = M_r^{-1} N_{\omega r}, \\ Q_{\xi\eta} & = & (D - \eta_k L_k)^{-1}\{(1-\xi_k)D + (\xi_k - \eta_k)L_k + \xi_k U_k\} = M_\eta^{-1} N_{\xi\eta}, \\ G_{GNBUSAOR} & = & \beta \sum_{k=1}^{\alpha} E_k \sum_{i=1}^{q(m,k)-1} (M_\eta M_r)^{-1} (N_{\omega r} N_{\xi\eta})^i (M_\eta M_r)^{-1} \omega_k \xi_k. \end{array} \tag{2.5}$$

It follows that when $q(m,k) = 1$, $\omega_k = \omega_2$, $r_k = r_2$, $\xi_k = \omega_1$ and $\eta_k = r_1$ for $1 < k < \alpha$, $m = 0, 1, 2, \ldots$, Algorithm 2.3 reduces to Algorithm 2.2.

3. Preliminaries

We shall use the following notations and lemmas. A matrix $A = (a_{ij})$ is called a *Z-matrix* if for any $i \neq j$, $a_{ij} \leq 0$. A *Z-matrix* is a nonsingular *M-matrix* if A is nonsingular and if $A^{-1} \geq 0$. Additionally, we denote the spectral radius of A by $\rho(A)$. It is well-known that if $A \geq 0$ and there exists a vector $x > 0$ such that $Ax < \alpha x$, then $\rho(A) < \alpha$. Let

$$\mathbb{L}_{n,I}(n_1, \ldots, n_N) = \{M = (M_{ij}) \in \mathbb{L}_n | M_{ii} \in \mathbb{R}^{n_i \times n_i} \text{ nonsingular}, \ i = 1, \ldots, N\}.$$

We will review the concepts of strictly block diagonally dominant matrix and the block H-matrix. Let $A \in \mathbb{L}_{n,I}$. Then its block comparison matrix $\langle A \rangle$ is defined by

$$\langle A \rangle_{ij} = \begin{cases} \|A_{ij}^{-1}\|^{-1}, & i = j, \\ -\|A_{ij}\|, & i \neq j, \end{cases} \qquad i, j = 1, \ldots, N$$

where $\|\cdot\|$ is a consistent matrix norm. If

$$\|A_{ii}^{-1}\|^{-1} > \sum_{i \neq j} \|A_{ij}\|, \ \ j = 1, 2, \ldots, N.$$

Then A is said to be a strictly block diagonally dominant matrix [13], if there exists a positive diagonally matrix X such that AX is a strictly block diagonally dominant matrix, then A is said to be a block H-matrix [14]. Clearly, a strictly block diagonally dominant matrix is certainly a block H-matrix.

Definition 3.1 [1]. Let $M \in \mathbb{L}_n$. We call $[M] = (\|M_{ij}\|) \in \mathbb{R}^{N \times N}$ the block absolute value of the blocked matrix M. The block absolute value $[x] \in \mathbb{R}^N$ of a blocked vector $x \in V_n$ is defined in an analogous way.

These kinds of block absolute values have the following important properties.

Lemma 3.1 [1]. *Let $L, M \in \mathbb{L}_n$, $x, y \in V_n$ and $r \in \mathbb{R}^1$. Then*
1. $|[L] - [M]| \leq [L + M] \leq [L] + [M]$ $(|[x] - [y]| \leq [x + y] \leq [x] + [y])$;
2. $[LM] \leq [L][M]$ $([xy] \leq [x][y])$;
3. $[rM] \leq |r|[M]$ $([rx] \leq |r|[x])$;
4. $\rho(M) \leq \rho(|M|) \leq \rho([M])$ *(here, $\|\cdot\|$ is either $\|\cdot\|_\infty$ or $\|\cdot\|_1$).*

Lemma 3.2 [1]. *Let $M \in \mathbb{L}_{n,I}$ be a strictly block diagonally dominant matrix, then*
1. M is nonsingular;
2. $[(M)^{-1}] \leq \langle M \rangle^{-1}$;
3. $\rho(J(\langle M \rangle)) < 1$.

4. Convergence

For Algorithms 1, 2 and 3, we give convergence theorems for block diagonally dominant matrices and block $H-$matrices.

Theorem 4.1. *Let A be a strictly block diagonally dominant matrix, and the collection of triples $(D-L_k,U_k,E_k)$ and $(D-U_k,L_k,E_k)$ $(k=1,2,\ldots,\alpha)$ are BMM of the blocked matrix $A\in\mathbb{L}_n$. Assume that*

$$\langle A\rangle=\langle D\rangle-[L_k]-[U_k]=\langle D\rangle-[B],\quad k=1,2,\ldots,\alpha,\quad 0<\omega_1,\omega_2<\frac{2}{1+\rho},\quad 0\le r_1\le\omega_1,\quad 0\le r_2\le\omega_2,$$

then LBUSAOR method converges for any initial vector $x^0\in V_N$, where $\rho=\rho(J(\langle A\rangle))=\rho(\langle D\rangle^{-1}[B])$.

Proof. By Lemma 3.1, we know $\rho(H_{LBUSAOR})\le\rho(|H_{LBUSAOR}|)\le\rho([H_{LBUSAOR}])$, and then, the iteration (2) converges for any initial vector $x^0\in V_N$ if and only if $\rho([H_{LBUSAOR}])<1$. Let $B=L_k+U_k$, by (7), we know that $[B]=[L_k]+[U_k]$, clearly, $D-rL_k$ $(k=1,2,\ldots,\alpha)$ are strictly block diagonally dominant matrix for $r>0$, and $\langle D\rangle-r[B]$, $(k=1,2,\ldots,\alpha)$ are strictly block diagonally dominant matrix for $0<r<\frac{2}{1+\rho}$ which follow from A is a strictly block diagonally dominant matrix. Since $\langle D\rangle-r[B]\le\langle D\rangle-r[U_k]\le\langle D\rangle$ for $0<r<\frac{2}{1+\rho}$, $k=1,2,\ldots,\alpha$, and $\langle A\rangle$ is a strictly diagonally dominant matrix, we have $\langle D\rangle-r[B]$ and $\langle D\rangle$ are strictly diagonally dominant M-matrices, for $0<r<\frac{2}{1+\rho}$, $k=1,2,\ldots,\alpha$. Therefore, $\langle D\rangle-r[U_k]$ are strictly diagonally dominant M-matrices, and then $D-rU_k$ are strictly block diagonally dominant matrices, for $0<r<\frac{2}{1+\rho}$, $k=1,2,\ldots,\alpha$.

Let $\bar{L_k}=D^{-1}L_k$ and $\bar{U_k}=D^{-1}U_k$, then $I-r\bar{L_k}$ and $I-r\bar{U_k}$ are also strictly block diagonally dominant matrices, for $0<r<\frac{2}{1+\rho}$, $k=1,2,\ldots,\alpha$. Thus, by Lemma 3.1 and (7), we have

$$[(I-r_1\bar{L_k})^{-1}]\le(\langle I-r_1\bar{L_k}\rangle)^{-1}=(I-r_1[\bar{L_k}])^{-1},$$
$$[(I-r_2\bar{U_k})^{-1}]\le(\langle I-r_2\bar{U_k}\rangle)^{-1}=(I-r_2[\bar{U_k}])^{-1}.$$

From (3), we have

$$\begin{aligned}[U_{\omega_2r_2}(k)] &= [(D-r_2U_k)^{-1}\{(1-\omega_2)D+(\omega_2-r_2)U_k+\omega_2L_k\}]\\ &= [(I-r_2\bar{U_k})^{-1}\{(1-\omega_2)I+(\omega_2-r_2)\bar{U_k}+\omega_2\bar{L_k}\}]\\ &\le (I-r_2[\bar{U_k}])^{-1}\{|1-\omega_2|I+(\omega_2-r_2)[\bar{U_k}]+\omega_2[\bar{L_k}]\}\\ &= I+(I-r_2[\bar{U_k}])^{-1}\{(|1-\omega_2|-1)I+\omega_2([\bar{U_k}]+[\bar{L_k}])\}.\end{aligned}$$

Since $\bar{L_k}=D^{-1}L_k$ and $\bar{U_k}=D^{-1}U_k$, we have $[\bar{L_k}]\le\langle D\rangle^{-1}[L_k]$ and $[\bar{U_k}]\le\langle D\rangle^{-1}[U_k]$ which follow from Lemma 3.1 and Lemma 3.2, and then

$$[\bar{U_k}]+[\bar{L_k}]\le\langle D\rangle^{-1}[U_k+L_k]=\langle D\rangle^{-1}[B]=J(\langle A\rangle),\ \ k=1,2,\ldots,\alpha.$$

Therefore, we have

$$[U_{\omega_2r_2}(k)]\le I-(I-r_2[\bar{U_k}])^{-1}(I-T(\omega_2)),$$

where $T(\omega_2)=|1-\omega_2|I+\omega_2J(\langle A\rangle)$. Note that $(I-r_2[\bar{U_k}(k)])^{-1}\ge I$, $k=1,2,\ldots,\alpha$, and then

$$[U_{\omega_2r_2}(k)]\le I-(I-T(\omega_2))=T(\omega_2).$$

Similar to the above proving process, we have

$$[L_{\omega_1r_1}(k)]\le I-(I-T(\omega_1))=T(\omega_1),$$

where $T(\omega_1)=|1-\omega_1|I+\omega_1J(\langle A\rangle)$.

