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Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
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83 Tat Chee Avenue
Kowloon,Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@math.cityu.edu.hk
Approximation Theory,
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11) Sever S.Dragomir
School of Computer Science and
Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001,AUSTRALIA
Tel. +61 3 9688 4437
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Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio,TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
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14) Augustine O.Esogbue
School of Industrial and Systems
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Georgia Institute of Technology
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GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

31) I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3 0651098283

32) Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock,Germany
manfred.tasche@mathematik.uni-
rostock.de
Numerical Fourier Analysis, Fourier
Analysis,Harmonic Analysis, Signal
Analysis, Spectral Methods, Wavelets,
Splines, Approximation Theory

33) Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
P.D.E, Control Theory, Functional
Analysis, rtrggani@memphis.edu

34) Gilbert G.Walter
Department Of Mathematical Sciences
University of Wisconsin-Milwaukee,Box
413,
Milwaukee,WI 53201-0413
414-229-5077
e-mail: ggw@csd.uwm.edu
Distribution Functions, Generalised
Functions, Wavelets

35) Xin-long Zhou
Fachbereich Mathematik, Fachgebiet
Informatik
Gerhard-Mercator-Universitat Duisburg

Princeton,NJ 08544-5263
609-258-4595(x4619 assistant)
e-mail: floudas@titan.princeton.edu
OptimizationTheory&Applications,
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Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152
901-678-3130
e-mail:jgoldste@memphis.edu
Partial Differential Equations,
Semigroups of Operators

17) H.H.Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail:gonska@informatik.uni-duisburg.de
Approximation Theory,
Computer Aided Geometric Design

18) John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales , control theory and their applications

19) Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Lotharstr.65,D-47048 Duisburg,Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis,Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

36) Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield,MO 65804-0094
417-836-5931
e-mail: xmy944f@missouristate.edu
Classical Approximation Theory,
Wavelets

37) Lotfi A. Zadeh
Professor in the Graduate School and Director,
Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
e-mail: zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

38) Ahmed I. Zayed
Department Of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

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Switching design for the robust stability of nonlinear uncertain stochastic switched discrete-time systems with interval time-varying delay

G. Rajchakit
Major of Mathematics, Faculty of Science
Maejo University, Chiangmai 50290, Thailand
Corresponding author: griengkrai@yahoo.com

Abstract

This paper is concerned with robust stability of nonlinear uncertain stochastic switched discrete time-delay systems with interval time-varying delay. The system to be considered is subject to interval time-varying delays, which allows the delay to be a fast time-varying function and the lower bound is not restricted to zero. Based on the discrete Lyapunov functional, a switching rule for the robust stability for the nonlinear uncertain stochastic switched discrete time-delay system with interval time-varying delay is designed via linear matrix inequalities.

Keywords. Switching design, nonlinear uncertain stochastic switched discrete system, time-varying delay, robust stability, Lyapunov function, linear matrix inequality.

1 Introduction

Time delay is often a source of instability and poor performance, and is encountered in various engineering systems, such as chemical processes and long transmission lines in pneumatic systems. Time-delay systems have received much attention in recent years, and various topics concerning time-delay systems have been addressed; see, e.g., [1-10], and the references cited therein. As an important class of hybrid systems, switched systems arise in many practical processes that cannot be described by exclusively continuous or exclusively discrete models, such as manufacturing, communication networks, automotive engineering control and chemical processes (see, e.g., [1–3] and the references therein). On the other hand, time-delay phenomena are very common in practical systems. A switched system with time-delay individual subsystems is called a switched time-delay system; in particular, when the subsystems are linear, it is then called a switched time-delay linear system. During the last decades, the stability analysis of switched linear continuous/discrete time-delay systems has attracted a lot of attention [4–8]. The main approach for stability analysis relies on the use of Lyapunov-Krasovskii functionals and linear matrix inequlity (LMI) approach for constructing a common Lyapunov function [9–11]. Although many important results have been obtained for switched linear continuous-time systems, there are few results concerning the stability of switched linear discrete systems with time-varying delays (see, e.g., [1–3] and the references therein). It was shown in [5, 7, 12] that when all subsystems are asymptotically stable, the switching system is asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems has been studied in [13], but the result was limited to constant delays. In [14], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the average dwell time scheme. To the best of the author's knowledge, the stability for linear discrete-time systems with both time-varying delays and nonlinear uncertain stochastic discrete switch system has not been fully investigated (see, e.g., [14–20] and the references therein), which are important in both theories and applications. This motivates our research.

This paper studies robust stability problem for nonlinear uncertain stochastic switched discrete-time delay systems with interval time-varying delays. Specifically, our goal is to develop a constructive way to design switching rule to robustly stable of the nonlinear uncertain stochastic switched discrete-time delay systems with interval time-varying delay. By using improved Lyapunov-Krasovskii functionals combined with LMIs technique, we propose new criteria for the robust stability of the nonlinear uncertain stochastic switched discrete-time delay system with interval time-varying delay. Compared to the existing results, our result has its own advantages. First, the time delay is assumed to be a time-varying function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, the delay function is bounded but not restricted to zero. Second, the approach allows us to design the switching rule for robust stability in terms of LMIs.

The paper is organized as follows: Section 2 presents definitions and some well-known technical propositions needed for the proof of the main results. Switching rule for the robust stability is presented in Section 3.

2 Preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all real non-negative numbers; R^n denotes the n-dimensional space with the scalar product of two vectors $\langle x, y\rangle$ or $x^T y$; $R^{n\times r}$ denotes the space of all matrices of $(n \times r)-$ dimension. N^+ denotes the set of all non-negative integers; A^T denotes the transpose of A; a matrix A is symmetric if $A = A^T$.

Matrix A is semi-positive definite $(A \geq 0)$ if $\langle Ax, x\rangle \geq 0$, for all $x \in R^n; A$ is positive definite $(A > 0)$ if $\langle Ax, x\rangle > 0$ for all $x \neq 0$; $A \geq B$ means $A - B \geq 0$. $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\min}(A) = \min\{Re\lambda : \lambda \in \lambda(A)\}$.

Consider a nonlinear uncertain stochastic switched discrete-time delay systems with interval time-varying delay of the form

$$\begin{aligned} x(k+1) &= (A_\gamma + \Delta A_\gamma(k))x(k) + (B_\gamma + \Delta B_\gamma(k))x(k-d(k)) + f(k, x(k-d(k))) \\ &\quad + \sigma(x(k), x(k-d(k)), k)\omega(k), \quad k \in N^+, \quad x(k) = v_k, \quad k = -d_2, -d_2+1, ..., 0, \end{aligned} \tag{2.1}$$

where $x(k) \in R^n$ is the state, $\gamma(.) : R^n \to \mathcal{N} := \{1, 2, \ldots, N\}$ is the switching rule, which is a function depending on the state at each time and will be designed. A switching function is a rule which determines a switching sequence for a given switching system. Moreover, $\gamma(x(k)) = i$ implies that the system realization is chosen as the i^{th} system, $i = 1, 2, ..., N$. It is seen that the system (2.1) can be viewed as an autonomous switched system in which the effective subsystem changes when the state $x(k)$ hits predefined boundaries. $A_i, B_i, i = 1, 2, ..., N$ are given constant matrices.

The nonlinear perturbations $f(k, x(k-d(k)))$ satisfies the following condition

$$f^T(k, x(k-d(k)))f(k, x(k-d(k))) \leq \beta^2 x^T(k-d(k))x(k-d(k)), \tag{2.2}$$

where β is positive constant. For simplicity, we denote $f(k, x(k-d(k))$ by f, respectively.

The time-varying uncertain matrices $\Delta A_i(k)$ and $\Delta B_i(k)$ are defined by:

$$\Delta A_i(k) = E_{ia}F_{ia}(k)H_{ia}, \quad \Delta B_i(k) = E_{ib}F_{ib}(k)H_{ib},$$

where $E_{ia}, E_{ib}, H_{ia}, H_{ib}$ are known constant real matrices with appropriate dimensions.
$F_{ia}(k), F_{ib}(k)$ are unknown uncertain matrices satisfying

$$F_{ia}^T(k)F_{ia}(k) \leq I, \quad F_{ib}^T(k)F_{ib}(k) \leq I, \quad k = 0, 1, 2, ..., \tag{2.3}$$

where I is the identity matrix of appropriate dimention, $\omega(k)$ is a scalar Wiener process (Brownian Motion) on $(\Omega, \mathcal{F}, \mathcal{P})$ with

$$E[\omega(k)] = 0, \quad E[\omega^2(k)] = 1, \quad E[\omega(i)\omega(j)] = 0 (i \neq j), \tag{2.4}$$

and σ: $R^n \times R^n \times R \to R^n$ is the continuous function, and is assumed to satisfy that

$$\begin{aligned}&\sigma^T(x(k), x(k-d(k)), k)\sigma(x(k), x(k-d(k)), k) \leq \rho_1 x^T(k)x(k) + \rho_2 x^T(k-d(k))x(k-d(k),\\ &x(k), x(k-d(k) \in R^n,\end{aligned} \tag{2.5}$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are known constant scalars. The time-varying function $d(k) : N^+ \to N^+$ satisfies the following condition:

$$0 < d_1 \leq d(k) \leq d_2, \quad \forall k \in N^+$$

Remark 2.1. It is worth noting that the time delay is a time-varying function belonging to a given interval, in which the lower bound of delay is not restricted to zero.

Definition 2.1. The nonlinear uncertain stochastic switched system with interval time-varying delay (2.1) is robustly stable in the mean square if there exists a positive definite scalar function $V(k, x(k) : R^n \times R^n \to R$ and a switching rule $\gamma(.)$ such that

$$E[\Delta V(k, x(k))] = E[V(k+1, x(k+1)) - V(k, x(k))] < 0,$$

along any trajectory of solution of the system (2.1) for all uncertainties which satisfy (2.3).

Definition 2.2. The system of matrices $\{J_i\}, i = 1, 2, \ldots, N$, is said to be strictly complete if for every $x \in R^n \backslash \{0\}$ there is $i \in \{1, 2, \ldots, N\}$ such that $x^T J_i x < 0$.

It is easy to see that the system $\{J_i\}$ is strictly complete if and only if

$$\bigcup_{i=1}^{N} \alpha_i = R^n \backslash \{0\},$$

where

$$\alpha_i = \{x \in R^n : \quad x^T J_i x < 0\}, i = 1, 2, ..., N.$$

Proposition 2.1. [22] *The system $\{J_i\}, i = 1, 2, \ldots, N$, is strictly complete if there exist $\delta_i \geq 0, i = 1, 2, \ldots, N, \sum_{i=1}^{N} \delta_i > 0$ such that*

$$\sum_{i=1}^{N} \delta_i J_i < 0.$$

If $N = 2$ then the above condition is also necessary for the strict completeness.

Proposition 2.2. (Cauchy inequality) *For any symmetric positive definite marix $N \in M^{n\times n}$ and $a, b \in R^n$ we have*

$$\pm a^T b \leq a^T N a + b^T N^{-1} b.$$

Proposition 2.3. [23] *Let E, H and F be any constant matrices of appropriate dimensions and $F^T F \leq I$. For any $\epsilon > 0$, we have*

$$EFH + H^T F^T E^T \leq \epsilon E E^T + \epsilon^{-1} H^T H.$$

3 Main results

Let us set

$$
\begin{aligned}
&W_i(S_1,S_2,P,Q)=\begin{bmatrix} W_{i11} & W_{i12} & W_{i13} & W_{i14}\\ * & W_{i22} & W_{i23} & W_{i24}\\ * & * & W_{i33} & W_{i34}\\ * & * & * & W_{i44}\end{bmatrix},\\
&W_{i11}=Q-P,\\
&W_{i12}=S_1-S_1A_i,\\
&W_{i13}=-S_1B_i,\\
&W_{i14}=-S_1-S_2A_i,\\
&W_{i22}=P+S_1+S_1^T+S_1E_{ib}E_{ib}^TS_1^T+H_{ia}^TH_{ia},\\
&W_{i23}=-S_1B_i,\\
&W_{i24}=S_2-S_1,\\
&W_{i33}=-Q+S_2E_{ib}E_{ib}^TS_2^T+H_{ia}^TH_{ia}+H_{ib}^TH_{ib}+\rho_2 I,\\
&W_{i34}=-S_2B_i,\\
&W_{i44}=-S_2-S_2^T+H_{ia}^TH_{ia}+H_{ib}^TH_{ib},\\
&J_i(S_1,S_2,Q)=(d_2-d_1)Q-S_1A_i-A_i^TS_1^T+S_1E_{ia}E_{ia}^TS_1^T\\
&\qquad\qquad +S_1E_{ib}E_{ib}^TS_1^T+S_2E_{ia}E_{ia}^TS_2^T+H_{ia}^TH_{ia}+\rho_1 I,\\
&\alpha_i=\{x\in R^n:\quad x^TJ_i(S_1,S_2,Q)x<0\},\ i=1,2,...,N,\\
&\bar\alpha_1=\alpha_1,\quad \bar\alpha_i=\alpha_i\setminus\bigcup_{j=1}^{i-1}\bar\alpha_j,\quad i=2,3,\dots,N.
\end{aligned}
\tag{3.1}
$$

The main result of this paper is summarized in the following theorem.

Theorem 3.1. *The nonlinear uncertain stochastic switched system with interval time-varying delay (2.1) is robustly stable if there exist symmetric positive definite matrices $P>0,Q>0$ and matrices S_1,S_2 satisfying the following conditions:*

(i) $\exists\delta_i\ge 0, i=1,2,\dots,N, \sum_{i=1}^N\delta_i>0:\ \sum_{i=1}^N\delta_iJ_i(S_1,S_2,Q)<0.$
(ii) $W_i(S_1,S_2,P,Q)<0,\quad i=1,2,...,N.$

The switching rule is chosen as $\gamma(x(k))=i$, whenever $x(k)\in\bar\alpha_i$.

Proof. Consider the following Lyapunov-Krasovskii functional for any ith system (2.1)

$$V(k)=V_1(k)+V_2(k)+V_3(k),$$

where

$$V_1(k)=x^T(k)Px(k),\quad V_2(k)=\sum_{i=k-d(k)}^{k-1}x^T(i)Qx(i),$$

$$V_3(k)=\sum_{j=-d_2+2}^{-d_1+1}\sum_{l=k+j+1}^{k-1}x^T(l)Qx(l),$$

We can verify that

$$\lambda_1\|x(k)\|^2\le V(k). \tag{3.2}$$

Let us set $\xi(k) = [x(k)\, x(k+1)\, x(k-d(k))\, f_i(k, x(k-d(k)))\, \omega(k)]^T$ and

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} P & 0 & 0 & 0 \\ I & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Then, the difference of $V_1(k)$ along the solution of the system (2.1) and taking the mathematical expectation, we obtained

$$\begin{aligned} E[\Delta V_1(k)] &= E[x^T(k+1)Px(k+1) - x^T(k)Px(k)] \\ &= E[\xi^T(k)H\xi(k) - 2\xi^T(k)G^T \begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \\ 0 \end{pmatrix}]. \end{aligned} \tag{3.3}$$

because of

$$\xi^T(k)H\xi(k) = x(k+1)Px(k+1),$$

$$2\xi^T(k)G^T \begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \\ 0 \end{pmatrix} = x^T(k)Px(k).$$