Let e denotes the vector $e=(1,1,\ldots,1)^T\in V_N$ and $J_\epsilon(\langle A\rangle)=J(\langle A\rangle)+\epsilon ee^T$, since $J(\langle A\rangle)$ is nonnegative, the matrix $J_\epsilon(\langle A\rangle)$ has only positive entries and irreducible for any $\epsilon>0$. By the Perron-Frobenius theorem for any $\epsilon>0$, there is a vector $x_\epsilon>0$ such that

$$J_\epsilon(\langle A\rangle)x_\epsilon=\rho(J_\epsilon(\langle A\rangle))x_\epsilon=\rho_\epsilon x_\epsilon,$$

where $\rho_\epsilon = \rho(J_\epsilon(\langle A\rangle))$. Moreover, if $\epsilon > 0$ is small enough, we have $\rho_\epsilon < 1$ by continuity of the spectral radius. Thus, we can get

$$|1-\omega_i| + \omega_i\rho_\epsilon < 1 \text{ for } 0 < \omega_i < \tfrac{2}{1+\rho},\ i = 1,2,$$

and then

$$\begin{array}{rcl}[H_{LBUSAOR}]x_\epsilon & \leq & \sum_{k=1}^{\alpha}[E_k][U_{\omega_2 r_2}(k)][L_{\omega_1 r_1}(k)]x_\epsilon \leq \sum_{k=1}^{\alpha}[E_k]T(\omega_2)T(\omega_1)x_\epsilon \\ & \leq & (|1-\omega_2|I + \omega_2 J_\epsilon(\langle A\rangle))(|1-\omega_1|I + \omega_1 J_\epsilon(\langle A\rangle))x_\epsilon \\ & = & (|1-\omega_1| + \omega_1\rho_\epsilon)(|1-\omega_2| + \omega_2\rho_\epsilon)x_\epsilon \\ & < & x_\epsilon,\end{array}$$

then $[H_{LBUSAOR}]x_\epsilon < x_\epsilon$ and $\rho([H_{LBUSAOR}]) < 1$. □

Theorem 4.2. *Let A be a block H-matrix, and the collection of triples $(D-L_k, U_k, E_k)$ and $(D-U_k, L_k, E_k)$ $(k=1,2,\ldots,\alpha)$ are BMM of the blocked matrix $A \in \mathbb{L}_n$. Assume that*

$$\langle A\rangle = \langle D\rangle - [L_k] - [U_k] = \langle D\rangle - [B], \quad k = 1,2,\ldots,\alpha, \quad 0 < \omega_1, \omega_2 < \frac{2}{1+\rho}, \quad 0 \leq r_1 \leq \omega_1, \quad 0 \leq r_2 \leq \omega_2,$$

then LBUSAOR method converges for any initial vector $x^0 \in V_N$, where $\rho = \rho(J(\langle A\rangle)) = \rho(\langle D\rangle^{-1}[B])$.

Proof. Since A is a block H-matrix, then, there exists a positive diagonally matrix X such that AX is a strictly block diagonally dominant matrix. Let $H_{LBUSAOR}(A)$ denote the iteration matrix of local parallel multisplitting blockwise relaxation method for blocked matrix A and $H_{LBUSAOR}(AX)$ denote the iteration matrix of local parallel multisplitting blockwise relaxation method for blocked matrix AX, respectively. By simple calculation, we have $H_{LBUSAOR}(A)$ and $H_{LBUSAOR}(AX)$ are similar. Since similar matrices have the same eigenvalues, it follows that $\rho(H_{LBUSAOR}(A)) = \rho(H_{LBUSAOR}(AX)) < 1$. □

Using GBUSAOR method, we can also get the following convergence results.

Theorem 4.3. *Let A be a strictly block diagonally dominant matrix, and the collection of triples $(D-L_k, U_k, E_k)$ and $(D-U_k, L_k, E_k)$ $(k=1,2,\ldots,\alpha)$ are BMM of the blocked matrix $A \in \mathbb{L}_n$. Assume that*

$$\langle A\rangle = \langle D\rangle - [L_k] - [U_k] = \langle D\rangle - [B],\ k = 1,2,\ldots,\alpha,$$

if

$$0 < \omega_1, \omega_2 < \frac{2}{1+\rho}, \quad 0 \leq r_1 \leq \omega_1, \quad 0 \leq r_2 \leq \omega_2, \quad 0 < \beta < \frac{2}{1+\theta^2},$$

then GBUSAOR method converges for any initial vector $x^0 \in V_N$, where $\rho = \rho(J(\langle A\rangle)) = \rho(\langle D\rangle^{-1}[B])$ and

$$\theta = max\{|1-\omega_1| + \omega_1\rho,\ |1-\omega_2| + \omega_2\rho\}.$$

Proof. Since $\rho(H_{GBUSAOR}) \leq \rho(|H_{GBUSAOR}|) \leq \rho([H_{GBUSAOR}])$, the iteration (4) converges for any initial vector $x^0 \in V_N$ if and only if $\rho([H_{GBUSAOR}]) < 1$. Let

$$\theta_\epsilon = max\{|1-\omega_1| + \omega_1\rho_\epsilon,\ |1-\omega_2| + \omega_2\rho_\epsilon\},$$

similar to the proof of Theorem 4.1, we have

$$\begin{array}{rcl}[H_{GBUSAOR}]x_\epsilon & \leq & \beta[\{|1-\omega_2|I + \omega_2 J_\epsilon(\langle A\rangle)\}\{|1-\omega_1|I + \omega_1 J_\epsilon(\langle A\rangle)\}]x_\epsilon + |1-\beta|x_\epsilon \\ & = & \beta(|1-\omega_1| + \omega_1\rho_\epsilon)\{|1-\omega_2|I + \omega_2 J_\epsilon(\langle A\rangle)\}x_\epsilon + |1-\beta|x_\epsilon \\ & = & \beta(|1-\omega_1| + \omega_1\rho_\epsilon)(|1-\omega_2| + \omega_2\rho_\epsilon)x_\epsilon + |1-\beta|x_\epsilon \\ & \leq & (\beta\theta^2 + |1-\beta|)x_\epsilon \\ & = & (\beta\theta_\epsilon^2 + |1-\beta|)x_\epsilon \\ & < & x_\epsilon\end{array}$$

then $[H_{GBUSAOR}]x_\epsilon < x_\epsilon$ and $\rho([H_{GBUSAOR}]) < 1$. □

Theorem 4.4. *Let A be a block H-matrix, and the collection of triples $(D - L_k, U_k, E_k)$ and $(D - U_k, L_k, E_k)$ $(k = 1, 2, \ldots, \alpha)$ are BMM of the blocked matrix $A \in \mathbb{L}_n$. Assume that*

$$\langle A\rangle = \langle D\rangle - [L_k] - [U_k] = \langle D\rangle - [B], \quad k = 1, 2, \ldots, \alpha, \tag{4.1}$$

if

$$0 < \omega_1, \omega_2 < \frac{2}{1+\rho}, \quad 0 \le r_1 \le \omega_1, \quad 0 \le r_2 \le \omega_2, \quad 0 < \beta < \frac{2}{1+\theta^2},$$

then GBUSAOR method converges for any initial vector $x^0 \in V_N$, where $\rho = \rho(J(\langle A\rangle)) = \rho(\langle D\rangle^{-1}[B])$ and

$$\theta = max\{|1-\omega_1| + \omega_1\rho,\ |1-\omega_2| + \omega_2\rho\}.$$

Proof. Similar to the proof of Theorem 4.2, we can prove Theorem 4.4. □

Remark. As some special case, for local parallel multisplitting blockwise relaxation USAOR method (LBUSAOR) and global parallel multisplitting blockwise relaxation USAOR method (GBUSAOR)(see Table 1), we also have the corresponding convergence results, when $\omega_1 = \omega_2$, $r_1 = r_2$, $\omega_1 = \omega_2 = r_1 = r_2$ and so on.