Using the expression of system (2.1)

$$\begin{aligned} 0 &= -S_1x(k+1) + S_1(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + S_1(B_i + E_{ib}F_{ib}(k)H_{ib})x(k-d(k)) + S_1 f \\ &\quad + S_1\sigma\omega(k), \\ 0 &= -S_2x(k+1) + S_2(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + S_2(B_i + E_{ib}F_{ib}(k)H_{ib})x(k-d(k)) + S_2 f \\ &\quad + S_2\sigma\omega(k), \\ 0 &= -\sigma^T x(k+1) + \sigma^T(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + \sigma^T(B_i + E_{ib}F_{ib}(k)H_{ib})x(k-d(k)) + \sigma^T f \\ &\quad + \sigma^T\sigma\omega(k), \end{aligned}$$

we have
$E[-2\xi^T(k)G^T$

$$\times \begin{pmatrix} 0.5x(k) \\ \begin{matrix}[-S_1x(k+1) + S_1(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + S_1(B_i + E_{ib}F_{ib}(k)H_{ib})x(k-d(k)) + S_1 f \\ +S_1\sigma\omega(k)]\end{matrix} \\ \begin{matrix}[-S_2x(k+1) + S_2(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + S_2(B_i + E_{ib}F_{ib}(k)H_{ib})x(k-d(k)) + S_2 f \\ +S_2\sigma\omega(k)]\end{matrix} \\ \begin{matrix}[-\sigma^T x(k+1) + \sigma^T(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + \sigma^T(B_i + E_{ib}F_{ib}(k)H_{ib})x(k-d(k)) + \sigma^T f \\ +\sigma^T\sigma\omega(k)]\end{matrix} \end{pmatrix}]$$

$$= E[-\xi^T(k)G^T \begin{pmatrix} 0.5I & 0 & 0 & 0 & 0 \\ S_1A_i + S_1E_{ia}F_{ia}(k)H_{ia} & -S_1 & S_1B_i + S_1E_{ib}F_{ib}(k)H_{ib} & S_1 & S_1\sigma \\ S_2A_i + S_2E_{ia}F_{ia}(k)H_{ia} & -S_2 & S_2B_i + S_2E_{ib}F_{ib}(k)H_{ib} & S_2 & S_2\sigma \\ \sigma^T A_i + \sigma^T E_{ia}F_{ia}(k)H_{ia} & -\sigma^T & \sigma^T B_i + \sigma^T E_{ib}F_{ib}(k)H_{ib} & \sigma^T & \sigma^T\sigma \end{pmatrix} \xi(k)$$

$$-\xi^T(k)\begin{pmatrix} 0.5I & 0 & 0 & 0 & 0 \\ S_1A_i+S_1E_{ia}F_{ia}(k)H_{ia} & -S_1 & S_1B_i+S_1E_{ib}F_{ib}(k)H_{ib} & S_1 & S_1\sigma \\ S_2A_i+S_2E_{ia}F_{ia}(k)H_{ia} & -S_2 & S_2B_i+S_2E_{ib}F_{ib}(k)H_{ib} & S_2 & S_2\sigma \\ \sigma^TA_i+\sigma^TE_{ia}F_{ia}(k)H_{ia} & -\sigma^T & \sigma^TB_i+\sigma^TE_{ib}F_{ib}(k)H_{ib} & \sigma^T & \sigma^T\sigma \end{pmatrix}^T G\xi(k)].$$

Therefore, from (3.3) it follows that

$$\begin{aligned} E[\Delta V_1(k)] = E[&x^T(k)[-P-S_1A_i-S_1E_{ia}F_{ia}(k)H_{ia}-A_i^TS_1^T-H_{ia}^TF_{ia}^T(k)E_{ia}S_1^T]x(k) \\ &+2x^T(k)[S_1-S_1A_i-S_1E_{ia}F_{ia}(k)H_{ia}]x(k+1) \\ &+2x^T(k)[-S_1B_i-S_1E_{ib}F_{ib}(k)H_{ib}]x(k-d(k)) \\ &+2x^T(k)[-S_1-S_2A_i-S_2E_{ia}F_{ia}(k)H_{ia}]f(k,x(k-d(k))) \\ &+2x^T(k)[-S_1\sigma-\sigma^TA_i-\sigma^TE_{ia}F_{ia}(k)H_{ia}]\omega(k) \\ &+x(k+1)[P+S_1+S_1^T]x(k+1) \\ &+2x(k+1)[-S_1B_i-S_1E_{ib}F_{ib}(k)H_{ib}]x(k-d(k)) \\ &+2x(k+1)[S_2-S_1]f(k,x(k-d(k))) \\ &+2x(k+1)[\sigma^T-S_1\sigma]\omega(k) \\ &+2x^T(k-d(k))[-S_3B_i-S_2E_{ib}F_{ib}(k)H_{ib}]f(k,x(k-d(k))) \\ &+2x^T(k-d(k))[-\sigma^TB_i-\sigma^TE_{ib}F_{ib}(k)H_{ib}]\omega(k) \\ &+f(k,x(k-d(k)))^T[-S_2-S_2^T]f(k,x(k-d(k))) \\ &+2f(k,x(k-d(k)))^T(k)[-S_2\sigma-\sigma^T]\omega(k) \\ &+\omega^T(k)[-\sigma^T\sigma]\omega(k)]. \end{aligned}$$

By asumption (2.4), we have

$$\begin{aligned} E[\Delta V_1(k)] = E[&x^T(k)[-P-S_1A_i-S_1E_{ia}F_{ia}(k)H_{ia}-A_i^TS_1^T-H_{ia}^TF_{ia}^T(k)E_{ia}S_1^T]x(k) \\ &+2x^T(k)[S_1-S_1A_i-S_1E_{ia}F_{ia}(k)H_{ia}]x(k+1) \\ &+2x^T(k)[-S_1B_i-S_1E_{ib}F_{ib}(k)H_{ib}]x(k-d(k)) \\ &+2x^T(k)[-S_1-S_2A_i-S_2E_{ia}F_{ia}(k)H_{ia}]f(k,x(k-d(k))) \\ &+x(k+1)[P+S_1+S_1^T]x(k+1) \\ &+2x(k+1)[-S_1B_i-S_1E_{ib}F_{ib}(k)H_{ib}]x(k-d(k)) \\ &+2x(k+1)[S_2-S_1]f(k,x(k-d(k))) \\ &+2x^T(k-d(k))[-S_2B_i-S_2E_{ib}F_{ib}(k)H_{ib}]f(k,x(k-d(k))) \\ &+f(k,x(k-d(k)))^T[-S_2-S_2^T]f(k,x(k-d(k))) \\ &+\omega^T(k)[-\sigma^T\sigma]\omega(k)]. \end{aligned}$$

Applying Propositon 2.2, Propositon 2.3, condition (2.3) and asumption (2.5), the following estimations hold

$$\begin{gathered} -S_1E_{ia}F_{ia}(k)H_{ia}-H_{ia}^TF_{ia}^T(k)E_{ia}^TS_1^T \le S_1E_{ia}E_{ia}^TS_1^T+H_{ia}^TH_{ia}, \\ -2x^T(k)S_1E_{ia}F_{ia}(k)H_{ia}x(k+1) \le x^T(k)S_1E_{ia}E_{ia}^TS_1^Tx(k)+x(k+1)^TH_{ia}^TH_{ia}x(k+1), \\ -2x^T(k)S_1E_{ib}F_{ib}(k)H_{ib}x(k-d(k)) \le x^T(k)S_1E_{ib}E_{ib}^TS_1^Tx(k)+x(k-d(k))^TH_{ib}^TH_{ib}x(k-d(k)), \\ -2x^T(k)S_2E_{ia}F_{ia}(k)H_{ia}f \le x^T(k)S_2E_{ia}E_{ia}^TS_2^Tx(k)+f^TH_{ia}^TH_{ia}f, \end{gathered}$$

$$-2x(k-d(k))^T(k)S_2E_{ib}F_{ib}(k)H_{ib}f \leq x(k-d(k))^T(k)S_2E_{ib}E_{ib}^TS_2^Tx(k-d(k)) + f^TH_{ib}^TH_{ib}f,$$
$$-2x^T(k+1)S_1E_{ib}F_{ib}(k)H_{ib}x(k-d(k)) \leq x^T(k+1)S_1E_{ib}E_{ib}^TS_1^Tx(k+1) + x(k-d(k))^TH_{ib}^TH_{ib}x(k-d(k)),$$
$$-\sigma^T(x(k),x(k-d(k)),k)\sigma(x(k),x(k-d(k)),k) \leq \rho_1x^T(k)x(k) + \rho_2x^T(k-d(k))x(k-d(k).$$

Therefore, we have

$$\begin{aligned}
E[\Delta V_1(k)] \leq E[&x^T(k)[-P - S_1A_i - A_i^TS_1^T + S_1E_{ia}E_{ia}^TS_1^T + S_1E_{ib}E_{ib}^TS_1^T \\
&+ S_2E_{ia}E_{ia}^TS_2^T + H_{ia}^TH_{ia} + \rho_1I]x(k) \\
&+ 2x^T(k)[S_1 - S_1A_i]x(k+1) \\
&+ 2x^T(k)[-S_1B_i]x(k-d(k)) \\
&+ 2x^T(k)[-S_1 - S_2A_i]f(k,x(k-d(k))) \\
&+ x(k+1)[P + S_1 + S_1^T + S_1E_{ib}E_{ib}^TS_1^T + H_{ia}^TH_{ia}]x(k+1) \qquad (3.4)\\
&+ 2x(k+1)[-S_1B_i]x(k-d(k)) \\
&+ 2x(k+1)[S_2 - S_1]f(k,x(k-d(k))) \\
&+ x^T(k-d(k))[S_2E_{ib}E_{ib}^TS_3^T + H_{ia}^TH_{ia} + H_{ib}^TH_{ib} + \rho_2I]x(k-d(k)) \\
&+ 2x^T(k-d(k))[-S_3B_i]f(k,x(k-d(k))) \\
&+ f(k,x(k-d(k)))^T[-S_2 - S_2^T + H_{ia}^TH_{ia} + H_{ib}^TH_{ib}]f(k,x(k-d(k)))].
\end{aligned}$$

The difference of $V_2(k)$ is given by

$$\begin{aligned}
E[\Delta V_2(k)] &= E[\sum_{i=k+1-d(k+1)}^{k} x^T(i)Qx(i) - \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i)] \\
&= E[\sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Qx(i) + x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) \qquad (3.5)\\
&\quad + \sum_{i=k+1-d_1}^{k-1} x^T(i)Qx(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Qx(i)].
\end{aligned}$$

Since $d(k) \geq d_1$ we have

$$\sum_{i=k+1-d_1}^{k-1} x^T(i)Qx(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Qx(i) \leq 0,$$

and hence from (3.5) we have

$$E[\Delta V_2(k)] \leq E[\sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Qx(i) + x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k))]. \qquad (3.6)$$

The difference of $V_3(k)$ is given by

$$\begin{aligned} E[\Delta V_3(k)] &= E[\sum_{j=-d_2+2}^{-d_1+1}\sum_{l=k+j}^{k} x^T(l)Qx(l) - \sum_{j=-d_2+2}^{-d_1+1}\sum_{l=k+j+1}^{k-1} x^T(l)Qx(l)] \\ &= E[\sum_{j=-d_2+2}^{-d_1+1}[\sum_{l=k+j}^{k-1} x^T(l)Qx(l) + x^T(k)Q(\xi)x(k) \\ &\quad - \sum_{l=k+j}^{k-1} x^T(l)Qx(l) - x^T(k+j-1)Qx(k+j-1)]] \\ &= E[\sum_{j=-d_2+2}^{-d_1+1}[x^T(k)Qx(k) - x^T(k+j-1)Qx(k+j-1)]] \\ &= E[(d_2-d_1)x^T(k)Qx(k) - \sum_{j=k+1-d_2}^{k-d_1} x^T(j)Qx(j)]. \end{aligned} \tag{3.7}$$

Since $d(k) \le d_2$, and

$$\sum_{i=k=1-d(k+1)}^{k-d_1} x^T(i)Qx(i) - \sum_{i=k+1-d_2}^{k-d_1} x^T(i)Qx(i) \le 0,$$

we obtain from (3.6) and (3.7) that

$$E[\Delta V_2(k) + \Delta V_3(k)] \le E[(d_2-d_1+1)x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k))]. \tag{3.8}$$

Therefore, combining the inequalities (3.4), (3.8) gives

$$E[\Delta V(k)] \le E[x^T(k)J_i(S_1,S_2,Q)x(k) + \psi^T(k)W_i(S_1,S_2,P,Q)\psi(k)], \tag{3.9}$$

where

$$\psi(k) = [x(k)\, x(k+1)\, x(k-d(k))\, f(k, x(k-d(k)))]^T,$$

$$W_i(S_1,S_2,P,Q) = \begin{bmatrix} W_{i11} & W_{i12} & W_{i13} & W_{i14} \\ * & W_{i22} & W_{i23} & W_{i24} \\ * & * & W_{i33} & W_{i34} \\ * & * & * & W_{i44} \end{bmatrix},$$

$$\begin{aligned} W_{i11} &= Q - P, \\ W_{i12} &= S_1 - S_1A_i, \\ W_{i13} &= -S_1B_i, \\ W_{i14} &= -S_1 - S_2A_i, \\ W_{i14} &= -S_1 - S_2A_i, \\ W_{i22} &= P + S_1 + S_1^T + S_1E_{ib}E_{ib}^TS_1^T + H_{ia}^TH_{ia}, \\ W_{i23} &= -S_1B_i, \\ W_{i24} &= S_2 - S_1, \\ W_{i33} &= -Q + S_2E_{ib}E_{ib}^TS_2^T + H_{ia}^TH_{ia} + H_{ib}^TH_{ib} + \rho_2 I, \\ W_{i34} &= -S_2B_i, \\ W_{i44} &= -S_2 - S_2^T + H_{ia}^TH_{ia} + H_{ib}^TH_{ib}, \\ J_i(S_1,S_2,Q) &= (d_2-d_1)Q - S_1A_i - A_i^TS_1^T + 2S_1E_{ia}E_{ia}^TS_1^T + S_1E_{ib}E_{ib}^TS_1^T \\ &\quad + S_2E_{ia}E_{ia}^TS_2^T + H_{ia}^TH_{ia} + \rho_1 I. \end{aligned}$$

Therefore, we finally obtain from (3.9) and the condition (ii) that

$$E[\Delta V(k)] < E[x^T(k)J_i(S_1, S_2, Q)x(k)], \quad \forall i = 1, 2, ..., N, k = 0, 1, 2, ...$$

We now apply the condition (i) and Proposition 2.1., the system $J_i(S_1, S_2, Q)$ is strictly complete, and the sets α_i and $\bar{\alpha}_i$ by (3.1) are well defined such that

$$\bigcup_{i=1}^{N} \alpha_i = R^n \backslash \{0\},$$

$$\bigcup_{i=1}^{N} \bar{\alpha}_i = R^n \backslash \{0\}, \quad \bar{\alpha}_i \cap \bar{\alpha}_j = \emptyset, i \neq j.$$

Therefore, for any $x(k) \in R^n, k = 1, 2, ...,$ there exists $i \in \{1, 2, \ldots, N\}$ such that $x(k) \in \bar{\alpha}_i$. By choosing switching rule as $\gamma(x(k)) = i$ whenever $x(k) \in \bar{\alpha}_i$, from the condition (3.9) we have

$$E[\Delta V(k)] \leq E[x^T(k)J_i(S_1, S_2, Q)x(k)] < 0, \quad k = 1, 2, ...,$$

which, combining the condition (3.2), and Definition 2.1., concludes the proof of the theorem in the mean square.

Remark 3.1. Note that the results proposed in [5, 7, 12] for switching systems to be asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems studied in [13] was limited to constant delays. In [21], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the averaged well time scheme.

4 Conclusion

This paper has proposed a switching design for the robust stability of nonlinear uncertain stochastic switched discrete time-delay systems with interval time-varying delays. Based on the discrete Lyapunov functional, a switching rule for the robust stability for the nonlinear uncertain stochastic switched discrete time-delay system with interval time-varying delay is designed via linear matrix inequalities.

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Stabilization of switched discrete-time systems with convex polytopic uncertainties

G. Rajchakit
Major of Mathematics, Faculty of Science
Maejo University, Chiangmai 50290, Thailand
Corresponding author: griengkrai@yahoo.com

Abstract

This paper is concerned with robust stabilization of switched discrete time-delay systems with convex polytopic uncertainties. The system to be considered is subject to interval time-varying delays, which allows the delay to be a fast time-varying function and the lower bound is not restricted to zero. Based on the discrete Lyapunov functional, a switching rule for the robust stabilization for the system with convex polytopic uncertainties is designed via linear matrix inequalities.

Keywords. Switching design, convex polytopic uncertainties, discrete system, robust stabilization, Lyapunov function, linear matrix inequality.

AMS (MOS) Subject Classification. 34D20, 93D20, 37C75.

1 Introduction

In many physical phenomena and practical applications, such as autonomous transmission systems, computer disc drivers, room temperature control, power electronics, chaos generators (see, e.g., [1–3] and the references therein), they are governed by more than one dynamical systems (differential or difference equations) governed by switching laws to determine which subsystem will be activated on a certain time interval. Such systems are called switched systems. On the other hand, time-delay phenomena are very common in practical systems. A switched system with time-delay individual subsystems is called a switched time-delay system; in particular, when the subsystems are linear, it is then called a switched time-delay linear system. During the last decades, the stability analysis of switched linear continuous/discrete time-delay systems has attracted a lot of attention [4–7]. The main approach for stability analysis relies on the use of Lyapunov-Krasovskii functionals and linear matrix inequlity (LMI) approach for constructing a common Lyapunov function [8–10]. Although many important results have been obtained for switched linear continuous-time systems, there are few results concerning the stability of switched linear discrete systems with time-varying delays. It was shown in [5, 7, 11] that when all subsystems are asymptotically stable, the switching system is asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems has been studied in [12], but the result was limited to constant delays. In [14], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the average dwell time scheme. To the best of our knowledge, the stabilization of discrete-time systems with both convex polytopic uncertainties and switch system, non-differentiable time-varying delays has not been fully studied yet (see, e.g., [1, 4–27] and the references therein), which are important in both theories and applications. This motivates our research.