Theorem 4.5. *Let A be a block H-matrix, and the collection of triples $(D - L_k, U_k, E_k)$ and $(D - U_k, L_k, E_k)$ $(k = 1, 2, \ldots, \alpha)$ are BMM of the blocked matrix $A \in \mathbb{L}_n$. Assume that*

$$\langle A\rangle = \langle D\rangle - [L_k] - [U_k] = \langle D\rangle - [B],\ k = 1, 2, \ldots, \alpha,$$

if

$$0 < \omega_k, \xi_k < \frac{2}{1+\rho}, \quad 0 \le r_k \le \omega_k, \quad 0 \le \eta_k \le \xi_k, \quad 0 < \beta < \frac{2}{1+\sigma^2},$$

then GNBUSAOR method converges for any initial vector $x^0 \in V_N$, where $\rho = \rho(J(\langle A\rangle)) = \rho(\langle D\rangle^{-1}[B])$, $q(m,k) \ge 1$, $k = 1, 2, \ldots, \alpha$, $m = 0, 1, 2, \ldots$ and

$$\sigma = \max_{1\le k\le\alpha}\{|1-\omega_k| + \omega_k\rho,\ |1-\xi_k| + \xi_k\rho\}.$$

Proof. We only need show $\rho([H_{GNBUSAOR}] < 1)$, when A is a strictly block diagonally dominant matrix. By the proof of Theorem 4.1, we know that

$$[(I - r_k\bar{L_k})^{-1}] \le (\langle I - r_k\bar{L_k}\rangle)^{-1} = (I - r_k[\bar{L_k}])^{-1},$$

$$[(I - \eta_k\bar{U_k})^{-1}] \le (\langle I - \eta_k\bar{U_k}\rangle)^{-1} = (I - \eta_k[\bar{U_k}])^{-1}.$$

From (6), we have

$$[P_{\omega r}] \le |1-\omega_k|I + \omega_k J_\epsilon(\langle A\rangle$$

and

$$[Q_{\xi\eta}] \le |1-\xi_k|I + \xi_k J_\epsilon(\langle A\rangle).$$

Let

$$\sigma_\epsilon = \max_{1\le k\le\alpha}\{|1-\omega_k| + \omega_k\rho_\epsilon, |1-\xi_k| + \xi_k\rho_\epsilon\}.$$

Similar to the above proving process, we get

$$\begin{array}{rcl}
[H_{GNBUSAOR}]x_\epsilon & \le & \beta[\sum_{k=1}^{\alpha} E_k(P_{\omega r}Q_{\xi\eta})^{q(m,k)}]x_\epsilon + |1-\beta|x_\epsilon \\
& \le & \beta\sum_{k=1}^{\alpha}[E_k]([P_{\omega r}][Q_{\xi\eta}])^{q(m,k)}x_\epsilon + |1-\beta|x_\epsilon \\
& \le & \beta(|1-\omega_k| + \omega_k\rho_\epsilon)(|1-\xi_k| + \xi_k\rho_\epsilon)x_\epsilon + |1-\beta|x_\epsilon \\
& = & (\beta\sigma^2 + |1-\beta|)x_\epsilon \\
& < & x_\epsilon,
\end{array}$$

then $[H_{GNBUSAOR}]x_\epsilon < x_\epsilon$ and $\rho([H_{GNBUSAOR}]) < 1$. □

5. Numerical example

We consider the blocked linear system [1,4]

$$Ax=b, \quad A\in\mathbb{L}_n,\ b\in V_n, \tag{5.1}$$

with $n_1=n_2=\ldots=n_N\equiv N$, and $n=N^2$, where

$$A=\begin{pmatrix} B & -I & & & \\ -I & B & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & B & -I \\ & & & -I & B \end{pmatrix},\quad B=\begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix}$$

and the right hand side vector b is chosen as

$$b^T=(1,\frac{1}{4},\ldots,\frac{1}{n^2})\in V_n.$$

Take $\alpha=2$ and two positive integers m_1 and m_2 satisfying $1<m_2<m_1<N$. Then, corresponding to the number sets $J_1=\{1,2,\ldots,m_1\}$ and $J_2=\{m_2,m_2+1,\ldots,N\}$, we determine BMM $(D-L_k,U_k,E_k)$ and $(D-U_k,L_k,E_k)$, $k=1,2$, of the blocked matrix A in accordance with the following way:

$D=\text{diag}(B,B,\ldots,B)\in\mathbb{L}_n$,

$$L_1=(\mathscr{L}_{ij}^{(1)})\in\mathbb{L}_n,\ \mathscr{L}_{ij}^{(1)}=\begin{cases} I & \text{for } j=i-1 \text{ and } 2\le i\le m_1,\\ 0 & \text{otherwise,}\end{cases}$$

$$U_1=(\mathscr{U}_{ij}^{(1)})\in\mathbb{L}_n,\ \mathscr{U}_{ij}^{(1)}=\begin{cases} I & \text{for } j=i-1 \text{ and } m_1+1\le i\le N,\\ I & \text{for } j=i+1 \text{ and } 1\le i\le N-1,\\ 0 & \text{otherwise,}\end{cases}$$

$$L_2=(\mathscr{L}_{ij}^{(2)})\in\mathbb{L}_n,\ \mathscr{L}_{ij}^{(2)}=\begin{cases} I & \text{for } j=i-1 \text{ and } m_2\le i\le N,\\ 0 & \text{otherwise,}\end{cases}$$

$$U_2=(\mathscr{U}_{ij}^{(2)})\in\mathbb{L}_n,\ \mathscr{U}_{ij}^{(2)}=\begin{cases} I & \text{for } j=i-1 \text{ and } 2\le i\le m_2-1,\\ I & \text{for } j=i+1 \text{ and } 1\le i\le N-1,\\ 0 & \text{otherwise,}\end{cases}$$

$$E_k=\text{diag}(E_{11}^{(k)},E_{22}^{(k)}\ldots,E_{NN}^{(k)})\in\mathbb{L}_n,\ k=1,2,\ E_{ii}^{(1)}=\begin{cases} I & \text{for } 1\le i\le m_2,\\ \frac{1}{2}I & \text{for } m_2\le i\le m_1,\\ 0 & \text{for } m_1<i\le N.\end{cases}\quad E_{ii}^{(2)}=\begin{cases} 0 & \text{for } \le i\le m_2,\\ \frac{1}{2}I & \text{for } m_2\le i\le m_1,\\ I & \text{for } m_1<i\le N.\end{cases}$$

In particular, we select the positive integer pair (m_1,m_2) to be
(a) $m_1=\text{Int}(\frac{2N}{3})$, $m_2=\text{Int}(\frac{N}{3})$;
(b) $m_1=\text{Int}(\frac{4N}{5})$, $m_2=\text{Int}(\frac{N}{5})$,
then we can get two kinds of concrete cases of the weighting matrices E_1 and E_2, here, $\text{Int}(\cdot)$ denotes the integer part of the corresponding real number.

In our numerical experiment, the initial approximation x^0 is taken as

$$x^0=(0.5,\ldots,0.5)^T\in V_n.$$

Let the convergence criterion be $\|x^{k+1}-x^k\|_\infty\le10^{-6}$. In Table 2 and Table 3, we report the number of iterations by IT.

From Table 2 and Table 3, we easily see that the multisplitting blockwise relaxation USAOR methods discussed in this paper substantially have better numerical behaviours than the multisplitting blockwise relaxation AOR methods studied in [1], which shows that our new methods are applicable and efficient.

Table 1: The iterations of LBUSAOR method for N=15

r_1	1	1.05	1.1	[0.92,1.1]	1.1	1.1	1.2
r_2	1.1	1.1	1.1	1.1	[1.0789,1.0881]	1.088	1.2
ω_1	1.406	1.406	1.6	[1.796,1.825]	1.8	1.8	1.9
ω_2	1.2	1.1	1.1	1.1	1.1	[1.096,1.1]	1.15
IT(a)	42	37	36	35	30	30	55
IT(b)	41	37	36	34	31	31	55

Table 2: The iterations of GBUSAOR method for N=15

r_1	1	1.05	1.1	[0.92,1.1]	1.1	1.1	1.2
r_2	1.1	1.1	1.1	1.1	[1.0789,1.0881]	1.088	1.2
ω_1	1.406	1.406	1.6	[1.796,1.825]	1.8	1.8	1.9
ω_2	1.2	1.1	1.1	1.1	1.1	[1.096,1.1]	1.15
β	0.8	1	1.05	1.1	1.2	[1.24,1.26]	1.3
IT(a)	51	37	35	27	25	24	90
IT(b)	52	37	35	28	26	25	64

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The properties and iterative algorithms of circulant matrices *

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Abstract

In this paper we investigate the structure and the iterative algorithms of circulant matrices. Firstly, we show that the reduced form of a circulant matrix A is an H-matrix if the matrix A be a circulant H-matrix, moreover, we can derive that the matrix A is a circulant H-matrix if some conditions are imposed on the reduced form of the matrix A. Secondly, by using the properties of the circulant matrix A, we present two new splittings of circulant M-matrices and obtain some efficacious iterative algorithms for solving a linear system $Ax = b$.

Key words: *Circulant matrix; H-matrix; M-matrix; reduced form; splitting; iterative algorithm*

1. Introduction

Patterned matrices just like circulant matrices, symmetric matrices, Jacobi matrices, centrosymmetric matrices etc. arise in many areas of physics, electromagnetics, signal processing, statistics and applied mathematics for the investigation of problems with periodicity properties. Also, the numerical solutions of certain types of elliptic and parabolic partial differential equations with periodic boundary conditions often involve the linear systems $Ax = b$ with a patterned matrix A [1-3]. The properties of patterned matrices have been extensively investigated [4-6]. For recent years many authors have paid attention to developing iteration algorithms for solving the linear systems with patterned matrices [7-9]. Circulant matrix is a kind of very important patterned matrices.

In this paper we investigate the structure and the iterative algorithms of circulant matrices. Firstly, we discuss the properties of the circulant matrix. We prove that the reduced form of a circulant matrix A is an H-matrix if the matrix A be a circulant H-matrix. Moreover, we derive that the matrix A is a circulant H-matrix if some conditions are imposed on the reduced form of A. Secondly, by means of the properties of the circulant matrix A, we present two new splittings of circulant M-matrices and obtain some efficacious iterative algorithms for solving a linear system $Ax = b$. This paper is organized as follows. In next section, we review some basic definitions and notations. In section 3, we show some properties of circulant matrices. In section 4, a new splitting scheme is constructed, which can be deprived from a random convergent splitting of a circulant matrix A, and two new splittings of the circulant M-matrix and H-matrix are presented. The convergence of the corresponding iterative sequences is also discussed. Finally, on the basis of the opposite triangular splittings and $GMRES$ algorithm, we give three iterative algorithms to solve the linear system $Ax = b$.

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2. Preliminaries

In this section we will review some basic notations which frequently used in the following. Let $a \in R^n$ and $a = (a_0, a_1, \cdots, a_{n-1})^T$. In a circulant matrix

$$Cir(a) := \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}$$

each row is a cyclic shift of the row above to the right. $Cir(a)$ is a special case of a Toeplitz matrix. It is evidently determined by its first row (or column).

Definition 2.1 A matrix $A = (a_{i,j})_{n\times n}$ is called an Z-matrix, if $a_{i,j} \leq 0 (i \neq j)$; an M-matrix if A is an Z-matrix and $A^{-1} \geq 0$. Let $\langle A\rangle = (\alpha_{i,j})_{n\times n}$, if $\alpha_{i,i} = |a_{i,i}|$, $\alpha_{i,j} = -|a_{i,j}| (i \neq j)$, then $\langle A\rangle$ is called a comparision matrix of A. A matrix A is called an H-matrix, if its comparison matrix $\langle A\rangle$ is an M-matrix.

Let $A = M - N$. Then the pairs of matrices (M, N) of A is called: a splitting of A if $\det(M) \neq 0$; a convergent splitting if $\rho(M^{-1}N) < 1$, where $\rho(M^{-1}N)$ denote the spectral radius of the matrix $M^{-1}N$; a regular splitting of A if $M^{-1} \geq 0$ and $N \geq 0$; a weak regular splitting of A if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

Lemma 2.2[2] Let $A = M - N$ be a weak regualr splitting of A, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$.

Now consider the circulant matrices.

Lemma 2.3[1] (1) Let $A \in R^{n\times n}$ be a circulant matrix, then $\langle A\rangle$ is a circulant matrix. Furthermore, if A is nonsingular, then A^{-1} is a circulant matrix; (2) A^T is a circulant matrix; (3) Let $B \in R^{n\times n}$ be a circulant matrix, then $A \pm B$ and AB are also circulant matrices.

Let $G = Cir(0, 1, 0, \cdots, 0) \in R^{n\times n}$.

Lemma 2.4 $A \in R^{n\times n}$ is a circulant matrix if and only if $G^T AG = A$.

All the formulas become slightly more complicated when n is odd. For simplicity, in this paper, when $n = 2m + 1$ we restrict the circulant matrix A to be symmetric, that is

$$A = Cir((a_0, a_1, a_2, \cdots, a_m, a_m, \cdots, a_1)^T). \tag{2.1}$$

Using the partition of the matrix, the circulant matrix can be described as follows.

(i) For the case $n = 2m$, a circulant matrix can be written as the form:

$$A = \begin{pmatrix} B & C \\ C & B \end{pmatrix}, \tag{2.2}$$

where each of the block matrices B and C is an $m \times m$ matrix.

(ii) For the case $n = 2m + 1$, a symmetric circulant matrix can be partitioned into the form:

$$A = \begin{pmatrix} B & J_m a & J_m C J_m \\ a^T J_m & \beta & a^T \\ C & a & J_m B J_m \end{pmatrix}, \tag{2.3}$$

where $J_m = (e_m, e_{m-1}, \cdots, e_1)$, e_i denotes the unit vector with ith entry 1, $B, C \in R^{m\times m}, a \in R^{m\times 1}$ and β is a scalar.

Lemma 2.5 Let A be a circulant matrix, then A is orthogonal similar to a block diagonal matrix. The block diagonal matrix can be described as follows.

(i) For $n = 2m$, let

$$P = \frac{\sqrt{2}}{2}\begin{pmatrix} I_m & I_m \\ -I_m & I_m \end{pmatrix},$$

then

$$P^T AP = \begin{pmatrix} B - C & \\ & B + C \end{pmatrix}, \tag{2.4}$$

where I_m is a mth unit matrix.

(ii) In terms of (2.1) if $n = 2m + 1$, we select the orthogonal matrix

$$P = \frac{\sqrt{2}}{2}\begin{pmatrix} I_m & & I_m \\ & \sqrt{2} & \\ -J_m & & J_m \end{pmatrix},$$

then

$$P^T AP = \begin{pmatrix} B - J_m C & & \\ & \beta & \sqrt{2}a^T J_m \\ & \sqrt{2}J_m a & B + J_m C \end{pmatrix}. \tag{2.5}$$

We call the matrix of the right side of (2.4) or (2.5) the reduced form of the circulant matrix A, corresponding to (2.2) or (2.3) respectively.

Lemma 2.6 Let A and C be M-matrices, if $A \le B \le C$, then B is also an M-matrix.

Lemma 2.7[3] Let $A^{-1} \ge 0$ and

$$A = \widetilde{M}_1 - \widetilde{N}_1 = \widetilde{M}_2 - \widetilde{N}_2$$

be weak regular splittings. In either of the following cases

$a)$ $\widetilde{N}_1 \le \widetilde{N}_2$

$b)$ $\widetilde{M}_1^{-1} \ge \widetilde{M}_2^{-1},\ N_1 \ge 0$

$c)$ $\widetilde{M}_1^{-1} \ge \widetilde{M}_2^{-1},\ N_2 \ge 0$

the inequality

$$\rho(\widetilde{M}_1^{-1}\widetilde{N}_1) \le \rho(\widetilde{M}_2^{-1}\widetilde{N}_2)$$

holds.

Lemma 2.8[4] Let A be nonsingular and $A^{-1} \ge 0$, $A = M_l - N_l (l = 1, 2, \cdots, k)$ are k weak regualar splittings of A. Then for any qualified $E_l (l = 1, 2, \cdots, k)$, $\rho(W) < 1$, where $W = \sum_{l=1}^{k} E_l M_l^{-1} N_l$, $\sum_{l=1}^{k} E_l = I$.

Lemma 2.9[4] Let A be an H-matrix, $A = M_l - N_l (l = 1, 2, \cdots, k)$ are k splittings of A and $\langle A\rangle = \langle M_l\rangle - |N_l|$, $diag(M_l) = diag(A)$, then $\rho(W) < 1$, where $W = \sum_{l=1}^{k} E_l M_l^{-1} N_l$, $\sum_{l=1}^{k} E_l = I$.