This paper studies robust stabilization problem for switched linear discrete systems with convex polytopic uncertainties with interval time-varying delays. Specifically, our goal is to develop a constructive way to design switching rule to robust stabilization the system. By using improved Lyapunov-Krasovskii functionals combined with LMIs technique, we propose new criteria for the robust stabilization of the system. Compared to the existing results, our result has its own advantages. First, the time delay is assumed to be a time-varying function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, the delay function is bounded but not restricted to zero. Second, the approach allows us to design the switching rule for robust stabilization in terms of of LMIs.

The paper is organized as follows: Section 2 presents definitions and some well-known technical propositions needed for the proof of the main results. Switching rule for the robust stabilization is presented in Section 3.

2 Preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all real non-negative numbers; R^n denotes the n-dimensional space with the scalar product of two vectors $\langle x, y\rangle$ or $x^T y$; $R^{n\times r}$ denotes the space of all matrices of $(n \times r)-$ dimension. A^T denotes the transpose of A; a matrix A is symmetric if $A = A^T$.
Matrix A is semi-positive definite $(A \geq 0)$ if $\langle Ax, x\rangle \geq 0$, for all $x \in R^n; A$ is positive definite $(A > 0)$ if $\langle Ax, x\rangle > 0$ for all $x \neq 0$; $A \geq B$ means $A - B \geq 0$. $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\min}(A) = \min\{Re\lambda : \lambda \in \lambda(A)\}$.

Consider a linear switched control discrete-time systems with convex polytopic uncertainties with interval time-varying delay of the form

$$\begin{aligned} x(k+1) &= A_{\gamma(x(k))}(\zeta)x(k) + B_{\gamma(x(k))}(\zeta)u(k), \quad k = 0, 1, 2, ... \\ x(k) &= v_k, \quad k = -d_2, -d_2+1, ..., 0, \end{aligned} \tag{2.1}$$

where $x(k) \in R^n$ is the state, $u(k) \in R^m, m \leq n$, is the control input, $\gamma(.) : R^n \to \mathcal{N} := \{1, 2, \dots, N\}$ is the switching rule, which is a function depending on the state at each time and will be designed. A switching function is a rule which determines a switching sequence for a given switching system.
We consider a delayed feedback control law

$$u(k) = C_{\gamma(x(k))}(\zeta)x(k - d(k)), k = -h_2, ..., 0, \tag{2.2}$$

and $C_{\gamma(x(k))}(\zeta)$ is the controller gain to be determined. Moreover, $\gamma(x(k)) = i$ implies that the system realization is chosen as the i^{th} system, $i = 1, 2, ..., N$. It is seen that the system (2.1) can be viewed as an autonomous switched system in which the effective subsystem changes when the state $x(k)$ hits predefined boundaries. $A_i(\zeta), B_i(\zeta), C_i(\zeta), i = 1, 2, ..., N$ are given constant matrices. The system matrices are subjected to uncertainties and belong to the polytope Ω given by

$$\Omega = \{[A_i, B_i, C_i](\zeta) := \sum_{j=1}^{N} \zeta_j [A_{ij}, B_{ij}, C_{ij}], \quad \sum_{j=1}^{N} \zeta_j = 1, \zeta_j \geq 0\},$$

where A_{ij}, B_{ij}, C_{ij}, $i, j = 1, 2, ..., N$, are given constant matrices with appropriate dimensions. The time-varying function $d(k)$ satisfies the following condition:

$$0 < d_1 \leq d(k) \leq d_2, \quad \forall k = 0, 1, 2, ...$$

Remark 2.1. It is worth noting that the time delay is a time-varying function belonging to a given interval, in which the lower bound of delay is not restricted to zero.

Applying the feedback controller (2.2) to the system (2.1), the closed-loop discrete time-delay system is

$$x(k+1) = A_i(\zeta)x(k) + B_i(\zeta)C_i(\zeta)x(k - d(k)), \quad k = 0, 1, 2, ... \tag{2.3}$$

Definition 2.1. The system (2.1) is robustly stablilizable if there exist a switching function $\gamma(.)$ and a delayed feedback control (2.2) such that the zero solution of the system (2.3) is asymptotically stable for all uncertainties in Ω.

Definition 2.2. The system of matrices $\{J_i\}, i = 1, 2, \ldots, N$, is said to be strictly complete if for every $x \in R^n \backslash \{0\}$ there is $i \in \{1, 2, \ldots, N\}$ such that $x^T J_i x < 0$.

It is easy to see that the system $\{J_i\}$ is strictly complete if and only if

$$\bigcup_{i=1}^{N} \alpha_i = R^n \backslash \{0\},$$

where

$$\alpha_i = \{x \in R^n : \quad x^T J_i x < 0\}, i = 1, 2, ..., N.$$

Proposition 2.1. [28] *The system $\{J_i\}, i = 1, 2, \ldots, N$, is strictly complete if there exist $\delta_i \geq 0, i = 1, 2, \ldots, N, \sum_{i=1}^{N} \delta_i > 0$ such that*

$$\sum_{i=1}^{N} \delta_i J_i < 0.$$

If $N = 2$ then the above condition is also necessary for the strict completeness.

Proposition 2.2. *For real numbers $\zeta_j \geq 0, j = 1, 2, ..., N$, $\sum_{j=1}^{N} \zeta_j = 1$, the following inequality hold*

$$(N-1)\sum_{j=1}^{N} \zeta_j^2 - 2\sum_{j=1}^{N-1}\sum_{l=j+1}^{N} \zeta_j \zeta_l \geq 0.$$

Proof. The proof is followed from the completing the square:

$$(N-1)\sum_{j=1}^{N} \zeta_j^2 - 2\sum_{j=1}^{N-1}\sum_{l=j+1}^{N} \zeta_j \zeta_l = \sum_{j=1}^{N-1}\sum_{l=j+1}^{N} (\zeta_j - \zeta_l)^2 \geq 0.$$

3 Main results

Let us set

$$\|x_k\| = \sup_{s \in [-d_2, 0]} \|x(k+s)\|,$$

$$W_{ijj}(P,Q,R) = \begin{pmatrix} Q_j - P_j & R_j^T - A_{ij}^T R_j & -R_j^T B_{ij} C_{ij} \\ R_j - R_j^T A_{ij} & P_j + R_j + R_j^T & -R_j^T B_{ij} C_{ij} \\ -C_{ij}^T B_{ij}^T R_j & -C_{ij}^T B_{ij}^T R_j & -Q_j \end{pmatrix},$$

$$W_{ijl}(P,Q,R) = \begin{pmatrix} Q_j - P_j & R_j^T - A_{il}^T R_j & -R_j^T B_{il} C_{il} \\ R_j - R_j^T A_{il} & P_j + R_j + R_j^T & -R_j^T B_{il} C_{il} \\ -C_{il}^T B_{il}^T R_j & -C_{il}^T B_{il}^T R_j & -Q_j \end{pmatrix},$$

$$W_{ilj}(P,Q,R) = \begin{pmatrix} Q_l - P_l & R_l^T - A_{ij}^T R_l & -R_l^T B_{ij} C_{ij} \\ R_l - R_l^T A_{ij} & P_l + R_l + R_l^T & -R_l^T B_{ij} C_{ij} \\ -C_{ij}^T B_{ij}^T R_l & -C_{ij}^T B_{ij}^T R_l & -Q_l \end{pmatrix},$$

$$\mathcal{R} = \begin{pmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P(\zeta) = \sum_{j=1}^{N} \zeta_j P_j, \quad Q(\zeta) = \sum_{j=1}^{N} \zeta_j Q_j, \quad R(\zeta) = \sum_{j=1}^{N} \zeta_j R_j, \quad \lambda_1 = \lambda_{\min}(P),$$

$$J_{ijj}(R,Q) := (d_2-d_1)Q_j - A_{ij}^T R_j - R_j^T A_{ij},$$

$$J_{ijl}(R,Q) := (d_2-d_1)Q_j - A_{il}^T R_j - R_j^T A_{il},$$

$$J_{ilj}(R,Q) := (d_2-d_1)Q_l - A_{ij}^T R_l - R_l^T A_{ij},$$

$$\begin{aligned}
\alpha_{ijj} =& \{x \in R^n : \quad x^T J_{ijj}(R,Q)x < 0, \}, \quad i=1,2,...,N, j=1,2,...,N,\\
\alpha_{ijl} =& \{x \in R^n : \quad x^T J_{ijl}(R,Q)x < 0, \}, \quad i=1,2,...,N, j=1,2,...,N-1; l=j+1,...,N,\\
\alpha_{ijl} =& \{x \in R^n : \quad x^T J_{ilj}(R,Q)x < 0, \}, \quad i=1,2,...,N, j=1,2,...,N-1; l=j+1,...,N,\\
\bar{\alpha}_{1jj} =& \alpha_{1jj}, \quad \bar{\alpha}_{ijj} = \alpha_{ijj} \setminus \bigcup_{i=1}^{i-1} \bar{\alpha}_{ijj}, \quad i=2,3,\ldots,N, j=1,2,\ldots,N,\\
\bar{\alpha}_{1jl} =& \alpha_{1jl}, \quad \bar{\alpha}_{ijl} = \alpha_{ijl} \setminus \bigcup_{i=1}^{i-1} \bar{\alpha}_{ijl}, \quad i=2,3,\ldots,N, j=1,2,...,N-1; l=j+1,...,N,\\
\bar{\alpha}_{1lj} =& \alpha_{1lj}, \quad \bar{\alpha}_{ilj} = \alpha_{ilj} \setminus \bigcup_{i=1}^{i-1} \bar{\alpha}_{ilj}, \quad i=2,3,\ldots,N, j=1,2,...,N-1; l=j+1,...,N.
\end{aligned} \tag{3.1}$$

The main result of this paper is summarized in the following theorem.

Theorem 3.1. *The switched control system with convex polytopic uncertainties (2.1) is stabilizable by the delayed feedback control (2.2) if there exist symmetric matrices* $P_i > 0, Q_i > 0, \mathcal{R} \geq 0, i = 1,2...,N$ *and matrix* $R_i, i = 1,2...,N$ *satisfying the following conditions*

(i) $\exists \delta_i \geq 0, \quad \sum_{i=1}^N \delta_i > 0 : \sum_{i=1}^N \delta_i J_{ijj} < 0,$ *and* $J_{ijj} + \mathcal{R} < 0, \quad i = 1,2,\ldots,N,$
$j = 1,2,\ldots,N.$

(ii) $\exists \delta_i \geq 0, \quad \sum_{i=1}^N \delta_i > 0 : \sum_{i=1}^N [\delta_i J_{ijl} + \delta_i J_{ilj}] < 0,$ *and* $J_{ijl} + J_{ilj} - \frac{2}{N-1}\mathcal{R} < 0,$
$i = 1,2,\ldots,N, \quad j = 1,2,\ldots,N-1, \quad l = j+1,\ldots,N.$

(iii) $W_{ijj} + \mathcal{R} < 0, \quad i = 1,2,...,N, \quad j = 1,2,...,N.$

(iv) $W_{ijl} + W_{ilj} - \frac{2}{N-1}\mathcal{R} < 0, \quad i = 1,2,...,N, \quad j = 1,2,...,N-1; \quad l = j+1,...,N.$

The switching rule is chosen as $\gamma(x(k)) = i$*, whenever* $x(k) \in \bar{\alpha}_{ijl}$.

Proof. Consider the following Lyapunov-Krasovskii functional for any ith system (2.1)

$$V(k) = V_1(k) + V_2(k) + V_3(k),$$

where

$$V_1(k) = x^T(k)P(\zeta)x(k), \quad V_2(k) = \sum_{i=k-d(k)}^{k-1} x^T(i)Q(\zeta)x(i),$$

$$V_3(k) = \sum_{j=-d_2+2}^{-d_1+1} \sum_{l=k+j+1}^{k-1} x^T(l)Q(\zeta)x(l),$$

We can verify that

$$\lambda_1 \|x(k)\|^2 \leq V(k). \tag{3.2}$$

Let us set $\xi(k) = [x(k)\, x(k+1)\, x(k-d(k))]^T$, and

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P(\zeta) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} P(\zeta) & 0 & 0 \\ R(\zeta) & R(\zeta) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, the difference of $V_1(k)$ along the solution of the system is given by

$$\begin{aligned} \Delta V_1(k) &= x^T(k+1)P(\zeta)x(k+1) - x^T(k)P(\zeta)x(k) \\ &= \xi^T(k)H(\zeta)\xi(k) - 2\xi^T(k)G^T(\zeta)\begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.3}$$

because of

$$\xi^T(k)H(\zeta)\xi(k) = x(k+1)P(\zeta)x(k+1).$$

Using the expression of system (2.3)

$$0 = -x(k+1) + A_i(\zeta)x(k) + B_i(\zeta)C_i(\zeta)x(k-d(k)),$$

we have

$$\begin{aligned} &-2\xi^T(k)G^T(\zeta)\begin{pmatrix} 0.5x(k) \\ -x(k+1) + A_i(\zeta)x(k) + B_i(\zeta)C_i(\zeta)x(k-d(k)) \\ 0 \end{pmatrix}\xi(k) \\ &= -\xi^T(k)G^T(\zeta)\begin{pmatrix} 0.5I & 0 & 0 \\ A_i(\zeta) & -I & B_i(\zeta)C_i(\zeta) \\ 0 & 0 & 0 \end{pmatrix}\xi(k) - \xi^T(k)\begin{pmatrix} 0.5I & A_i(\zeta)^T & 0 \\ 0 & -I & 0 \\ 0 & (B_i(\zeta)C_i(\zeta))^T & 0 \end{pmatrix}G(\zeta)\xi(k). \end{aligned}$$

Therefore, from (3.3) it follows that

$$\Delta V_1(k) = \xi^T(k)W_i(P(\zeta), Q(\zeta), R(\zeta))\xi(k), \tag{3.4}$$

where

$$\begin{aligned} W_i(P(\zeta), Q(\zeta), R(\zeta)) = &\begin{pmatrix} 0 & 0 & 0 \\ 0 & P(\zeta) & 0 \\ 0 & 0 & 0 \end{pmatrix} - G^T(\zeta)\begin{pmatrix} 0.5I & 0 & 0 \\ A_i(\zeta) & -I & B_i(\zeta)C_i(\zeta) \\ 0 & 0 & 0 \end{pmatrix} \\ &- \begin{pmatrix} 0.5I & A_i^T(\zeta) & 0 \\ 0 & -I & 0 \\ 0 & (B_i(\zeta)C_i(\zeta))^T & 0 \end{pmatrix}G(\zeta). \end{aligned}$$

The difference of $V_2(k)$ is given by

$$\begin{aligned} \Delta V_2(k) &= \sum_{i=k+1-d(k+1)}^{k} x^T(i)Q(\zeta)x(i) - \sum_{i=k-d(k)}^{k-1} x^T(i)Q(\zeta)x(i) \\ &= \sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Q(\zeta)x(i) + x^T(k)Q(\zeta)x(k) - x^T(k-d(k))Q(\zeta)x(k-d(k)) \\ &\quad + \sum_{i=k+1-d_1}^{k-1} x^T(i)Q(\zeta)x(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Q(\zeta)x(i). \end{aligned} \tag{3.5}$$

Since $d(k) \geq d_1$ we have

$$\sum_{i=k+1-d_1}^{k-1} x^T(i)Q(\zeta)x(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Q(\zeta)x(i) \leq 0,$$

and hence from (3.5) we have

$$\Delta V_2(k) \leq \sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Q(\zeta)x(i) + x^T(k)Q(\zeta)x(k) - x^T(k-d(k))Q(\zeta)x(k-d(k)). \tag{3.6}$$

The difference of $V_3(k)$ is given by

$$\begin{aligned}
\Delta V_3(k) &= \sum_{j=-d_2+2}^{-d_1+1} \sum_{l=k+j}^{k} x^T(l)Q(\zeta)x(l) - \sum_{j=-d_2+2}^{-d_1+1} \sum_{l=k+j+1}^{k-1} x^T(l)Q(\zeta)x(l) \\
&= \sum_{j=-d_2+2}^{-d_1+1} [\sum_{l=k+j}^{k-1} x^T(l)Q(\zeta)x(l) + x^T(k)Q(\zeta)(\xi)x(k) \\
&\quad - \sum_{l=k+j}^{k-1} x^T(l)Q(\zeta)x(l) - x^T(k+j-1)Q(\zeta)x(k+j-1)] \\
&= \sum_{j=-d_2+2}^{-d_1+1} [x^T(k)Q(\zeta)x(k) - x^T(k+j-1)Q(\zeta)x(k+j-1)] \\
&= (d_2-d_1)x^T(k)Q(\zeta)x(k) - \sum_{j=k+1-d_2}^{k-d_1} x^T(j)Q(\zeta)x(j).
\end{aligned} \tag{3.7}$$

Since $d(k) \leq d_2$, and

$$\sum_{i=k=1-d(k+1)}^{k-d_1} x^T(i)Q(\zeta)x(i) - \sum_{i=k+1-d_2}^{k-d_1} x^T(i)Q(\zeta)x(i) \leq 0,$$

we obtain from (3.6) and (3.7) that

$$\Delta V_2(k) + \Delta V_3(k) \leq (d_2-d_1+1)x^T(k)Q(\zeta)x(k) - x^T(k-d(k))Q(\zeta)x(k-d(k)). \tag{3.8}$$

Therefore, combining the inequalities (3.4), (3.8) gives

$$\Delta V(k) \leq x^T(k)J_i(R(\zeta), Q(\zeta))x(k) + \xi^T(k)W_i(P(\zeta), Q(\zeta), R(\zeta))\xi(k), \tag{3.9}$$

where

$$W_i(P(\zeta), Q(\zeta), R(\zeta)) = \begin{pmatrix} Q(\zeta)-P(\zeta) & R^T(\zeta)-A_i^T(\zeta)R(\zeta) & -R^T(\zeta)B_i(\zeta)C_i(\zeta) \\ R(\zeta)-R^T(\zeta)A_i(\zeta) & P(\zeta)+R(\zeta)+R^T(\zeta) & -R^T(\zeta)B_i(\zeta)C_i(\zeta) \\ -C_i^T(\zeta)B_i^T(\zeta)R(\zeta) & -C_i^T(\zeta)B_i^T(\zeta)R(\zeta) & -Q(\zeta) \end{pmatrix},$$

and

$$J_i(R(\zeta), Q(\zeta)) = (d_2-d_1)Q(\zeta) - A_i^T(\zeta)R(\zeta) - R^T(\zeta)A_i(\zeta).$$

Let us denote

$$W_{ijj}(P,Q,R) = \begin{pmatrix} Q_j-P_j & R_j^T-A_{ij}^TR_j & -R_j^TB_{ij}C_{ij} \\ R_j-R_j^TA_{ij} & P_j+R_j+R_j^T & -R_j^TB_{ij}C_{ij} \\ -C_{ij}^TB_{ij}^TR_j & -C_{ij}^TB_{ij}^TR_j & -Q_j \end{pmatrix},$$

$$W_{ijl}(P,Q,R) = \begin{pmatrix} Q_j - P_j & R_j^T - A_{il}^T R_j & -R_j^T B_{il} C_{il} \\ R_j - R_j^T A_{il} & P_j + R_j + R_j^T & -R_j^T B_{il} C_{il} \\ -C_{il}^T B_{il}^T R_j & -C_{il}^T B_{il}^T R_j & -Q_j \end{pmatrix},$$

$$W_{ilj}(P,Q,R) = \begin{pmatrix} Q_l - P_l & R_l^T - A_{ij}^T R_l & -R_l^T B_{ij} C_{ij} \\ R_l - R_l^T A_{ij} & P_l + R_l + R_l^T & -R_l^T B_{ij} C_{ij} \\ -C_{ij}^T B_{ij}^T R_l & -C_{ij}^T B_{ij}^T R_l & -Q_l \end{pmatrix},$$

$$J_{ijj}(R,Q) := (d_2 - d_1)Q_j - A_{ij}^T R_j - R_j^T A_{ij},$$

$$J_{ijl}(R,Q) := (d_2 - d_1)Q_j - A_{il}^T R_j - R_j^T A_{il},$$

$$J_{ilj}(R,Q) := (d_2 - d_1)Q_l - A_{ij}^T R_l - R_l^T A_{ij},$$

$$(A_i^T R)_{jl} := A_{il}^T R_j + A_{ij}^T R_l, \quad (R^T A_i)_{jl} = R_j^T A_{il} + R_l^T A_{ij},$$

$$(R^T B_i C_i)_{jl} = R_j^T B_{il} C_{il} + R_l^T B_{ij} C_{ij}, \quad (C_i^T B_i^T R)_{jl} = C_{il}^T B_{il}^T R_j + C_{ij}^T B_{ij}^T R_l,$$

$$P_{jl} = P_j + P_l, \quad Q_{jl} = Q_j + Q_l, \quad R_{jl} = R_j + R_l.$$

From the convex combination of the expression of $P(\zeta), Q(\zeta), R(\zeta), A(\zeta), B(\zeta), C(\zeta)$, we have

$$\begin{aligned} W_i(P(\zeta), Q(\zeta), R(\zeta)) &= \sum_{j=1}^N \zeta_j^2 \begin{pmatrix} Q_j - P_j & R_j^T - A_{ij}^T R_j & -R_j^T B_{ij} C_{ij} \\ R_j - R_j^T A_{ij} & P_j + R_j + R_j^T & -R_j^T B_{ij} C_{ij} \\ -C_{ij}^T B_{ij}^T R_j & -C_{ij}^T B_{ij}^T R_j & -Q_j \end{pmatrix} \\ &+ \sum_{j=1}^{N-1} \sum_{l=j+1}^N \zeta_j \zeta_l \begin{pmatrix} Q_j - P_j + Q_l - P_l & R_{jl}^T - (A_i^T R)_{jl} & -(R^T B_i C_i)_{jl} \\ R_{jl} - (R^T A_i)_{jl} & P_{jl} + R_{jl} + R_{jl}^T & -(R^T B_i C_i)_{jl} \\ -(C_i^T B_i^T R)_{jl} & -(C_i^T B_i^T R)_{jl} & -Q_{jl} \end{pmatrix} \\ &= \sum_{j=1}^N \zeta_j^2 W_{ijj}(P,Q,R) + \sum_{j=1}^{N-1} \sum_{l=j+1}^N \zeta_j \zeta_l [W_{ijl}(P,Q,R) + W_{ilj}(P,Q,R)]. \end{aligned}$$

$$\begin{aligned} J_i(R(\zeta), Q(\zeta)) &= \sum_{j=1}^N \zeta_j^2 (d_2 - d_1) Q_j - A_{ij}^T R_j - R_j^T A_{ij} \\ &+ \sum_{j=1}^{N-1} \sum_{l=j+1}^N \zeta_j \zeta_l (d_2 - d_1) Q_{jl} - (A_i^T R)_{jl} - (R^T A_i)_{jl} \\ &= \sum_{j=1}^N \zeta_j^2 J_{ijj}(Q,R) + \sum_{j=1}^{N-1} \sum_{l=j+1}^N \zeta_j \zeta_l [J_{ijl}(Q,R) + J_{ilj}(Q,R)]. \end{aligned}$$

Then the conditions (i)-(iv) give

$$W_i(P(\zeta), Q(\zeta), R(\zeta)) < -\sum_{j=1}^N \zeta_j^2 \mathcal{R} + \frac{2}{N-1} \sum_{j=1}^{N-1} \sum_{l=j+1}^N \zeta_j \zeta_l \mathcal{R} \leq 0,$$

$$J_i(R(\zeta), Q(\zeta)) < -\sum_{j=1}^N \zeta_j^2 \mathcal{R} + \frac{2}{N-1} \sum_{j=1}^{N-1} \sum_{l=j+1}^N \zeta_j \zeta_l \mathcal{R} \leq 0,$$

because of Proposition 2.2:

$$(N-1) \sum_{j=1}^N \zeta_j^2 - 2 \sum_{j=1}^{N-1} \sum_{l=j+1}^N \zeta_j \zeta_l = \sum_{j=1}^{N-1} \sum_{l=j+1}^N (\zeta_j - \zeta_l)^2 \geq 0.$$

Therefore, we finally obtain from (3.9) and the condition (iii), (iv) that

$$\Delta V(k) < x^T(k)J_i(R(\zeta), Q(\zeta))x(k), \quad \forall i = 1, 2, ..., N, k = 0, 1, 2, ...$$

We now apply the condition (i), (ii), and Proposition 2.1., the system $J_i(R(\zeta), Q(\zeta))$ is strictly complete, and the sets α_{ijl} and $\bar{\alpha}_{ijl}$ by (3.1) are well defined such that

$$\bigcup_{i=1}^{N} \alpha_{ijl} = R^n \backslash \{0\},$$

$$\bigcup_{i=1}^{N} \bar{\alpha}_{ijl} = R^n \backslash \{0\}, \quad \bar{\alpha}_{ijl} \cap \bar{\alpha}_{tjl} = \emptyset, i \neq t.$$

Therefore, for any $x(k) \in R^n, k = 0, 1, 2, ...$, there exists $i \in \{1, 2, \ldots, N\}$ such that $x(k) \in \bar{\alpha}_{ijl}$. By choosing switching rule as $\gamma(x(k)) = i$ whenever $x(k) \in \bar{\alpha}_{ijl}$, from the condition (3.9) we have

$$\Delta V(k) \leq x^T(k)J_i(R(\zeta), Q(\zeta))x(k) < 0, \quad k = 1, 2, ...,$$

which, combining the condition (3.2) and the Lyapunov stability theorem [29], concludes the proof of the theorem.

Remark 3.1. Note that theresult sproposed in [4,5,6] for switching systems to be asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems studied in [9] was limited to constant delays. In [10], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the averaged well time scheme.

4 Conclusion

This paper has proposed a switching design for the robust stabilization of switched linear discrete-time systems with convex polytopic uncertainties with interval time-varying delays. Based on the discrete Lyapunov functional, a switching rule for the robust stabilization for the system with convex polytopic uncertainties is designed via linear matrix inequalities.

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A preconditioner for block two-by-two symmetric indefinite matrices *

Chun Wen †, Ting-Zhu Huang‡

School of Mathematical Sciences,

University of Electronic Science and Technology of China,

Chengdu, Sichuan, 611731, P.R. China

Abstract

A new preconditioner for the numerical solution of block two-by-two symmetric indefinite matrices is presented in this paper. The proposed preconditioner is constructed as the product of two fairly simple preconditioners: one is the famous block Jacobi preconditioner, and the other is the popular constraint preconditioner. Here, we call it the product preconditioner. Results concerning the eigenvalue distribution and form of the eigenvectors of the product preconditioned matrix are analyzed. Numerical experiments are used to illustrate the efficiency of the proposed product preconditioner.

Key words: Product preconditioner; Symmetric indefinite matrices; Krylov subspace method

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1 Introduction

Recently, a large amount of work has been devoted to the problem of solving linear systems in saddle point form. Here, our concern is to construct a new preconditioner for the numerical solution of block two-by-two symmetric indefinite matrices whose (1,1) and (2,2) block are nonsingular. Often this kind of linear systems in saddle point form is likely to generate from a wide range of applications, such as the Helmholtz equation

$$\begin{cases} \Delta u + (2\pi)^2 u = f(x, y), & (x, y) \in \Omega \cup \Re_2^+, \\ u = 0, & (x, y) \in \partial(\Omega \cup \Re_2^+), \end{cases} \tag{1}$$

with radiation boundary condition

$$\lim_{r \to \infty} r\left(\frac{\partial u}{\partial \eta} - \imath 2\pi u\right) = 0, \tag{2}$$

where $\Omega = [0, 1] \times [-1, 0]$ is a unit square domain, $\Re_2^+$ denotes the upper half-space and $\imath$ is the imaginary unit in (2), see [1, 2] for details.

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†E-mail: wchun17@163.com

‡E-mail: tingzhuhuang@126.com

By using a finite difference discretization to the Helmholtz equation (1) on the uniform grid of Ω, we obtain the linear system in saddle point form

$$\mathcal{A}u = \begin{pmatrix} A & B \\ B^{\mathrm{T}} & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} = b, \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times m}$ are nonsingular, $B \in \mathbb{R}^{n\times m}$, $u = [x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n+m}$ and $b = [f^{\mathrm{T}}, g^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n+m}$, with $x, f \in \mathbb{R}^n$ and $y, g \in \mathbb{R}^m$, are the unknown and given right-hand side vectors, respectively. Then the coefficient matrix $\mathcal{A} \in \mathbb{R}^{(n+m)\times(n+m)}$ is a nonsingular, symmetric and possibly indefinite matrix, and our main aim is to solve the linear system (3) of $n+m$ linear equations with $n+m$ unknowns.

Iterative procedure is a convenient numerical solution method for computing the linear system (3). Often we have Uzawa's algorithms [3, 4] and multigrid methods [5, 6]. In particular, Krylov subspace methods have become more and more popular for solving the linear system (3), such as the conjugate gradient (CG) and biconjugate gradient stabilized (Bi-CGSTAB) methods, minimal residual method (MINRES), generalized minimal residual (GMRES) and quasi-minimal residual (QMR) methods which have been considered in [7–14].

However, these iterative methods are all likely to suffer from slow convergence for some large linear systems which come from many practical applications like the computational fluid dynamics and structural mechanics. Thus it is necessary to use the idea of preconditioning such that the preconditioned matrix has a tightly clustered eigenvalues, see [1, 15–22] and the references therein.

More precisely, we see that a kind of triangular preconditioner has been proposed by Elman and Silvester [14] and Elman [23] when the (2,2) block matrix $C = 0$. These triangular preconditioners were extended by Kay, Loghin and Wathen [24], Cao [25] and Simoncini [26] to the case where C is symmetric positive or negative semidefinite. In addition, Keller, Gould and Wathen [18] presented a constraint preconditioner for the case $C = 0$, in which they discussed the eigenvalue distribution and form of the eigenvectors of the constraint preconditioned matrix and its minimal polynomial. Thereafter, Dollar and Wathen [19] and Dollar [22] studied an approximation factorization constraint preconditioner by combining with the conjugate gradient method, and extended the idea of [18] by allowing the matrix C to be symmetric and positive semidefinite. Furthermore, we found block diagonal, triangular and constraint preconditioners had been discussed by Siefert and De Sturler [17], Murphy, Golub and Wathen [15], De Sturler and Liesen [16], and Cao [20, 21] for the numerical solution of nonsymmetric or generalized saddle point problems. More preconditioning techniques for solving the linear system in saddle point form can be found in an excellent survey written by Benzi, Golub and Liesen [1].

In this paper, we are concerned with investigating a new preconditioner for the symmetric indefinite linear system (3). The proposed preconditioner is constructed as the product of two fairly simple preconditioners: one is the famous block Jacobi preconditioner, and the other is the popular constraint preconditioner [22]. We call it the product preconditioner. The idea used to develop the product preconditioner can trace back to [27]. Benzi has used the idea in [27] to solve Markov chain problems, see [28, 29]. Results concerning the eigenvalue distribution and form of the eigenvectors of the product preconditioned matrix are given in this paper. Numerical experiments with preconditioned GMRES method [30] on certain problem serve to illustrate the efficiency and stability of the proposed product preconditioner.

The remainder of this paper is organized as follows. In Section 2, we first briefly introduce the background material on stationary iterations and matrix splittings, and then construct the

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product preconditioner. In Section 3, we analyze the eigensolution distribution of the product preconditioned matrix. Numerical experiments with various preconditioned GMRES methods are presented in Section 4. Finally, conclusions are made in Section 5.

2 Background and product preconditioner

In this section, we first briefly introduce the background material on stationary iterations and matrix splittings from [27, 28, 31], and then construct the product preconditioner.

2.1 Stationary iterations and matrix splittings

Consider the solution of a large sparse linear system of the form $\mathcal{A}u = b$, where $\mathcal{A}$ is a square and nonsingular, symmetric indefinite matrix, and b is the given right-hand vector. Stationary iterative method is likely to be an attractive method by using a splitting of the coefficient matrix $\mathcal{A}$, denoted as

$$\mathcal{A} = M - N,$$

where M is a nonsingular matrix. Then the splitting gives rise to the stationary iterative method

$$u_{k+1} = Tu_k + c, \quad k = 0, 1, \cdots, \tag{4}$$

where $T = M^{-1}N$ is called the iterative matrix, $c = M^{-1}b$, and u_0 is a given initial guess. It is well known that the iterative method (4) converges for any initial guess u_0 if and only if its spectral radius $\rho(T) < 1$ [31].