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3. Some properties of circulant matrices

In this section we will give some new properties of the reduced form of some special circulant matrices which consist in the original matrices. The following theorem is evident.

Theorem 3.1 The reduced form of a circulant matrix A is nonsingular or positive definite, respectively, if and only if A itself is nonsingular or positive definite, respectively.

Theorem 3.2 The reduced form of a circulant matrix A is an H-matrix, if the matrix A is a circulant H-matrix.

Proof Let A be a circulant H-matrix. Then, for the case of $n = 2m$ we will prove that both $B + C$ and $B - C$ are also H-matrices.

From

$$\langle B\rangle - |C| \le \langle B - C\rangle \le |D| \tag{3.1}$$

and

$$\langle B\rangle - |C| \le \langle B + C\rangle \le |D| \tag{3.2}$$

where $|D|$ is the diagonal part of the matrix $|B| + |C|$, we obtain that $\langle B\rangle - |C|$ is an M-matrix.

Since A is an H-matrix, according to the properties of the H-matrices, the comparison matrix $\langle A\rangle$ is an M-matrix, and $\langle B\rangle$ is an M-matrix too. By Lemma 2.1, $\langle A\rangle$ can be represented as

$$\langle A\rangle = \begin{pmatrix} \langle B\rangle & -|C| \\ -|C| & \langle B\rangle \end{pmatrix}. \tag{3.3}$$

Let us consider the block Gauss-Seidel splitting of the matrix $\langle A\rangle$:

$$\langle A\rangle = \begin{pmatrix} \langle B\rangle & 0 \\ -|C| & \langle B\rangle \end{pmatrix} - \begin{pmatrix} 0 & |C| \\ 0 & 0 \end{pmatrix}. \tag{3.4}$$

In terms of

$$\begin{pmatrix} \langle B\rangle & 0 \\ -|C| & \langle B\rangle \end{pmatrix}^{-1} = \begin{pmatrix} \langle B\rangle^{-1} & 0 \\ \langle B\rangle^{-1}|C|\langle B\rangle^{-1} & 0 \end{pmatrix} \ge 0$$

and $\begin{pmatrix} 0 & |C| \\ 0 & 0 \end{pmatrix} \ge 0$, it follows that the formula (3.4) is a regular splitting. By Lemma 2.2, the above splitting of the matrix $\langle A\rangle$ is convergent, thus

$$\begin{aligned} \rho\left(\begin{pmatrix} \langle B\rangle & 0 \\ -|C| & \langle B\rangle \end{pmatrix}^{-1} \begin{pmatrix} 0 & |C| \\ 0 & 0 \end{pmatrix}\right) &= \rho\left(\begin{pmatrix} 0 & \langle B\rangle^{-1}|C| \\ 0 & (\langle B\rangle^{-1}|C|)^2 \end{pmatrix}\right) \\ &= \rho^2(\langle B\rangle^{-1}|C|) < 1. \end{aligned}$$

It is evident that $\rho(\langle B\rangle^{-1}|C|) < 1$. We obtain that

$$(\langle B\rangle - |C|)^{-1} = (I - \langle B\rangle^{-1}|C|)^{-1}\langle B\rangle^{-1} \ge 0.$$

According to the definition of M-matrix, $\langle B\rangle - |C|$ is an M-matrix. By Lemma 2.6, $\langle B - C\rangle$ and $\langle B+C\rangle$ are M-matrices, by the definition of H-matrix, both $B-C$ and $B+C$ are H-matrices.

Now we show that for the case of $n = 2m + 1$, both matrix $B - J_m C$ and

$$\begin{pmatrix} \beta & \sqrt{2}a^T J_m \\ \sqrt{2}J_m a & B + J_m C \end{pmatrix}$$

are H-matrices.

On the base of (2.3) we get that

$$\langle A \rangle = \begin{pmatrix} \langle B \rangle & -J_m|a| & -J_m|C|J_m \\ -|a^T|J_m & |\beta| & -|a^T| \\ -|C| & -|a| & J_m\langle B \rangle J_m \end{pmatrix}. \tag{3.5}$$

As A is an H-matrix, both $\langle A \rangle$ and $\langle B \rangle$ are M-matrices. It is easy to verify that

$$\begin{pmatrix} |\beta| & -\sqrt{2}|a^T|J_m \\ -\sqrt{2}J_m|a| & \langle B \rangle - J_m|C| \end{pmatrix} \leq \begin{pmatrix} |\beta| & -\sqrt{2}|a^T|J_m \\ -\sqrt{2}J_m|a| & \langle B + J_m|C| \rangle \end{pmatrix} \leq \begin{pmatrix} |\beta| & \\ & |D| \end{pmatrix}$$

where $|D|$ is the diagonal part of the matrix $|B| + J_m|C|$.

Using the equivalence conditions of M-matrices ([2]), we can easily prove that there exists a positive vector $x \in R^{2m+1}$ such that $\langle A \rangle x > 0$. Partitioning x in the form of (2.3), denoted by $(y^T, \gamma, (J_m y)^T)^T$, then $\langle A \rangle x > 0$ implies that

$$\begin{pmatrix} \langle B \rangle & -J_m|a| & -J_m|C|J_m \\ -|a^T|J_m & |\beta| & -|a^T| \\ -|C| & -|a| & J_m\langle B \rangle J_m \end{pmatrix} \begin{pmatrix} y \\ \gamma \\ J_m y \end{pmatrix} > 0$$

Then

$$(\langle B \rangle - J_m|C| - (2/|\beta|)J_m|a||a^T|J_m)y > 0.$$

Using the equivalence of an M-matrix again, we have that $(\langle B \rangle - J_m|C| - (2/|\beta|)J_m|a||a^T|J_m)$ is an M-matrix.

By

$$(\langle B \rangle - J_m|C| - (2/|\beta|)J_m|a||a^T|J_m) \leq \langle B \rangle - J_m|C| \leq |D|,$$

using Lemma 2.6, we find that $\langle B \rangle - J_m|C|$ is an M-matrix.

It follows from (3.5) that

$$P^T\langle A \rangle P = \begin{pmatrix} \langle B \rangle + J_m|C| & & \\ & |\beta| & -\sqrt{2}|a^T|J_m \\ & -\sqrt{2}J_m|a| & \langle B \rangle - J_m|C| \end{pmatrix}.$$

As $\langle A \rangle$ is an M-matrix, all the real eigenvalues of $\langle A \rangle$ are positive, thus all the real eigenvalues of the matrix

$$\begin{pmatrix} |\beta| & -\sqrt{2}|a^T|J_m \\ -\sqrt{2}J_m|a| & \langle B \rangle - J_m|C| \end{pmatrix}$$

are positive too. According to the equivalence conditions of M-matrices, the above matrix is an M-matrix, we obtain that

$$\begin{pmatrix} \beta & \sqrt{2}a^T J_m \\ \sqrt{2}J_m a & B + J_m C \end{pmatrix}$$

is an H-matrix. We have completed the proof.

Note that converse of Theorem 3.2 does not hold in general. For example, the circulant matrix

$$A = \begin{pmatrix} 5 & 3 & 1 & 2 \\ 2 & 5 & 3 & 1 \\ 1 & 2 & 5 & 3 \\ 3 & 1 & 2 & 5 \end{pmatrix}$$

is not an H-matrix, but its reduced form (2.4)

$$\begin{pmatrix} 4 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 5 & 6 \end{pmatrix}$$

is an H-matrix.

By the above example, we find that reduced form of the circulant matrix A is an H-matrix does not imply that A is an H-matrix itself. However, if some conditions are imposed on the matrices B and C in the reduced form of A, then we can derive that the matrix A is a circulant H-matrix.

Theorem 3.3 Let A be an $n \times n$ circulant matrix.

(i) For $n = 2m$, if $\langle B \rangle - |C|$ is an M-matrix, then A is an H-matrix.

(ii) For $n = 2m+1$, if $\begin{pmatrix} |\beta| & -\sqrt{2}|a^T|J_m \\ -\sqrt{2}J_m|a| & \langle B \rangle - J_m|C| \end{pmatrix}$ is an M-matrix, then A is an H-matrix.

Proof First consider the case of $n = 2m$. From the assumption that $\langle B \rangle - |C|$ is an $m \times m$ M-matrix, and

$$\langle B \rangle - |C| \le \langle B \rangle \le D_{\langle B \rangle},$$

where $D_{\langle B \rangle}$ denotes the diagonal part of the matrix $\langle B \rangle$, by Lemma 2.6, we find that the matrix $\langle B \rangle$ is an M-matrix.

Since $\langle B \rangle$ is an M-matrix, according to the definition of regular splitting, then $\langle B \rangle - |C|$ is a regular splitting of the M-matrix $(\langle B \rangle - |C|)$, there holds $\rho(\langle B \rangle^{-1}|C|) < 1$. By means of the proof of Theorem 3.2, the bolck Gauss-Seidel splitting of

$$\langle A \rangle = \begin{pmatrix} \langle B \rangle & 0 \\ -|C| & \langle B \rangle \end{pmatrix} - \begin{pmatrix} 0 & |C| \\ 0 & 0 \end{pmatrix}$$

is a regular splitting. Note that

$$\rho\left(\begin{pmatrix} \langle B\rangle & 0 \\ -|C| & \langle B\rangle \end{pmatrix}^{-1}\begin{pmatrix} 0 & |C| \\ 0 & 0 \end{pmatrix}\right) = \rho^2(\langle B\rangle^{-1}|C|) < 1.$$

It is well known, a Z-matrix is an M-matrix if and only if it has a convergent regular splitting, so the matrix $\langle A\rangle$ is an M-matrix, and thus A is an H-matrix.

We now turn to consider the case $n = 2m+1$. From the hypothesis, we have that the matrix

$$\begin{pmatrix} |\beta| & -\sqrt{2}|a^T|J_m \\ -\sqrt{2}J_m|a| & \langle B\rangle - J_m|C| \end{pmatrix}$$

is an M-matrix. Therefore, the matrices $\langle B\rangle - J_m|C|, \langle B\rangle, \langle B - J_mC\rangle$ and $\langle B + J_mC\rangle$ are all M-matrices too. Moreover, the Schur complement of the matrix

$$\begin{pmatrix} |\beta| & -\sqrt{2}|a^T|J_m \\ -\sqrt{2}J_m|a| & \langle B\rangle - J_m|C| \end{pmatrix}$$

is $(\langle B\rangle - J_m|C| - (2/|\beta|)J_m|a||a^T|J_m)$, which is still an M-matrix by using the property of the Schur complement of an M-matrix. Utilizing the equivalence of an M-matrix again, there exist a positive vector y such that

$$(\langle B\rangle - J_m|C| - (2/|\beta|)J_m|a||a^T|J_m)y > 0,$$

that is,

$$(\langle B\rangle - J_m|C|)y > (2/|\beta|)(|a^T|J_my)J_m|a|.$$

Select a positive scalar α such that

$$(\langle B\rangle - J_m|C|)y > \alpha J_m|a| > (2/\beta)(|a^T|J_my)J_m|a|,$$

we obtain that

$$\begin{pmatrix} \langle B\rangle & -J_m|a| & -J_m|C|J_m \\ -|a^T|J_m & |\beta| & -|a^T| \\ -|C| & -|a| & J_m\langle B\rangle J_m \end{pmatrix}\begin{pmatrix} y \\ \alpha \\ J_my \end{pmatrix} > 0,$$

which mean that the matrix $\langle A\rangle$ is an M-matrix. Thus we yield the desired results.

4. The iterative methods for circulant matrices

4.1 Construction of an arithmetic mean splitting

We will give a new splitting scheme of a circulant matrix A, which is called the arithmetic mean splitting. Let the circulant matrix $A = M - N$ be a random convergent splitting of the matrix A. An iterative sequence derived from the splitting is defined by

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b. \tag{4.1.1}$$

Since $A = G^T AG$, we can get another convergent iterative sequence:

$$x_{k+1} = G^T M^{-1} NGx_k + G^T M^{-1} Gb. \tag{4.1.2}$$

In terms of (4.1.1) and (4.1.2), we can obtain a new iterative sequence:

$$x_{k+1} = \frac{1}{2}(M^{-1}N + G^T M^{-1}NG)x_k + \frac{1}{2}(M^{-1} + G^T M^{-1}G)b. \tag{4.1.3}$$

Denote $F = \frac{1}{2}(M^{-1} + G^T M^{-1}G)$ and $H = \frac{1}{2}(M^{-1}N + G^T M^{-1}NG)$, then (4.1.3) can be written as

$$x_{k+1} = Hx_k + Fb. \tag{4.1.4}$$

If det(F)$\neq 0$, a new splitting of A can be expressed as

$$A = F^{-1} - F^{-1}H. \tag{4.1.5}$$

(4.1.5) is called the arithmetic mean splitting of the matrix A. By (4.1.3) we can derive a splitting from a random convergent splitting of the circulant matrix A.

Theorem 4.1.1 Let A be a circulant matrix and

$$A = M - N$$

be a weak regular splitting, then

$$A = G^T AG = G^T MG - G^T NG$$

is also a weak regular splitting.

Proof Since $A = M - N$ is a weak regular splitting, there holds $M^{-1} \geq 0$ and $M^{-1}N \geq 0$. Consider that G is a permutation matrix and use the fact $G^{-1} = G^T$, then

$$(G^T MG)^{-1} = G^T M^{-1}G \geq 0,$$

and

$$(G^T MG)^{-1}(G^T NG) = G^T M^{-1}NG \geq 0,$$

Therefore $A = G^T MG - G^T NG$ is a weak regular splitting.

According to Lemma 2.8 and Theorem 4.1.1, we can get the following iterative convergent theorem.

Theorem 4.1.2 Let A be a circulant M-matrix, and $A = M - N$ be a weak regular of A, then the iterative sequence

$$x_{k+1} = \frac{1}{2}(M^{-1}N + G^T M^{-1}NG)x_k + \frac{1}{2}(M^{-1} + G^T M^{-1}G)b$$

is convergent.

Using Lemma 2.9 we can get the following result.

Theorem 4.1.3 Let A be a circulant H-matrix, $A = F - Q$ be a splitting of the matrix A and $\langle A\rangle = \langle F\rangle - |Q|$, then the iterative sequence

$$x_{k+1} = \frac{1}{2}(F^{-1}J + G^T F^{-1}JG)x_k + \frac{1}{2}(F^{-1} + G^T F^{-1}G)b$$

is convergent.

4.2 Two new splittings of circulant M-matrices

Now we will present two new splittings of the circulant matrix A and investigate their convergence.

(1) Opposite triangular splitting I:

(i) For $n = 2m$, $A = F_1 - Q_1$, where

$$F_1 = \begin{pmatrix} \hat{B}_1 & \hat{C}_1 \\ \hat{C}_1 & \hat{B}_1 \end{pmatrix}, \; Q_1 = \begin{pmatrix} B_1^* & C_1^* \\ C_1^* & B_1^* \end{pmatrix},$$

here $\hat{B}_1$ and $\hat{C}_1$ is the left lower triangular matrix of B and C respectively, and B_1^* and C_1^* is the strictly right upper triangular matrix of $-B$ and $-C$ respectively.

(ii) For $n = 2m+1$, $A = F_2 - Q_2$, where

$$F_2 = \begin{pmatrix} \hat{B}_2 & J_m a & J_m \hat{C}_2 J_m \\ 0 & \beta & 0 \\ \hat{C}_2 & a & J_m \hat{B}_2 J_m \end{pmatrix}, \; Q_2 = \begin{pmatrix} B_2^* & 0 & J_m C_2^* J_m \\ -a^T J_m & 0 & -a^T \\ C_2^* & 0 & J_m B_2^* J_m \end{pmatrix},$$

here $\hat{B}_2$ is the left lower triangular matrix of B, $\hat{C}_2$ is the left upper triangular matrix of C, B_2^* is the strictly right upper triangular matrix of $-B$, and C_2^* is the strictly right lower triangular matrix of $-C$.

(2) Opposite triangular splitting II:

(i) For $n = 2m$, $A = R_1 - V_1$, where

$$R_1 = \begin{pmatrix} E_1 & H_1 \\ H_1 & E_1 \end{pmatrix}, \; V_1 = \begin{pmatrix} E_1^* & H_1^* \\ H_1^* & E_1^* \end{pmatrix},$$

here E_1 and H_1 is the right upper triangular matrix of B and C respectively, and E_1^* and H_1^* is the strictly left lower triangular matrix of $-B$ and $-C$ respectively.