Recently, Benzi and Szyld have defined a related approach by the alternating iterations

$$\begin{cases} u_{k+1/2} = M_1^{-1}N_1u_k + M_1^{-1}b, \\ u_{k+1} = M_2^{-1}N_2u_{k+1/2} + M_2^{-1}b, \end{cases} \quad k = 0, 1, \cdots, \tag{5}$$

in an excellent paper [27], where $\mathcal{A} = M_1 - N_1 = M_2 - N_2$ are splittings of $\mathcal{A}$, both M_1 and M_2 matrices are nonsingular, and u_0 is defined as above. Not only the existence and uniqueness of splittings for stationary iterative methods with applications to alternating methods were proved, but also the convergence theory of some alternating iterations were analyzed in [27]. In addition, Benzi and Szyld have constructed a splitting $\mathcal{A} = M - N$ based on the nonsingular matrix M_1 and M_2. The splitting is given by (see Eq. (10) in [27])

$$M^{-1} = M_2^{-1}(M_1 + M_2 - \mathcal{A})M_1^{-1}. \tag{6}$$

Evidently, the matrix $M_1 + M_2 - \mathcal{A}$ must be nonsingular for (6) to be well defined.

2.2 Product preconditioner

Now, we construct the product preconditioner as the multiplication of two fairly simple preconditioners from the derivation of the alternating iterations in [27]. The first preconditioner is the famous block Jacobi preconditioner

$$M_{bj} = \begin{pmatrix} A & O \\ O & -C \end{pmatrix}. \tag{7}$$

Note that M_{bj} is nonsingular since both A and C are invertible.

The second preconditioner is the popular nonsingular constraint preconditioner

$$M_{sc} = \begin{pmatrix} G & B \\ B^{\mathrm{T}} & -C \end{pmatrix} \tag{8}$$

discussed in [22], where $G \in \mathbb{R}^{n\times n}$ is an approximation of A, but is not equal to A. In practice, G is often taken to be the diagonal matrix formed with the diagonal entries of A, i.e., $G = \mathrm{diag}(\mathrm{diag}(A))$. Note that the Schur complement matrices $-(C+B^{\mathrm{T}}A^{-1}B)$ and $-(C+B^{\mathrm{T}}G^{-1}B)$ are nonsingular since matrix $\mathcal{A}$ in (3) and M_{sc} in (8) are nonsingular (proof can be found in [20]).

According to the alternating iterations (5) and equation (6), the product preconditioner M_{ps} is given by

$$M_{ps}^{-1} = M_{sc}^{-1}(M_{bj} + M_{sc} - \mathcal{A})M_{bj}^{-1}, \tag{9}$$

where the matrix

$$M_{bj} + M_{sc} - \mathcal{A} = \begin{pmatrix} G & O \\ O & -C \end{pmatrix}$$

is invertible. Hence, M_{ps}^{-1} is well defined. From equation (9), we have the product preconditioner

$$M_{ps} = M_{bj}(M_{bj} + M_{sc} - \mathcal{A})^{-1}M_{sc} = \begin{pmatrix} A & AG^{-1}B \\ B^{\mathrm{T}} & -C \end{pmatrix}. \tag{10}$$

Also, we can rewrite

$$M_{ps} = \begin{pmatrix} A & AG^{-1}B \\ B^{\mathrm{T}} & -C \end{pmatrix} = \begin{pmatrix} I & O \\ B^{\mathrm{T}}A^{-1} & I \end{pmatrix}\begin{pmatrix} A & AG^{-1}B \\ O & -(C+B^{\mathrm{T}}G^{-1}B) \end{pmatrix},$$

then, we have

$$M_{ps}^{-1} = \begin{pmatrix} A^{-1} - G^{-1}B(C+B^{\mathrm{T}}G^{-1}B)^{-1}B^{\mathrm{T}}A^{-1} & G^{-1}B(C+B^{\mathrm{T}}G^{-1}B)^{-1} \\ (C+B^{\mathrm{T}}G^{-1}B)^{-1}B^{\mathrm{T}}A^{-1} & -(C+B^{\mathrm{T}}G^{-1}B)^{-1} \end{pmatrix}.$$

Finally, the product preconditioned matrix $M_{ps}^{-1}\mathcal{A}$ can be expressed as

$$M_{ps}^{-1}\mathcal{A} = \begin{pmatrix} I & A^{-1}B - G^{-1}B(C+B^{\mathrm{T}}G^{-1}B)^{-1}(C+B^{\mathrm{T}}A^{-1}B) \\ O & (C+B^{\mathrm{T}}G^{-1}B)^{-1}(C+B^{\mathrm{T}}A^{-1}B) \end{pmatrix}. \tag{11}$$

3 Properties of the preconditioned matrix $M_{ps}^{-1}\mathcal{A}$

In this section, we focus on analyzing the eigenvalue distribution and form of the eigenvectors of the product preconditioned matrix $M_{ps}^{-1}\mathcal{A}$.

3.1 Eigenvalue distribution

In this section, we consider the eigenvalue distribution of the product preconditioned matrix $M_{ps}^{-1}\mathcal{A}$. It is well known that the convergence of an iterative method has close relation to the distribution of the eigenvalues of the coefficient matrix for symmetric matrix systems. Hence, a desired eigenvalue distribution is wished to obtain by the applications of preconditioning techniques. We prove a result of this type as follows.

Theorem 1. *Let $\mathcal{A} \in \mathbb{R}^{(n+m)\times(n+m)}$ defined in (3) be a nonsingular and symmetric indefinite matrix. Preconditioning $\mathcal{A}$ by the product preconditioner*

$$M_{ps} = \begin{pmatrix} A & AG^{-1}B \\ B^{\mathrm{T}} & -C \end{pmatrix},$$

where $G \in \mathbb{R}^{n\times n}$ is an approximation of A, $G \neq A$, $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times m}$ are nonsingular, $B \in \mathbb{R}^{n\times m}$. Then the product preconditioned matrix $M_{ps}^{-1}\mathcal{A}$ has

- *an eigenvalue at 1 with multiplicity n;*
- *m eigenvalues which are defined by the generalized eigenvalue problem $(C + B^{\mathrm{T}}A^{-1}B)y = \lambda(C + B^{\mathrm{T}}G^{-1}B)y$.*

Proof. Suppose λ is the eigenvalue of $M_{ps}^{-1}\mathcal{A}$, and $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}} \neq 0$ is the corresponding eigenvector. Besides, from (11), we have the preconditioned matrix

$$M_{ps}^{-1}\mathcal{A} = \begin{pmatrix} I & A^{-1}B - G^{-1}B(C + B^{\mathrm{T}}G^{-1}B)^{-1}(C + B^{\mathrm{T}}A^{-1}B) \\ O & (C + B^{\mathrm{T}}G^{-1}B)^{-1}(C + B^{\mathrm{T}}A^{-1}B) \end{pmatrix},$$

where $A^{-1}B - G^{-1}B(C + B^{\mathrm{T}}G^{-1}B)^{-1}(C + B^{\mathrm{T}}A^{-1}B)$ is irrelevant to the results in Theorem 1. Hence, by making use of the related knowledge in linear algebra, we obtain the results in Theorem 1 immediately. $\square$

3.2 Eigenvector distribution

To our knowledge, the termination of a Krylov subspace method is not only related to the distribution of eigenvalues of the preconditioned matrix, but also to the number of corresponding linearly independent eigenvectors. Hence, for completeness of this paper, we establish the relationship between eigenvalues and eigenvectors of the preconditioned matrix $M_{ps}^{-1}\mathcal{A}$ and discuss its eigenvector distribution. The following analysis is similar to the discussions in [4, 18, 22].

We start this part from the generalized eigenvalue problem

$$\begin{pmatrix} A & B \\ B^{\mathrm{T}} & -C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} A & AG^{-1}B \\ B^{\mathrm{T}} & -C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{12}$$

where λ is the eigenvalue of $M_{ps}^{-1}\mathcal{A}$, and $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}} \neq 0$ is the corresponding eigenvector. By calculations, we obtain

$$Ax + By = \lambda Ax + \lambda AG^{-1}By \tag{13}$$

and

$$B^{\mathrm{T}}x - Cy = \lambda(B^{\mathrm{T}}x - Cy). \tag{14}$$

From (14), we obtain $(1-\lambda)(B^{\mathrm{T}}x - Cy) = 0$. Hence, either $\lambda = 1$ or $B^{\mathrm{T}}x - Cy = 0$ holds true. In the former case, we have

$$By = AG^{-1}By. \tag{15}$$

Evidently, equation (15) is satisfied by $y = 0$, and thus there are n linearly independent eigenvectors of the form $(x^{\mathrm{T}}, 0^{\mathrm{T}})^{\mathrm{T}}$ associated with the unit eigenvalue. On the other hand, there may exist $y \neq 0$ which satisfies (15). Then, without loss of generality, we suppose that there are i $(0 \leq i \leq m)$ linearly independent eigenvectors of the form $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}}$, where the components y result from the eigenvalue problem $By = AG^{-1}By$.

Now, suppose $\lambda \neq 1$, then we have $B^{\mathrm{T}}x - Cy = 0$, which implies $y = C^{-1}B^{\mathrm{T}}x$ since C is nonsingular. Substituting this into equation (13), we get the generalized eigenvalue problem

$$(A + BC^{-1}B^{\mathrm{T}})x = \lambda(A + AG^{-1}BC^{-1}B^{\mathrm{T}})x, \tag{16}$$

where x is impossible to be equal to a zero vector. Since if $x = 0$, then we have $y = 0$, which is conflict with the known condition $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}} \neq 0$. Therefore, we suppose there exist j $(0 \leq j \leq n)$ linearly independent eigenvectors of the form $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}}$, where components x arise from the eigenvalue problem (16) with $y = C^{-1}B^{\mathrm{T}}x$.

We conclude this subsection with the following theorem.

Theorem 2. *Let $\mathcal{A} \in \mathbb{R}^{(n+m)\times(n+m)}$ defined in (3) be a nonsingular and symmetric indefinite matrix. Preconditioning $\mathcal{A}$ by the product preconditioner*

$$M_{ps} = \begin{pmatrix} A & AG^{-1}B \\ B^{\mathrm{T}} & -C \end{pmatrix},$$

where $G \in \mathbb{R}^{n\times n}$ is an approximation of A, $G \neq A$, $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times m}$ are nonsingular, $B \in \mathbb{R}^{n\times m}$. Then the product preconditioned matrix $M_{ps}^{-1}\mathcal{A}$ has $n+m$ eigenvalues as given in Theorem 1 and $n+i+j$ linearly independent eigenvectors. There are

- *n eigenvectors of the form $[x^{\mathrm{T}}, 0^{\mathrm{T}}]^{\mathrm{T}}$ that correspond to case $\lambda = 1$;*
- *$\exists$ i $(0 \leq i \leq m)$ eigenvectors of the form $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}}$, where the components y construct a basis of the generalized eigenvalue problem $By = AG^{-1}By$ and $\lambda = 1$;*
- *$\exists$ j $(0 \leq j \leq n)$ eigenvectors of the form $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}}$ that correspond to case $\lambda \neq 1$.*

Proof. According to the analysis above, we have obtained the specific form of the eigenvectors of the preconditioned matrix $M_{ps}^{-1}\mathcal{A}$. Now, our aim is to prove that the $n+i+j$ eigenvectors are linearly independent, that is, we need to show that

$$\begin{pmatrix} x_1^{(1)} & \cdots & x_n^{(1)} \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_1^{(1)} \\ \vdots \\ a_n^{(1)} \end{pmatrix} + \begin{pmatrix} x_1^{(2)} & \cdots & x_i^{(2)} \\ y_1^{(2)} & \cdots & y_i^{(2)} \end{pmatrix} \begin{pmatrix} a_1^{(2)} \\ \vdots \\ a_i^{(2)} \end{pmatrix}$$

$$+ \begin{pmatrix} x_1^{(3)} & \cdots & x_j^{(3)} \\ y_1^{(3)} & \cdots & y_j^{(3)} \end{pmatrix} \begin{pmatrix} a_1^{(3)} \\ \vdots \\ a_j^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{17}$$

implies that the vectors $a^{(k)}$ $(k = 1, 2, 3)$ are zero vectors. Multiplying (17) by the preconditioned matrix $M_{ps}^{-1}\mathcal{A}$, and recalling that the first matrix in (17) arises from the case $\lambda_k = 1$ $(k = 1, \cdots, n)$, the second matrix from the case $\lambda_k = 1$ $(k = 1, \cdots, i)$, where the components y are basis vectors of the generalized eigenvalue problem $By = \lambda AG^{-1}By$, and the last matrix from the case $\lambda_k \neq 1$ $(k = 1, \cdots, j)$. We have

$$\begin{pmatrix} x_1^{(1)} & \cdots & x_n^{(1)} \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_1^{(1)} \\ \vdots \\ a_n^{(1)} \end{pmatrix} + \begin{pmatrix} x_1^{(2)} & \cdots & x_i^{(2)} \\ y_1^{(2)} & \cdots & y_i^{(2)} \end{pmatrix} \begin{pmatrix} a_1^{(2)} \\ \vdots \\ a_i^{(2)} \end{pmatrix}$$

$$+ \begin{pmatrix} x_1^{(3)} & \cdots & x_j^{(3)} \\ y_1^{(3)} & \cdots & y_j^{(3)} \end{pmatrix} \begin{pmatrix} \lambda_1^{(3)} a_1^{(3)} \\ \vdots \\ \lambda_j^{(3)} a_j^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{18}$$

Subtracting (17) from (18), we obtain

$$\begin{pmatrix} x_1^{(3)} & \cdots & x_j^{(3)} \\ y_1^{(3)} & \cdots & y_j^{(3)} \end{pmatrix} \begin{pmatrix} (\lambda_1^{(3)} - 1) a_1^{(3)} \\ \vdots \\ (\lambda_j^{(3)} - 1) a_j^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since the components $x_k^{(3)}$ $(k = 1, \cdots, j)$ are linearly independent eigenvectors which arise from the generalized eigenvalue problem (16) and $y_k^{(3)} = C^{-1}B^{\mathrm{T}}x_k^{(3)}$ $(k = 1, \cdots, j)$. Thus we have

$$(\lambda_k - 1)a_k^{(3)} = 0, \quad k = 1, \cdots, j.$$

As a result of the eigenvalues λ_k $(k = 1, \cdots, j)$ are nonunit. We obtain $a_k^{(3)} = 0$ $(k = 1, \cdots, j)$. In addition, we know the components $y_k^{(2)}$ $(k = 1, \cdots, i)$ are basis vectors of the equation $By = AG^{-1}By$, which implies that $y_k^{(2)}$ $(k = 1, \cdots, i)$ are linearly independent. Thus we have $a_k^{(2)} = 0$ $(k = 1, \cdots, i)$.

Therefore, substituting $a_k^{(2)} = 0$ $(k = 1, \cdots, i)$ and $a_k^{(3)} = 0$ $(k = 1, \cdots, j)$ into (17), then equation (17) simplifies to

$$\begin{pmatrix} x_1^{(1)} & \cdots & x_n^{(1)} \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_1^{(1)} \\ \vdots \\ a_n^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Clearly, $a_k^{(1)} = 0$ $(k = 1, \cdots, n)$ follows from the linear independence of $x_k^{(1)}$ $(k = 1, \cdots, n)$. Summarizing the discussions above, we obtain $a^{(k)} = 0$ $(k = 1, 2, 3)$. $\square$

4 Numerical experiments

In this section, we report on numerical results obtained with a Matlab 7. 0.1 implementation on a Window-XP with 2.93GHz 64-bit processor and 2GB memory. The main goal is to test the product preconditioner (PS) defined in (10) and to compare it with the block diagonal preconditioner (BD)

$$M_{bd} = \begin{pmatrix} G & O \\ O & -(C + B^{\mathrm{T}}G^{-1}B) \end{pmatrix}, \tag{19}$$

presented in [16, 17, 24, 25], the block triangular preconditioner (BT)

$$M_{bt} = \begin{pmatrix} G & B \\ O & -(C + B^{\mathrm{T}}G^{-1}B) \end{pmatrix}, \tag{20}$$

considered in [11, 16, 23–26] and the constraint (SC) preconditioners given in (8) by the computing time (CPU), iteration step (IT) and relative residual error (RES).

There are various strategies to choose G in PS, SC, BD and BT preconditioners. In our computations, we not only take G to be the diagonal matrix formed with the diagonal entries of A, i.e., $G = \mathrm{diag}(\mathrm{diag}(A))$, but also to be the tridiagonal matrix of the (1,1) block matrix of A, that is, $G = \mathrm{tridiag}(A)$. As a representative iterative solver we used GMRES [30] with the right preconditioning in our experiments. All iterations are started from the zero vector, and terminated when $\mathrm{RES} = \|b - Au\|_2/\|b\|_2 \leq 10^{-9}$.