(ii) For $n = 2m+1$,$A = R_2 - V_2$, where

$$R_2 = \begin{pmatrix} E_2 & 0 & J_m H_2 J_m \\ a^T J_m & \beta & a^T \\ H_2 & 0 & J_m E_2 J_m \end{pmatrix}, \; V_2 = \begin{pmatrix} E_2^* & -J_m a & J_m H_2^* J_m \\ 0 & 0 & 0 \\ H_2^* & -a & J_m E_2^* J_m \end{pmatrix},$$

here E_2 is the right upper triangular matrix of B, H_2 is the right lower triangular matrix of C, E_2^* is the strictly left lower triangular matrix of $-B$, and H_2^* is the strictly left upper triangular matrix of $-C$.

In terms of the opposite triangular splitting I, we can get the following two SOR iterative sequences [5], at the same time, we can also get the similar conclusion by means of the opposite triangular splitting II.

(i) For $n = 2m$,

(a) Global SOR sequence

$$F_1 x_{k+1} = ((1-\omega)F_1 + \omega Q_1)x_k + \omega b. \tag{4.2.1}$$

(b) Part SOR sequence

$$F_1 x_{k+1} = Q_1 x_k + b. \tag{4.2.2}$$

Thus we have

$$P^T F_1 P P^T x_{k+1} = P^T Q_1 P P^T x_k + P^T b, \tag{4.2.3}$$

$$\hat{F}_1 = P^T F_1 P = \begin{pmatrix} \hat{B}_1 - \hat{C}_1 & 0 \\ 0 & \hat{B}_1 + \hat{C}_1 \end{pmatrix} = \begin{pmatrix} \hat{T}_1 & 0 \\ 0 & \hat{T}_2 \end{pmatrix},$$

$$\hat{G}_1 = P^T Q_1 P = \begin{pmatrix} B_1^* - C_1^* & 0 \\ 0 & B_1^* + C_1^* \end{pmatrix} = \begin{pmatrix} \hat{H}_1 & 0 \\ 0 & \hat{H}_2 \end{pmatrix},$$

where $\hat{T}_1 = \hat{B}_1 - \hat{C}_1$ and $\hat{T}_2 = \hat{B}_1 + \hat{C}_1$ are $m \times m$ left lower triangular matrices, $\hat{H}_1 = B_1^* - C_1^*$ and $\hat{H}_2 = B_1^* + C_1^*$ are $m \times m$ strictly right upper triangular matrices.

Let $P^T x_{k+1} = y_{k+1},\ P^T x_k = y_k,\ P^T b = \hat{b}$, then (4.2.3) becomes

$$\begin{pmatrix} \hat{T}_1 & 0 \\ 0 & \hat{T}_2 \end{pmatrix} \begin{pmatrix} y_{k+1}^{(1)} \\ y_{k+1}^{(2)} \end{pmatrix} = \begin{pmatrix} \hat{H}_1 & 0 \\ 0 & \hat{H}_2 \end{pmatrix} \begin{pmatrix} y_k^{(1)} \\ y_k^{(2)} \end{pmatrix} + \begin{pmatrix} \hat{b}^{(1)} \\ \hat{b}^{(2)} \end{pmatrix}.$$

Thus

$$\begin{cases} \hat{T}_1 y_{k+1}^{(1)} = \hat{H}_1 y_k^{(1)} + \hat{b}_1, \\ \hat{T}_2 y_{k+1}^{(2)} = \hat{H}_2 y_k^{(2)} + \hat{b}_2. \end{cases} \tag{4.2.4}$$

From (4.2.4), we can get Part SOR sequence:

$$\begin{cases} \hat{T}_1 y_{k+1}^{(1)} = ((1-\omega_1)\hat{T}_1 + \omega_1 \hat{H}_1) y_k^{(1)} + \omega_1 \hat{b}_1, \\ \hat{T}_2 y_{k+1}^{(2)} = ((1-\omega_2)\hat{T}_2 + \omega_2 \hat{H}_2) y_k^{(2)} + \omega_1 \hat{b}_2. \end{cases} \tag{4.2.5}$$

(ii) For $n = 2m+1$

(aa) Global SOR sequence

$$F_2 x_{k+1} = ((1-\omega)F_2 + \omega Q_2) x_k + \omega b. \tag{4.2.6}$$

(bb) Part SOR sequence

$$F_2 x_{k+1} = Q_2 x_k + b. \tag{4.2.7}$$

We can get the similar Part SOR iterative sequence:

$$\begin{cases} T_1^* y_{k+1}^{(1)} = ((1-\omega_1)T_1^* + \omega_1 H_1^*) y_k^{(1)} + \omega_1 b_1^*, \\ T_2^* y_{k+1}^{(2)} = ((1-\omega_2)T_2^* + \omega_2 H_2^*) y_k^{(2)} + \omega_1 b_2^*, \end{cases} \tag{4.28}$$

where $T_1^* = \hat{B}_2 - J_m\hat{C}_2$ and $T_2^* = \begin{pmatrix} \beta & 0 \\ \sqrt{2}J_m a & \hat{B}_2 + J_m\hat{C}_2 \end{pmatrix}$ are $m \times m$ left lower triangular matrices, $H_1^* = B_2^* - J_mC_2^*$ and $H_2^* = \begin{pmatrix} 0 & -\sqrt{2}a^T J_m \\ 0 & B_2^* + J_mC_2^* \end{pmatrix}$ are $m \times m$ strictly right upper triangular matrices.

Now we will discuss the convergence of the two splittings of circulant matrices and the SOR iterative sequence above.

Theorem 4.2.1 Let A be a circulant M-matrix, and $A = F - Q$ be opposite triangular splitting I or II of the matrix A, then $\rho(F^{-1}Q) < 1$.

Proof It can easily get that

$$A \le F \le |D|,$$

where D is the diagonal part of the matrix A. By Lemma 2.6, F is also an M-matrix, then $F^{-1} \ge 0$. On the other hand, it is evident that $Q \ge 0$. By the definition of the regular splitting, $A = F - Q$ is a regular splitting of the matrix A. Using Lemma 2.2 we have $\rho(F^{-1}Q) < 1$.

Theorem 4.2.2 Let A be a circulant M-matrix, and $A = F_1 - Q_1$ be opposite triangular splitting I of A, then

(1) if $\omega \in \left(0, \frac{2}{1+\rho(F_1^{-1}Q_1)}\right)$; Global SOR sequence is convergent,

(2) if $\omega_1 \in \left(0, \frac{2}{1+\rho(\hat{T}_1^{-1}\hat{H}_1)}\right)$, $\omega_2 \in \left(0, \frac{2}{1+\rho(\hat{T}_2^{-1}\hat{H}_2)}\right)$, Part SOR sequence is convergent.

Proof (1) By Theorem 4.2.1, $\rho(F_1^{-1}Q_1) < 1$. Using Lemma 6 in [6], when $\omega \in \left(0, \frac{2}{1+\rho(F_1^{-1}Q_1)}\right)$, $\rho(H(\omega)) < 1$, where $H(\omega) = (1-\omega)I + \omega F_1^{-1}Q_1$ is the iterative matrix of global SOR sequence.

(2) Since $\rho(F_1^{-1}Q_1) < 1$, then $\rho(\hat{F}_1^{-1}\hat{Q}_1) < 1$. From (4.2.3), $\rho(\hat{T}_1^{-1}\hat{H}_1) < 1$, and $\rho(\hat{T}_2^{-1}\hat{H}_2) < 1$. By Lemma 6 of [6], when $\omega_1 \in \left(0, \frac{2}{1+\rho(\hat{T}_1^{-1}\hat{H}_1)}\right)$ and $\omega_2 \in \left(0, \frac{2}{1+\rho(\hat{T}_2^{-1}\hat{H}_2)}\right)$, the Part SOR sequence is convergent. Similarly, (4.2.7) and (4.2.8) are convergent.

It is easy to find that the proof of the case of $n = 2m + 1$ is similar to above.

Using the same methods, we can obtain the related results of the splitting II. We will make a comparison of convergence rate of the iterative sequences. From Lemma 2.10, we can get the following two theorems.

Theorem 4.2.3 Let A be a circulant M-matrix, $A = D - (L + U)$ be Jacobi′s splitting of A and $A = F - Q$ be opposite triangular splitting I of A, then $\rho(F^{-1}Q) \le \rho(D^{-1}(L + U))$.

Proof It can easily get that $A^{-1} \ge 0$, $A = F - Q$ and $A = D - (L + U)$ are the regular splittings of the matrix A and $0 \le Q \le L + U$, then by Lemma 2.7 there holds

$$\rho(F^{-1}Q) \le \rho(D^{-1}(L + U)).$$

Example 4.2.4 Consider the circulant M-matrix

$$A = \begin{pmatrix} 2 & -1 & -0.5 & 0 \\ 0 & 2 & -1 & -0.5 \\ -0.5 & 0 & 2 & -1 \\ -1 & -0.5 & 0 & 2 \end{pmatrix}.$$

The iterative matrices of Jacobi's splitting and the opposite triangular splitting I of the matrix A can be expressed by G_J and G_I, respectively. We have $\rho(G_J) = 0.7500$ and $\rho(G_I) = 0.4444$. Thus $\rho(G_I) < \rho(G_J)$.