The test problem is the Helmholtz equation (1), together with radiation boundary condition (2), see [2, 9] for details. By using a finite difference discretization to equation (1) on the uniform grid of Ω, we obtain the nonsingular and symmetric indefinite linear system (3), where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times m}$ are nonsingular, $B \in \mathbb{R}^{n\times m}$. To be more precise, we have matrix

$$A = K \otimes I + I \otimes K + I \otimes D, \quad B = -(I \otimes e_n), \quad C = I - hT,$$

with $K = \mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{p\times p}$, $D = -4\pi^2h^2I$, $I \in \mathbb{R}^{p\times p}$ an identity matrix, $e_n = [0, 0, \cdots, 0, 1]^{\mathrm{T}} \in \mathbb{R}^p$, $h = 1/(p+1)$, and $T \in \mathbb{R}^{p\times p}$ a Toeplitz matrix which results from the generating function $f(\theta) = 2|\theta|(\theta^2 - 1)$. Hence, we have $n = p^2$, $m = p$, and the order of the coefficient matrix $\mathcal{A}$ is $n+m$. Moreover, we choose the right-hand vector $b = [f^{\mathrm{T}}, g^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n+m}$ such that the exact solution of system (3) is $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}} = [1, 1, \cdots, 1]^{\mathrm{T}}$, and GMRES(50) with at most 50 restarts is used in our experiments thus the number 2500 in Table 1 and Table 2 means that the corresponding preconditioned GMRES method does not converge in 2500 iterations.

h		1/32	1/48	1/64	1/80	1/90
$n+m$		992	2256	4032	6320	8010
BD	IT	123	325	794	530	1108
	CPU	0.4530	2.5470	10.7350	12.6410	33.2660
	RES	8.4763e-10	9.7414e-10	9.7214e-10	9.9784e-10	9.9993e-10
BT	IT	98	165	554	785	731
	CPU	0.3440	1.3750	7.7040	18.0790	22.9220
	RES	6.9060e-10	9.6006e-10	9.6952e-10	9.8453e-10	9.9223e-10
SC	IT	98	217	341	850	976
	CPU	0.3280	1.5930	4.4210	17.6250	27.5470
	RES	7.4747e-10	9.4590e-10	9.7868e-10	9.9417e-10	9.9499e-10
PS	IT	8	9	10	10	10
	CPU	0.1250	0.4530	1.2650	2.7180	4.0620
	RES	9.2136e-10	4.2030e-10	9.2256e-11	1.5617e-10	1.9037e-10

Table 1: IT, CPU and RES of the BD, BT, SC and PS preconditioned GMRES methods for this Helmholtz equation when $G = \mathrm{tridiag}(A)$.

h		1/32	1/48	1/64	1/80	1/90
$n+m$		992	2256	4032	6320	8010
BD	IT	261	615	1085	1316	2500
	CPU	0.4220	2.0320	5.8590	11.6560	28.7190
	RES	8.7031e-10	9.9129e-10	9.9364e-10	9.9653e-10	8.1554e-09
BT	IT	211	603	969	2003	1403
	CPU	0.3280	2.0160	5.2500	18.4530	16.2660
	RES	9.5016e-10	9.7251e-10	9.9961e-10	9.9769e-10	9.9885e-10
SC	IT	200	410	1115	1025	1470
	CPU	0.3430	1.3900	6.3750	9.5320	18.0310
	RES	9.8438e-10	9.6321e-10	9.9669e-10	9.9912e-10	9.9463e-10
PS	IT	9	9	10	10	10
	CPU	0.0780	0.1720	0.3430	0.5320	0.7190
	RES	6.2956e-11	6.5957e-10	1.2888e-10	2.1624e-10	2.6212e-10

Table 2: IT, CPU and RES of the BD, BT, SC and PS preconditioned GMRES methods for this Helmholtz equation when $G = \text{diag}(\text{diag}(A))$.

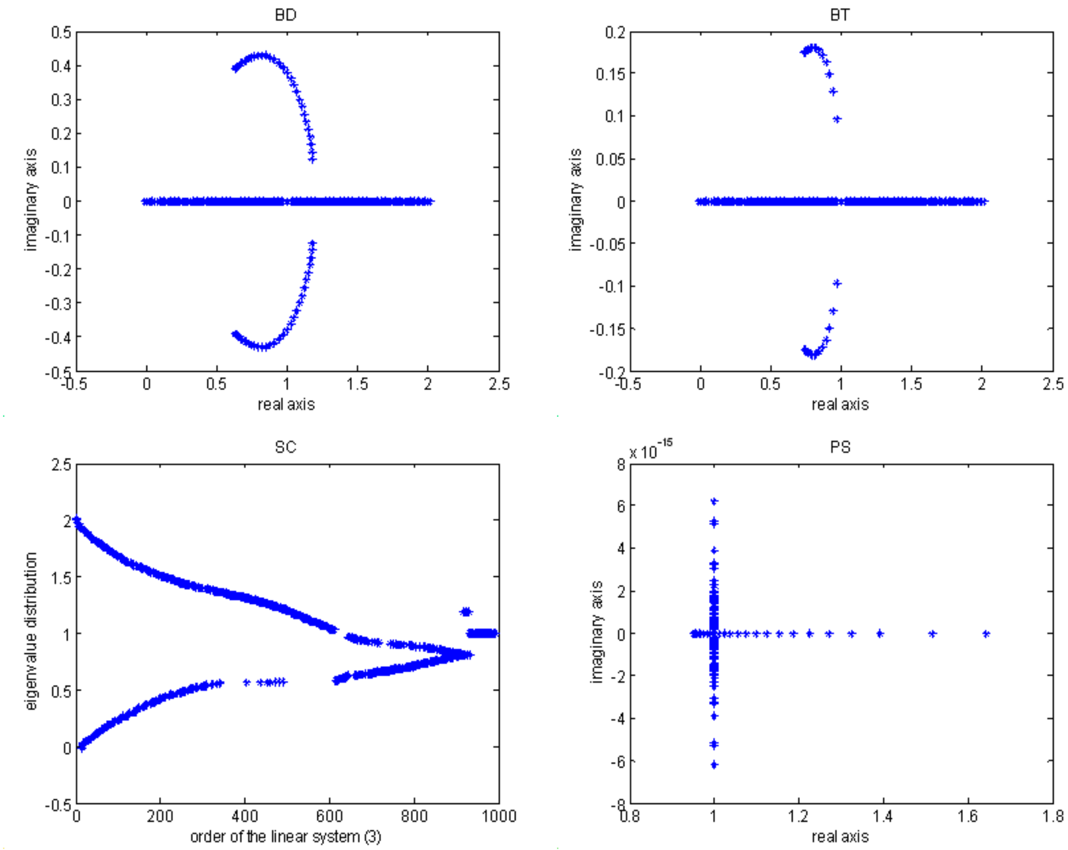

Figure 1: Comparisons of the eigenvalue distribution of the BD, BT, SC and PS preconditioned matrices for this Helmholtz equation when $G = \text{tridiag}(A)$ and $n + m = 992$.

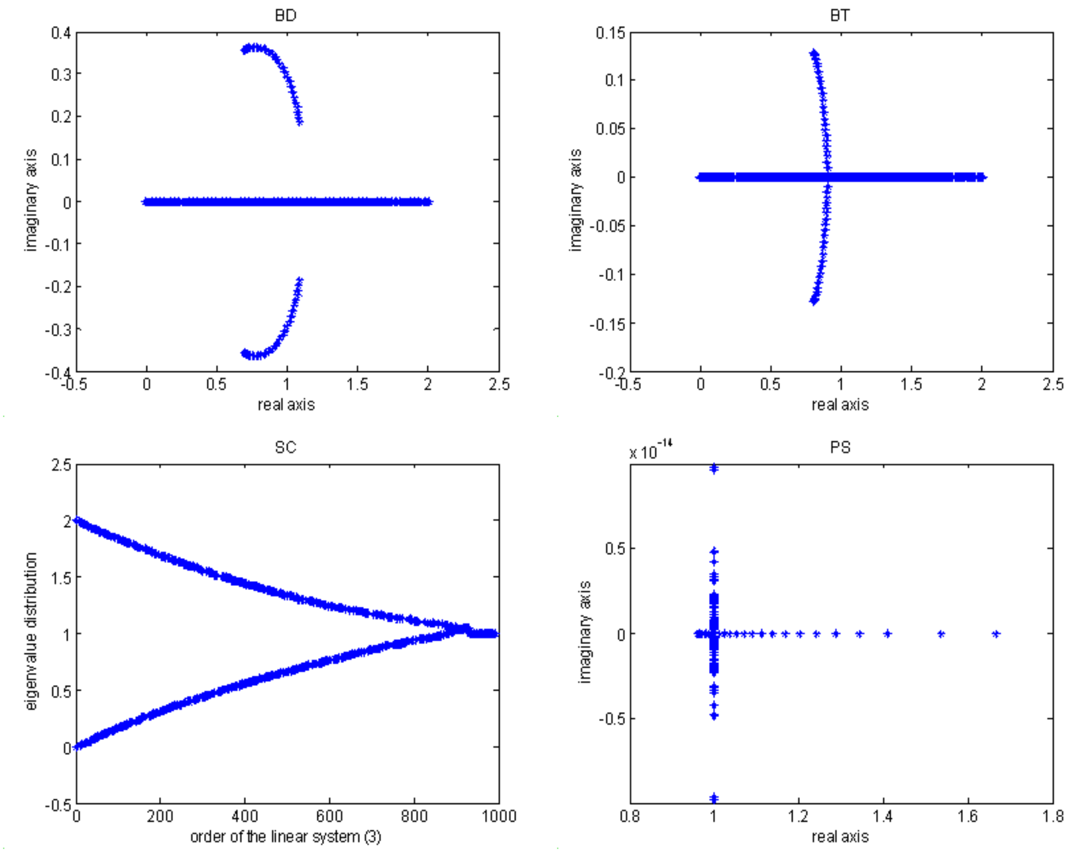

Figure 2: Comparisons of the eigenvalue distribution of the BD, BT, SC and PS preconditioned matrices for this Helmholtz equation when $G = \text{diag}(\text{diag}(A))$ and $n + m = 992$.

Table 1 supplies the IT, CPU and RES of the BD, BT, SC and PS preconditioned GMRES methods for this Helmholtz equation when $G = \text{tridiag}(A)$. As we have seen from Table 1, the PS preconditioned GMRES method has given the best iteration counts. For the BD, BT and SC preconditioned GMRES methods, their iteration counts have been reduced by around 96%. In terms of the computing time, the PS preconditioned GMRES method costs much less than these of the BD, BT and SC preconditioned GMRES methods. In addition, the precision of the relative residual error for the PS preconditioned GMRES method is higher than these of the BD, BT and SC preconditioned GMRES methods, except for the case that $n + m = 992$.

Table 2 provides the IT, CPU and RES of the BD, BT, SC and PS preconditioned GMRES methods for this Helmholtz equation when $G = \text{diag}(\text{diag}(A))$. From Table 2, it is not difficult to find that, for this approximate (1,1) block matrix G, all the iteration counts, computing time and the relative residual error of the PS preconditioned GMRES method are better than these of the BD, BT and SC preconditioned GMRES methods.

Both the numerical results in Table 1 and Table 2 have shown that the PS preconditioned GMRES method is superior to the BD, BT and SC preconditioned GMRES methods in obtaining a considerable reduction of iteration counts. These results have confirmed our theoretical analysis in previous sections. That is, the convergence of a Krylov subspace method under preconditioning has relation to the spectral properties of the preconditioned matrix.

For obtaining an intuitive comparison, Figure 1 and Figure 2 have plotted the eigenvalue distribution of the BD, BT, SC and PS preconditioned matrices for $G = \mathrm{tridiag}(A)$ and $G = \mathrm{diag}(\mathrm{diag}(A))$ with the chosen order of the nonsingular and symmetric indefinite linear system (3) is 992, respectively.

5 Conclusions

We have proposed and investigated a new preconditioner for the numerical solution of block two-by-two symmetric indefinite matrices whose (1,1) and (2,2) blocks are nonsingular. As we have seen in this paper, the proposed preconditioner is constructed as the product of two fairly simple preconditioners: one is the famous block Jacobi preconditioner, and the other is the popular constraint preconditioner. Here, we call it the product preconditioner, and denote it as PS preconditioner. Results concerning the eigenvalue distribution and form of the eigenvectors of the preconditioned matrix $M_{ps}^{-1}\mathcal{A}$ are discussed in Section 3, respectively. Numerical experiments with preconditioned GMRES method on the problem (1) are used to illustrate the efficiency and stability of the proposed product preconditioner. Moreover, we have confirmed our theoretical analysis by comparing the IT, CPU and RES of the BD, BT, SC and PS preconditioned GMRES methods in Table 1 and Table 2.

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HYERS-ULAM STABILITY OF A GENERAL DIAGONAL SYMMETRIC FUNCTIONAL EQUATION

CHOONKIL PARK AND HAMID REZAEI*

Abstract. Using the direct method and the fixed point method, we prove the Hyers-Ulam stability for the symmetric functional equation $f(\varphi_1(x,y,z)) = \varphi_2(f(x), f(y), f(z))$ in Banach spaces. As a consequence, we obtain some stability results in the sense of Hyers-Ulam-Rassias.

1. Introduction

The stability theory of functional equations originated from the well-known Ulam's problem [15], concerning the stability of homomorphisms in metric groups. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Let X_1 and X_2 be Banach spaces. Assume that $f : X_1 \to X_2$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X_1$ and some $\varepsilon > 0$. Then there exists a unique additive mapping $T : X_1 \to X_2$ such that $\|f(x) - T(x)\| \le \varepsilon$ for all $x \in X_1$. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [14] for linear mappings, considering the Cauchy difference to be unbounded.

Theorem 1.1. ([14]) *Let X_1 be a normed space and X_2 a Banach space. Let $f : X_1 \to X_2$ satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in X_1$, where $\theta > 0$ and $p \in [0, 1)$. Then there exists a unique additive mapping $A : X_1 \to X_2$ such that $\|f(x) - A(x)\| \le \frac{2\theta}{2-2^p}\|x\|^p$ for all $x \in X_1$.

A generalization of the Th.M. Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias' approach. J.M. Rassias [13] followed the innovative approach of the Th.M. Rassias Theorem [14] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q = 1$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3, 5, 8, 9]).

In this paper, we introduce the following functional equation

$$f(\varphi_1(x,y,z)) = \varphi_2(f(x), f(y), f(z)). \tag{1.2}$$

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*Corresponding author.

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Using the direct method and the fixed point method, we prove the Hyers-Ulam stability of the functional equation (1.2) in Banach spaces.

2. Hyers-Ulam stability of (1.2): direct method

In this section, we prove the Hyers-Ulam stability of the functional equation (1.2), where $\varphi_i : X_i \times X_i \times X_i \to X_i, i = 1, 2$, are mappings such that

$$\varphi_i(\varphi_i(x, x, x), \varphi_i(y, y, y)) = \varphi_i(\varphi_i(x, y, z), \varphi_i(x, y, z), \varphi_i(x, y, z)). \tag{2.1}$$

Let us call such mappings *diagonal symmetric* on X_i. For example

(1) Let X be a vector space, and $\varphi : X \times X \times X \to X$ be a function that

$$\varphi(\lambda x, \lambda y, \lambda z) = \lambda\varphi(x, y, z) \quad (x, y, z \in X)$$

for every scalar λ and $\varphi(x, x, x) = \alpha x$ for some scalar α, then φ is diagonal symmetric on X.

(2) Let X be a vector space, and $\varphi : X\times X\times X \to X$ defined by $\varphi(x, y, z) = ax+by+cz+d$, where a, b, c, d are scalars and $x, y, z \in X$. Then it is easy to check that φ is diagonal symmetric.

Theorem 2.1. *Assume that X_1 is a normed space and X_2 is a Banach space and that φ_1, φ_2 are continuous diagonal symmetric mappings on X_1, X_2, respectively. Put $T_i(x) := \varphi_i(x, x, x)$ for $i = 1, 2$ and suppose that T_2 is an invertible bounded linear operator on X_2. Let $\beta : X_1 \times X_1 \times X_1 \to [0, +\infty)$ be a function with this property that there exists some $0 < \lambda < 1$ such that*

$$\|T_2^{-1}\|\beta(T_1x, T_1y, T_1z) \le \lambda\beta(x, y, z)$$

for all $x, y, z \in X_1$. If $f : X_1 \to X_2$ is a mapping satisfying

$$\|f(\varphi_1(x, y, z)) - \varphi_2(f(x), f(y), f(z))\| < \beta(x, y, z) \tag{2.2}$$

for all $x, y, z \in X_1$, then there exists a unique mapping $A : X_1 \to X_2$ such that

$$\|f(x) - A(x)\| \le \frac{\|T_2^{-1}\|\beta(x, x, x)}{1 - \lambda}, \tag{2.3}$$

$$A(\varphi_1(x, y, z)) = \varphi_2(A(x), A(y), A(z)) \tag{2.4}$$

for all $x, y, z \in X_1$.