Theorem 4.2.5 Let A be a circulant M-matrix, $A = D-(L+U)$ and $A = R-V$ be Jacobi's splitting and opposite triangular splitting II of the matrix A respectively, then $\rho(R^{-1}V) \leq \rho(D^{-1}(L+U))$.

Proof The proof is similar to that of Theorem 4.2.3.

Example 4.2.6 Consider the circulant M-matrix

$$A = \begin{pmatrix} 8 & -1 & -2 & -4 \\ -4 & 8 & -1 & -2 \\ -2 & -4 & 8 & -1 \\ -1 & -2 & -4 & 8 \end{pmatrix}.$$

Let G_{II} be the iterative matrix of opposite triangular splitting II. We get $\rho(G_J) = 0.8750$, and $\rho(G_{II}) = 0.6944$. Thus $\rho(R^{-1}V) \leq \rho(D^{-1}(L+U))$.

Example 4.2.7 Consider the circulant M-matrix

$$A = \begin{pmatrix} 8 & -1 & -2 & -3 \\ -3 & 8 & -1 & -2 \\ -2 & -3 & 8 & -1 \\ -1 & -2 & -3 & 8 \end{pmatrix}.$$

Let the iterative matrix of Gauss-Seidel splitting be G_G. We get $\rho(G_G) = 0.5111$, and $\rho(G_I) = \rho(G_{II}) = 0.4444$. Then $\rho(F_1^{-1}J_1) = \rho(F_2^{-1}J_2) \leq \rho(D-L)^{-1}U)$, which mean that in this example, the opposite triangular splitting I and II have a better convergence rate than that of Gauss-Seidel splitting.

In fact, we can get the similar conclusion for the case of $n = 2m+1$.

4.3 Several splittings of circulant H-matrices

In this subsection, we also give two new splittings which are similar to those in Subsection 4.2. Now we only discuss their convergence, their costs of computation and store are analogous with those of Subsection 4.2.

Theorem 4.3.1 Let A be a circulant H-matrix, $A = F - Q$ be opposite triangular splitting I(II) of the matrix A, then $\rho(F^{-1}Q) < 1$.

Proof There holds $\langle A\rangle = \langle F\rangle - |Q|$, by Lemma 2.9, thus we get $\rho(F^{-1}Q) < 1$.

Theorem 4.3.2 Let A be a circulant H-matrix, $A = F - Q$ be opposite triangular splitting I(II) of A, then

(1) if $\omega \in \left(0, \frac{2}{1+\rho(F_1^{-1}Q_1)}\right)$, then Global SOR sequence is convergent;

(2) if $\omega_1 \in \left(0, \frac{2}{1+\rho(\hat{T}_1^{-1}\hat{H}_1)}\right)$, $\omega_2 \in \left(0, \frac{2}{1+\rho(\hat{T}_2^{-1}\hat{H}_2)}\right)$, then Part SOR sequence is convergent.

Proof The proof is similar to Theorem 4.2.2.

4.4 Three algorithms for the solution of $Ax = b$

Finally, we will construct three algorithms for the linear system $Ax = b$. The following algorithms 1 and 2 are based on the opposite triangular splittings in Subsections 4.2 and 4.3, and GMRES(m) algorithm is applied when the matrix A is very large and sparse.

Algorithm 1 (opposite triangular splitting I)

Step 1: Select an arbitrary starting point x_0 and a stopping criteria ε.

Step 2: Let $A = F - Q$ be the opposite triangular splitting I. Its iterative sequence is $Fx_k = Qx_{k-1} + b$, where A is a circulant M-matrix or a circulant H-matrix. By Lemma 2.5, there exists an orthogonal matrix P such that

$$P^T FPP^T x_{k+1} = P^T QPP^T x_k + P^T b.$$

It is easy to know that $\hat{F} = P^T FP$ and $\hat{Q} = P^T QP$ are a left lower triangular matrix and a right strictly upper triangular matrix, respectively. Let

$$\begin{aligned} \hat{x}_k = &\quad P^T x_k = (\hat{x}_{k,1}\hat{x}_{k,2}, \cdots, \hat{x}_{k,n})^T, \\ \hat{b} = &\quad P^T b = (\hat{b}_1\hat{b}_2, \cdots, \hat{b}_n)^T. \end{aligned}$$

Step 3: For $k = 1, 2, \cdots$, and for $j = 1$ to n, construct

$$\hat{x}_{k,i} = \frac{1}{\hat{F}_{i,i}} \left(\hat{b}_i - \sum_{s=1}^{i-1} \hat{F}_{i,s}\hat{x}_{k,s} + \sum_{s=i+1}^{n} \hat{Q}_{i,s}\hat{x}_{k-1,s} \right).$$

Step 4: If $\|\hat{x}_k - \hat{x}_{k-1}\| < \varepsilon$, then stop, let $x = P\hat{x}_k$, which is an approximate solution to the linear system $Ax = b$; Otherwise set $k = k + 1$ and return to step 3.

Algorithm 2 (opposite triangular splitting II)

Step 1: Select an arbitrary starting point x_0 and a stopping criteria ε.

Step 2: Let $A = R - V$ be the opposite triangular splitting II and its iterative sequence be $Rx_k = Vx_{k-1} + b$, where A is a circulant M-matrix or a circulant H-matrix. By Lemma 2.5, there exists an orthogonal matrix P:

$$P^T RPP^T x_{k+1} = P^T VPP^T x_k + P^T b.$$

It is easy to know that $\hat{R} = P^T RP$ and $\hat{V} = P^T VP$ are an right upper triangular matrix and a strictly left lower triangular matrix respectively. Let

$$\begin{aligned} \hat{x}_k = &\quad P^T x_k = (\hat{x}_{k,1}\hat{x}_{k,2}, \cdots, \hat{x}_{k,n})^T, \\ \hat{b} = &\quad P^T b = (\hat{b}_1\hat{b}_2, \cdots, \hat{b}_n)^T. \end{aligned}$$

Step 3: For $k = 1, 2, \cdots$, and for $j = 1$ to n, construct

$$\hat{x}_{k,i} = \frac{1}{\hat{R}_{i,i}} \left(\hat{b}_i + \sum_{s=1}^{i-1} \hat{V}_{i,s}\hat{x}_{k-1,s} - \sum_{s=i+1}^{n} \hat{R}_{i,s}\hat{x}_{k,s} \right).$$

Step 4: If $\|\hat{x}_k - \hat{x}_{k-1}\| < \varepsilon$, then stop, let $x = P\hat{x}_k$, which is an approximate solution to the linear system $Ax = b$; Otherwise set $k = k + 1$ and return to step 3.

When the circulant matrix A is very large and sparse, the GMRES(m) algorithm is very useful to solve the linear system $Ax = b$. Using the circulant property of the matrix A, we can reduce a large of the cost of computation and store by means of GMRES(m) algorithm.

Algorithm 3 (GMRES(m) algorithm)

Step 1: Reduce the linear system $Ax = b$ to

$$P^T APP^T x = P^T b.$$

Let $\tilde{A} = P^T AP$, $\tilde{x} = P^T x$, and $\tilde{b} = P^T b$.

Step 2: Choose $\tilde{x}_0 \in R^n$, calculate $r_0 = \tilde{b} - \tilde{A}\tilde{x}_0$ and $v_1 = r_0/\|r_0\|_2$.

Step 3: Choose an appropriate m, obtain $\{v_i\}_{i=1}^m$ and $\tilde{H}_m$ by the Arnoldi process.

Step 4: Calculate $y_m = min_{y\in R^k}\|\beta e_1 - \tilde{H}_m y\|_2$.

Step 5: Obtain $\tilde{x}_m = \tilde{x}_0 + V_m y_m$.

Step 6: Calculate $\|r_m\| = \|\tilde{b} - \tilde{A}\tilde{x}_m\|$. For a given $\varepsilon > 0$, if $\|r_m\| < \varepsilon$, then stop, and we can obtain the approximate solution: $x = P\tilde{x}$.

Step 7: Otherwise let $\tilde{x}_0 = \tilde{x}_m$, and $v_1 = r_m/\|r_m\|_2$, return to step 3.

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