Proof. Letting $z = y = x$ (2.2), we get

$$\|fT_1(x) - T_2f(x)\| \le \beta(x, x)$$

for all $x \in X_1$. It follows from (2.1) that

$$\varphi_i(T_ix, T_iy, T_iz) = T_i(\varphi_i(x, y, z)) \tag{2.5}$$

for all $x, y, z \in X_i$ and $i = 1, 2$. Let $q_n(x) := T_2^{-n}f(T_1^n x)$ for all $n \ge 1$ and all $x \in X_1$. Then

$$\begin{aligned}\|q_{n+1}(x) - q_n(x)\| &= \|T_2^{-n-1}f(T_1^{n+1}x) - T_2^{-n}f(T_1^n x)\| \\ &\le \|T_2^{-n-1}\|\|fT_1(T_1^n x) - T_2f(T_1^n x)\| \\ &\le \|T_2^{-1}\|^{n+1}\beta(T_1^n x, T_1^n x) \le \|T_2^{-1}\|\lambda^n\beta(x, x, x).\end{aligned}$$

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Here, in the last inequality, the contractive property of β is used. Hence

$$\|q_{n+1}(x) - q_n(x)\| \le \|T_2^{-1}\|\lambda^n \beta(x,x,x),$$

and so the sequence $\{q_n(x)\}$ is a Cauchy sequence for each x. Since X_2 is complete, there exists a limit mapping $A(x) := \lim_{n\to\infty} q_n(x)$. Now by induction on n, we prove that

$$\|q_n(x) - f(x)\| \le \sum_{i=0}^{n-1} \|T_2^{-1}\|\lambda^i \beta(x,x,x) \tag{2.6}$$

for all $n \in \mathbb{N}$ and all $x \in X_1$. Fix $x \in X_1$. Note that

$$\begin{aligned} \|q_1(x) - f(x)\| &= \|T_2^{-1} f(T_1(x)) - f(x)\| \\ &\le \|T_2^{-1}\|\|f(T_1(x)) - T_2(f(x))\| \le \|T_2^{-1}\|\beta(x,x,x). \end{aligned}$$

Now suppose (2.6) holds for a fixed n. Then

$$\begin{aligned} \|q_{n+1}(x) - f(x)\| &\le \|q_{n+1}(x) - q_n(x)\| + \|q_n(x) - f(x)\| \\ &\le \|T_2^{-1}\|\lambda^n \beta(x,x) + \sum_{i=0}^{n-1} \|T_2^{-1}\|\lambda^i \beta(x,x,x) \\ &= \sum_{i=0}^{n} \|T_2^{-1}\|\lambda^i \beta(x,x,x). \end{aligned}$$

Letting $n \to +\infty$ in (2.6), we get

$$\|A(x) - f(x)\| \le \frac{\|T_2^{-1}\|\beta(x,x,x)}{1-\lambda}$$

for all $x \in X_1$.

Now we prove that A satisfies (2.4). Replacing x, y, z in (2.2) with $T_1^n x, T_1^n y, T_1^n z$, respectively, we get

$$\begin{aligned} \|f(\varphi_1(T_1^n x, T_1^n y, T_1^n y)) - \varphi_2(f(T_1^n x), f(T_1^n y), f(T_1^n z))\| \\ \le \beta(T_1^n x, T_1^n y, T_1^n z). \end{aligned} \tag{2.7}$$

It follows from (2.5) that

$$\varphi_1(T_1^n x, T_1^n y, T_1^n z) = T_1^n(\varphi_1(x,y,z)) \tag{2.8}$$

for all $x, y, z \in X_1$, and

$$\varphi_2(T_2^n x, T_2^n y, T_1^n z) = T_2^n(\varphi_2(x,y,z))$$

for all $x, y, z \in X_2$. Replacing x, y, z by $T_2^{-n}x, T_2^{-n}y, T_1^n z$, respectively, in the last above relation, we get

$$\varphi_2(x,y,z) = T_2^n(\varphi_2(T_2^{-n}x, T_2^{-n}y, T_2^{-n}z))$$

and then replacing x, y, z by $f(T_1^n x), f(T_1^n y), f(T_1^n z)$, respectively, we get

$$\varphi_2(f(T_1^n x), f(T_1^n y), f(T_1^n z)) = T_2^n(\varphi_2(T_2^{-n}(f(T_1^n x)), T_2^{-n}(f(T_1^n y)), T_2^{-n}(f(T_1^n z)))).$$

By the definition of q_n, we obtain

$$\varphi_2(f(T_1^n x), f(T_1^n y), f(T_1^n z)) = T_2^n(\varphi_2(q_n(x), q_n(y), q_n(z))). \tag{2.9}$$

It follows from (2.7), (2.8) and (2.9) that

$$\begin{aligned}
&\|q_n(\varphi_1(x,y,z)) - \varphi_2(q_n(x), q_n(y), q_n(z))\| \\
&= \|T_2^{-n} f(T_1^n \varphi_1(x,y,z)) - \varphi_2(q_n(x), q_n(y), q_n(z))\| \\
&\leq \|T_2^{-1}\|^n \|f(T_1^n \varphi_1(x,y,z)) - T_2^n \varphi_2(q_n(x), q_n(y), q_n(z))\| \\
&= \|T_2^{-1}\|^n \|f(\varphi_1(T_1^n x, T_1^n y, T_1^n z)) - \varphi_2(f(T_1^n x), f(T_1^n y), f(T_1^n z))\| \\
&\leq \|T_2^{-1}\|^n \beta(T_1^n x, T_1^n y, T_1^n z) \\
&\leq \lambda^n \beta(x,x,x).
\end{aligned}$$

Therefore,

$$\|q_n(\varphi_1(x,y,z)) - \varphi_2(q_n(x), q_n(y), q_n(z))\| \leq \lambda^n \beta(x,x,x)$$

for all $x, y \in X_1$ and all $n \in \mathbb{N}$. Applying the continuity of φ, considering $0 < \lambda < 1$, and letting $n \to +\infty$ in the last inequality, we obtain (2.4).

Now we prove that A is a unique mapping satisfying (2.3) and (2.4). Assume that there exists another mapping $A' : X \to X$ satisfying (2.3) and (2.4). Letting $y = x$ in (2.4), we get $AT_1(x) = T_2A(x)$ and $A'T_1(x) = T_2A'(x)$ and more generally

$$AT_1^n(x) = T_2^n A(x) \text{ and } A'T_1^n(x) = T_2^n A'(x).$$

Hence

$$A(x) = T_2^{-n} A(T_1^n(x)) \text{ and } A'(x) = T_2^{-n} A'(T_1^n(x))$$

for all $x \in X$ and $n \in \mathbb{N}$. By the triangle inequality, (2.3) and (2.10), we obtain

$$\begin{aligned}
\|A(x) - A'(x)\| &= \|T_2^{-n} A(T_1^n x) - T_2^{-n} A'(T_1^n x)\| \\
&\leq \|T_2^{-1}\|^n \|A(T_1^n x) - A'(T_1^n x)\| \\
&\leq \|T_2^{-1}\|^n (\|A(T_1^n x) - f(T_1^n x)\| + \|f(T_1^n x) - A'(T_1^n x)\|) \\
&\leq \|T_2^{-1}\|^n (2\frac{\|T_2^{-1}\|\beta(T_1^n x, T_1^n x)}{1-\lambda}) \\
&\leq 2\|T_2^{-1}\|(\frac{\|T_2^{-1}\|^n \beta(T_1^n x, T_1^n x)}{1-\lambda}) \\
&\leq 2\|T_2^{-1}\|\frac{\lambda^n \beta(x,x)}{1-\lambda}
\end{aligned}$$

for all $x \in X_1$ and all $n \in \mathbb{N}$. Letting $n \to +\infty$, we get $A(x) = A'(x)$ for all $x \in X_1$. □

The proof of the following theorem is similar and we omit it:

Theorem 2.2. *Assume that X_1 is a normed space and X_2 is a Banach space and that φ_1, φ_2 are continuous diagonal symmetric mappings on X_1, X_2, respectively. Put $T_i(x) := \varphi_i(x,x,x)$ for $i = 1, 2$ and suppose that T_2 is a bounded linear operator on X_2 and T_1 is invertible on X_1. Let $\beta : X_1 \times X_1 \times X_1 \to [0, +\infty)$ be a function with this property that there exists some $0 < \lambda < 1$ such that*

$$\|T_2\|\beta(T_1^{-1}x, T_1^{-1}y, T_1^{-1}z) \leq \lambda\beta(x,y,z)$$

for all $x, y, z \in X_1$. If $f : X_1 \to X_2$ is a mapping satisfying

$$\|f(\varphi_1(x,y,z)) - \varphi_2(f(x), f(y), f(z))\| < \beta(x,y,z)$$

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for all $x, y, z \in X_1$, then there exists a unique mapping $A : X_1 \to X_2$ such that

$$\|f(x) - A(x)\| \le \frac{\|T_1^{-1}\|\beta(x,x,x)}{1-\lambda}$$
$$A(\varphi_1(x,y,z)) = \varphi_2(A(x), A(y), A(z))$$

for all $x, y, z \in X_1$.

Corollary 2.3. *Assume that X_1 is a normed space and X_2 is a Banach space and that φ_1, φ_2 are continuous diagonal symmetric mappings on X_1, X_2, respectively. Put $T_i(x) := \varphi_i(x,x,x)$ for $i = 1, 2$ and suppose that T_2 is an invertible bounded linear operator on X_2. Let $f : X_1 \to X_2$ be a mapping for which there exist some $\theta_1, \theta_2 > 0$, and $p \ge 0$ such that*

$$\|f(\varphi_1(x,y,z)) - \varphi_2(f(x), f(y), f(z))\| < \theta_1(\|x\|^p + \|y\|^p + \|z\|^p) + \theta_2(\|x\|^{p/3}\|y\|^{p/3}\|z\|^{p/3})$$

for all $x, y, z \in X_1$. If $\|T_2^{-1}\|\|T_1\|^p < 1$, then there exists a unique mapping $A : X_1 \to X_2$ such that

$$\|f(x) - A(x)\| \le \theta\|T_2^{-1}\|\frac{(2\theta_1 + \theta_2)\|x\|^p}{1 - \|T_2^{-1}\|\|T_1\|^p},$$

$$A(\varphi_1(x,y,z)) = \varphi_2(A(x), A(y), A(z))$$

for all $x, y, z \in X_1$.

Proof. Let

$$\beta(x,y,z) := \theta_1(\|x\|^p + \|y\|^p + \|z\|^p) + \theta_2(\|x\|^{p/3}\|y\|^{p/3}\|z\|^{p/3})$$

for $x, y \in X_1$, and $\lambda := \|T_2^{-1}\|\|T_1\|^p$. Then

$$\|T_2^{-1}\|\beta(T_1x, T_1y) \le \lambda\beta(x,y,z)$$

for all $x, y, z \in X_1$. This completes the proof. □

Consider the following choices of $\varphi_1, \varphi_2, T_1$ and T_2:
(1) $\varphi_1(x,y,z) = \varphi_2(x,y,z) = \frac{x+y+z}{2}$ and $T_1(x) = T_2(x) = \frac{3x}{2}$;

(2) $\varphi_1(x,y,z) = xyz$, $\varphi_2(x,y,z) = x + y + z$,$T_1(x) = x^3$ and $T_2(x) = 3x$;
to deduce the following corollary:

Corollary 2.4. *The following functional equations has Hyers-Ulam stability in the sense of Theorem 2.1 and 2.2:*

(i) $2f(\frac{x+y+z}{2}) = f(x) + f(y) + f(z)$, $f : X_1 \to X_2$ where X_1 is a vector space and X_2 is a Banach space.

(ii) $f(xyz) = f(x) + f(y) + f(z)$, $f : X_1 \to X_2$ where X_1 is any abelian semigroup and X_2 is a Banach space.

Let A be a C^*-algebra and $a \in A$ a self-adjoint element, i.e., $a = a^*$. Then a is said to be positive if it is of the form $a = bb^*$ for some $a \in A$. The set of positive elements of A is denoted by A^+.

Note that A^+ is a closed convex cone (see [4]).

It is well-known that for a positive element a and a positive integer n there exists a unique positive element $x \in A^+$ such that $a = x^n$. We denote x by $\sqrt[n]{a}$. Then the functional equation (1.1) is Hyers-Ulam stable in the sense of Theorem 2.1 and 2.2 when the mapping $\varphi : A^+ \times A^+ \times A^+ \to A^+$ is one of the following choices:

(1) $\varphi(x, y, z) = \sqrt[n]{ax^2 + by^2 + cz^2}$ where $a + b + c > 1$,
(2) $\varphi(x, y, z) = \sqrt[n]{x^n + y^n + z^n}$,

3. Hyers-Ulam stability of (1.2): fixed point method

We now introduce one of the fundamental results of the fixed point theory.

For a nonempty set X, we introduce the definition of the generalized metric on X. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$ for all $x, y \in X$,
- $d(x, z) \le d(x, y) + d(y, z)$,

for all $x, y, z \in X$.

Using the fixed point method, we prove the Hyers-Ulam stability of the functional equation (1.2) in Banach spaces.

Theorem 3.1. ([10]) *Let $(\mathcal{X}, d)$ be a generalized complete metric space. Assume that $\Lambda : \mathcal{X} \to \mathcal{X}$ is a strictly contractive operator with the Lipschitz constant $L < 1$, i.e.,*

$$d(\Lambda g, \Lambda h) \le L d(g, h)$$

for all $g, h \in \mathcal{X}$. If there exists a nonnegative integer n_0 such that $d(\Lambda^{n_0+1} f, \Lambda^{n_0} f) < +\infty$ for some $f \in \mathcal{X}$, then the following statements are true:

(1) The sequence $\{\Lambda^n f\}$ converges to a fixed point A of Λ;

(2) A is the unique fixed point of Λ in $\mathcal{X}^* = \{g \in \mathcal{X} : d(\Lambda^{n_0} f, g) < +\infty\}$;

(3) If $g \in X^*$, then

$$d(g, A) \le \frac{1}{1 - L} d(\Lambda g, g).$$

Radu [12] proved the Hyers-Ulam stability of the additive Cauchy equation (1.1) by using fixed point method (see [2]).

In the following, Theorem 2.1 is proved by the fixed point method.

Theorem 3.2. *Let $X_1, X_2, \varphi_1, \varphi_2, T_1, T_2, \beta$ be given as in Theorem 2.1. If $f : X_1 \to X_2$ is a mapping satisfying (2.2), then there exists a unique mapping $A : X_1 \to X_2$ satisfying (2.3) and $AT_1(x) = T_2 A(x)$ for all $x \in X_1$.*

Proof. Letting $y = x$ in (2.2), we get

$$\|fT_1(x) - T_2 f(x)\| \le \beta(x, x, x)$$

for all $x \in X_1$. Consider the set $\mathcal{X} := \{f : f : X_1 \to X_2 \text{ is a function } \}$ and define the generalized metric on $\mathcal{X}$ by

$$d(g, h) = \inf \{\mu \in (0, +\infty) : \|g(x) - h(x)\| \le \mu \beta(x, x) \text{ for all } x \in X_1\}.$$

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where, as usual, $\inf = +\infty$. It is easy to show that $(\mathcal{X}, d)$ is complete (see [11]). Now we consider the linear mapping $\Lambda : \mathcal{X} \to \mathcal{X}$ such that

$$\Lambda g(x) = T_2^{-1} g(T_1 x)$$

for all $x \in X_1$. For given $g, h \in \mathcal{X}$,

$$\|\Lambda g(x) - \Lambda f(x)\| = \|T_2^{-1} g(T_1 x) - T_2^{-1} h(T_1 x)\| \le \|T_2^{-1}\| \beta(T_1 x, T_1 x, T_1 x) \le \lambda \beta(x, x, x)$$

for all $x \in X_1$. By the definition of d,

$$d(\Lambda f, \Lambda g) \le \lambda d(f, g).$$

Note that

$$\begin{aligned} \|f(x) - \Lambda f(x)\| &= \|f(x) - T_2^{-1} f(T_1 x)\| \\ &\le \|T_2^{-1}\| \|T_1 f(x) - f(T_1 x)\| \le \|T_2^{-1}\| \beta(x, x, x) \end{aligned}$$

for all $x \in X_1$, and so $d(\Lambda f, f) \le \|T_2^{-1}\| < +\infty$. By the preceding theorem, there exists a mapping $A : X_1 \to X_2$ satisfying the following conditions:

(1) A is a fixed point of Λ, i.e., $T_2^{-1} A T_1 = \Lambda A = A$ whence $A(T_1(x)) = T_2(A(x))$ for all $x \in X_1$. Moreover, A is a unique fixed point of Λ in the set $\mathcal{X}^* := \{g \in \mathcal{X} : d(f, g) < +\infty\}$ which implies that

$$\|f(x) - A(x)\| \le \mu \beta(x, x, x).$$

(2) $d(\Lambda^n f, A) \to 0$ as $n \to +\infty$, i.e., $A(x) = \lim_n T_2^{-n} f(T_1^n x)$.

(3) By (3) of the preceding theorem, we conclude that

$$d(f, A) \le \frac{1}{1-\lambda} d(f, \Lambda f) < \frac{1}{1-\lambda} \|T_2^{-1}\|,$$

and so

$$\|f(x) - A(x)\| \le \frac{\|T_2^{-1}\| \beta(x, x, x)}{1-\lambda},$$

as desired. In order to prove that A satisfies (2.4), we can proceed exactly as in the proof of Theorem 2.2 to show that $A : X_1 \to X_2$ is indeed a mapping satisfying (2.4). □

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Choonkil Park

Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea

E-mail address: baak@hanyang.ac.kr

Hamid Rezaei

Department of Mathematics, College of Sciences, Yasouj University, Yasouj-75914-74831, Iran

E-mail address: rezaei@yu.ac.ir

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GENERAL DECAY OF SOLUTIONS FOR A SINGULAR NONLOCAL VISCOELASTIC PROBLEM WITH NONLINEAR DAMPING AND SOURCE

YUN SUN, GANG LI, AND WENJUN LIU

Abstract. This paper deals with a singular nonlocal viscoelastic problem with nonlinear damping and source terms. We establish a general decay rate result without imposing any restrictive growth assumption on the damping term.

1. Introduction

In this paper, we investigate the following one-dimensional viscoelastic equation

$$\begin{cases} u_{tt} - \dfrac{1}{x}(xu_x)_x + \displaystyle\int_0^t g(t-s)\dfrac{1}{x}(xu_x(x,s))_x \mathrm{d}s + h(u_t) = b|u|^{p-2}u, \ x \in (0,\ell), t \in (0,\infty), \\ u(\ell,t) = 0, \displaystyle\int_0^\ell xu(x,t)\mathrm{d}x = 0 \qquad\qquad t \in [0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad\qquad x \in [0,\ell], \end{cases} \tag{1.1}$$

where $\ell < \infty$, $b > 0$, $p > 2$, g and h are specific functions which will be given later.

This type of evolution problems are generally encountered when the data on the boundary can not be measured directly, but their average values are known. For the case of singular type, we can refer to [8, 9, 10, 11, 14] for the existence, uniqueness and blow-up results. Here, it is worth mentioning that many results concerning decay have been extensively studied for the case of classical conditions. Under the condition $-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t)$, the exponential or polynomial decay results were obtained in [3, 4, 5, 6]. Later, some authors relaxed these conditions by considering only $g'(t) \leq -\xi g(t)$ or $g'(t) \leq -\xi g^r(t)$, for all $t \geq 0$ and some $\xi > 0$ (see [1, 2, 15]). In [12, 13], the condition has been replaced by $g'(t) \leq -\xi(t)g(t)$, where $\xi(t)$ is a positive function. This allows the authors to obtain general rates of decay than just exponential or polynomial type.

Motivated by [11, 13], we study problem (1.1) in this paper and intend to establish a general decay result under certain conditions, without imposing any restrictive growth assumption on the damping term. The paper is organized as follows. In Section 2 we present some assumptions and known results needed for our work. Section 3 is devoted to the proof of some lemmas and the decay result: Theorem 2.4 .

2. Preliminaries and main result

In this section we first introduce some functional spaces and present some assumptions and known results which will be used throughout this paper, and then state our main result.

Let $L_x^p = L_x^p(0,\ell)$ be the weighted Banach space equipped with the norm $\|u\|_p = \left(\int_0^\ell x|u|^p \mathrm{d}x\right)^{\frac{1}{p}}$. In particular, when $p = 2$, we denote $H = L_x^2(0,\ell)$ to be the weighted Hilbert space of square

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integrable functions having the finite norm $\|u\|_H = \left(\int_0^\ell xu^2 \mathrm{d}x\right)^{\frac{1}{2}}$. We take $V = V_x^{1,1}(0,\ell)$ to be the weighted Hilbert space equipped with the norm $\|u\|_V = \left(\|u\|_H^2 + \|u_x\|_H^2\right)^{\frac{1}{2}}$, and $V_0 = \{v \in V \text{ such that } v(\ell) = 0\}$.

For the functionals g and h we give the following assumptions as in [13]:

(H1) $g(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function such that

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)\mathrm{d}s = l > 0,$$

and there exists a nondecreasing differentiable function $\xi(t)$ such that

$$g'(t) \le -\xi(t)g(t), \quad t \ge 0 \qquad \textit{and} \qquad \int_0^{+\infty} \xi(t)\mathrm{d}t = \infty$$

(H2) $h : \mathbb{R} \mapsto \mathbb{R}$ is a nondecreasing C^0 function such that there exists a strictly increasing function $h_0 \in C^1([0,+\infty))$, with $h_0(0) = 0$, and positive constants c_1, c_2, and ϵ such that

$$h_0(|s|) \le |h(s)| \le h_0^{-1}(s), \quad \forall\ |s| \le \epsilon, \tag{2.1}$$

$$c_1|s| \le |h(s)| \le c_2|s|, \quad \forall\ |s| \ge \epsilon. \tag{2.2}$$

Remark 1. *Hypothesis* (H2) *implies that $sh(s) > 0$, for all $s \neq 0$.*

Lemma 2.1. ([11]) *For any v in V_0, we have*

$$\int_0^\ell x(v(x))^2 \mathrm{d}x \le C_* \int_0^\ell x\left(v_x(x)\right)^2 \mathrm{d}x.$$

Lemma 2.2. ([11]) *For any v in V_0, $2 < p < 4$, we have*

$$\int_0^\ell x(v(x))^p \mathrm{d}x \le C_p \|v_x\|_H^p,$$

where C_p is a constant depending on p only.

Lemma 2.3. ([11, Theorem 2.3]) *Suppose that $2 < p < 3$ and* (H1) *and* (H2) *hold. Then for any u_0 in V_0 and u_1 in H, problem* (1.1) *has a unique local solution*

$$u \in C(0, t_*; V_0) \cap C^1(0, t_*; H)$$

for $t_* > 0$ small enough.

Now we introduce the functionals for $I(t)$ and $E(t)$:

$$I(t) := I(u(t)) = \Big(1 - \int_0^t g(s)\,\mathrm{d}s\Big)\int_0^\ell xu_x^2 \mathrm{d}x + (g \circ u_x)(t) - b\int_0^\ell x|u(t)|^p \mathrm{d}x, \tag{2.3}$$

$$\begin{aligned} E(t) := E(u(t)) = {} & \frac{1}{2}\Big(1 - \int_0^t g(s)\,\mathrm{d}s\Big)\int_0^\ell xu_x^2 \mathrm{d}x + \frac{1}{2}(g \circ u_x)(t) \\ & - \frac{b}{p}\int_0^\ell x|u(t)|^p \mathrm{d}x + \frac{1}{2}\int_0^\ell xu_t^2 \mathrm{d}x, \end{aligned} \tag{2.4}$$

where

$$(g \circ u_x)(t) = \int_0^\ell \int_0^t xg(t-s)|u_x(x,t) - u_x(x,s)|^2 \mathrm{d}s\mathrm{d}x.$$

Remark 2. *Multiplying Eq.* (1.1) *by xu_t and integrating over $(0,\ell)$, we can easily get*

$$E'(t) = \frac{1}{2}(g' \circ u_x)(t) - \frac{1}{2}g(t)\int_0^\ell xu_x^2(x,t)\mathrm{d}x - \int_0^\ell xu_t h(u_t)\mathrm{d}x \le 0, \quad \forall\ t \ge 0. \tag{2.5}$$

Our main result of this paper reads as follows.

Theorem 2.4. *Suppose that* (H1) *and* (H2) *hold,* $2 < p < 3$*, if* $(u_0, u_1) \in V_0 \times H$ *such that*

$$\beta = \frac{bC_p}{l}\left(\frac{2p}{(p-2)l}E(u_0,u_1)\right)^{\frac{p-2}{2}} < 1, \quad I(u_0) > 0. \tag{2.6}$$

Then, there exists a constant $C > 0$ *such that, for* t *large, the solution of* (1.1) *satisfies*

$$E(t) \le C\left(H_0^{-1}\left(\frac{1}{\int_0^t \xi(s)\mathrm{d}s}\right)\right)^2 \quad where \quad H_0(s) = sh_0(s). \tag{2.7}$$

Moreover, if we define $J(s) = \frac{h_0(s)}{s}$*, which is strictly increasing with* $J(0) = 0$*, then we can improve* (2.7) *to the following estimate:*

$$E(t) \le C\left(h_0^{-1}\left(\frac{1}{\int_0^t \xi(s)\mathrm{d}s}\right)\right)^2. \tag{2.8}$$

For the proof of the above theorem, we use the following lemma.

Lemma 2.5. ([7]) *Let* $E : \mathbb{R}^+ \to \mathbb{R}^+$ *be a nonincreasing function and* $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ *be a strictly increasing* C^1 *function, with* $\sigma(t) \to +\infty$ *as* $t \to +\infty$*. Assume that there exist* $p, q \ge 0$ *and* $c > 0$ *such that*

$$\int_S^\infty \sigma'(t)E(t)^{1+p}\mathrm{d}t \le cE(s)^{1+p} + \frac{cE(s)}{\sigma^q}, \quad 1 \le S < +\infty.$$

Then there exist positive constants κ *and* ω *such that*

$$E(t) \le \kappa e^{-\omega\sigma(t)} \qquad \forall\, t \ge 1, if p = q = 0$$
$$E(t) \le \frac{\kappa}{\sigma(t)^{\frac{1+q}{p}}} \qquad \forall\, t \ge 1, if p > 0.$$

3. General decay of solutions

In this section we prove our main result. For this purpose we establish several lemmas.

Lemma 3.1. ([11, Lemma 4.1 and Lemma 4.2]) *Under the assumptions of Theorem 2.4, we conclude that* $I(u(t)) > 0, \forall\, t > 0$ *and the solution is global and bounded. Furthermore, the following inequality holds*

$$l\int_0^\mu xu_x^2\mathrm{d}x \le \left(\frac{2p}{p-2}\right)E(u_0,u_1), \quad \forall t > 0. \tag{3.1}$$

Lemma 3.2. *For all* $u \in V_0$*, there exists* $C_* > 0$ *such that*

$$\int_0^\ell x\left(\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right)^2\mathrm{d}x \le (1-l)C_*(g\circ u_x)(t).$$

Proof. Using Cauchy-Schwarz's inequality, (H1) and Lemma 2.1, we can easily obtain the result.

We define the following functionals

$$\mathcal{L}(t) := N_1E(t) + N_2K(t) + \chi(t), \tag{3.2}$$

where

$$K(t) := -\int_0^\ell xu_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x,$$
$$\chi(t) := \int_0^\ell xuu_t\mathrm{d}x,$$

N_1 and N_2 are positive constants to be chosen later.

Lemma 3.3. *Suppose that* (H1) *holds and* $p > 2$. *Let* u *be the solution of problem* (1.1). *Then there exist positive constants* $\alpha_1, \alpha_2 > 0$ *such that*

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t). \tag{3.3}$$

Proof. Straightforward computations, Young's inequality and Lemma 3.2 lead to

$$\begin{aligned}\mathcal{L}(t) \leq & \left[\frac{N_1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)+\frac{C_*}{2}\right]\int_0^\ell xu_x^2\mathrm{d}x+\left(\frac{N_1}{2}+\frac{N_2}{2}+\frac{1}{2}\right)\int_0^\ell xu_t^2\mathrm{d}x \\ & +\left[\frac{N_1}{2}+\frac{N_2}{2}(1-l)C_*\right](g\circ u_x)(t)-\frac{bN_1}{p}\int_0^\ell x|u|^p\mathrm{d}x \\ \leq & \alpha_1\left(\int_0^\ell xu_x^2\mathrm{d}x+\int_0^\ell xu_t^2\mathrm{d}x+(g\circ u_x)(t)-\frac{b}{p}\int_0^\ell x|u|^p\mathrm{d}x\right),\end{aligned} \tag{3.4}$$

for some $\alpha_1 > 0$. On the other hand,

$$\begin{aligned}\mathcal{L}(t) \geq & \frac{1}{2}(N_1 l - C_*)\int_0^\ell xu_x^2\mathrm{d}x+\frac{1}{2}(N_1-N_2-1)\int_0^\ell xu_t^2\mathrm{d}x \\ & +\frac{1}{2}\left[N_1-N_2(1-l)C_*\right](g\circ u_x)(t)-\frac{bN_1}{p}\int_0^\ell x|u|^p\mathrm{d}x.\end{aligned} \tag{3.5}$$

Choose $N_2 > 1$ and then take N_1 satisfying

$$N_1 > \max\left\{\frac{C_*}{l}, N_2+1, N_2(1-l)C_*\right\}. \tag{3.6}$$

Then we completes the proof.

Lemma 3.4. *Suppose that* (H1) *and* (H2) *hold and* $p > 2$, *let* $(u_0, u_1) \in V_0 \times H$ *be given. If* u *is the solution of* (1.1), *then we have*

$$\chi'(t) \leq -\frac{l}{2}\int_0^\ell xu_x^2\mathrm{d}x+\int_0^\ell xu_t^2\mathrm{d}x+C(g\circ u_x)(t)+C\int_0^\ell xh^2(u_t)\mathrm{d}x+b\int_0^\ell x|u|^p\mathrm{d}x. \tag{3.7}$$

Proof. By exploiting problem (1.1) and integrating by parts, we get

$$\begin{aligned}\chi'(t) = & \int_0^\ell xu_t^2\mathrm{d}x-\int_0^\ell xu_x^2\mathrm{d}x+\left(\int_0^t g(s)\mathrm{d}s\right)\int_0^\ell xu_x^2\mathrm{d}x \\ & +\int_0^\ell xu_x\int_0^t g(t-s)\left(u_x(s)-u_x(t)\right)\mathrm{d}s\mathrm{d}x-\int_0^\ell xuh(u_t)\mathrm{d}x+b\int_0^\ell x|u|^p\mathrm{d}x.\end{aligned} \tag{3.8}$$

Using Young's and Poincaré's inequalities and Lemma 3.2, we obtain

$$\begin{aligned}& \int_0^\ell xu_x\int_0^t g(t-s)\left(u_x(s)-u_x(t)\right)\mathrm{d}s\mathrm{d}x \\ \leq & \delta\int_0^\ell xu_x^2\mathrm{d}x+\frac{1}{4\delta}\int_0^\ell x\left(\int_0^t g(t-s)\left(u_x(s)-u_x(t)\right)\mathrm{d}s\right)^2\mathrm{d}x \\ \leq & \delta\int_0^\ell xu_x^2\mathrm{d}x+\frac{C}{\delta}(g\circ u_x)(t),\end{aligned} \tag{3.9}$$

$$-\int_0^\ell xuh(u_t)\mathrm{d}x \leq \delta\int_0^\ell xu^2\mathrm{d}x+\frac{1}{4\delta}\int_0^\ell xh^2(u_t)\mathrm{d}x \leq \delta C_*\int_0^\ell xu_x^2\mathrm{d}x+\frac{1}{4\delta}\int_0^\ell xh^2(u_t)\mathrm{d}x. \tag{3.10}$$

Combining (3.8)-(3.10), and choosing δ small enough such that $\delta \leq \frac{l}{2(1+C_*)}$, then (3.7) is obtained.

Lemma 3.5. *Under the assumptions* (H1) *and* (H2), *suppose* $2 < p < 3$, *then the functional* K *satisfies, along the solution, the estimate*

$$K'(t) \le -\left(\int_0^t g(s)\mathrm{d}s - \delta\right)\int_0^\ell x u_t^2 \mathrm{d}x + \left(\delta + \frac{\delta l^2}{bC_p}\right)\int_0^\ell x u_x^2 \mathrm{d}x + \left(C + \frac{C}{\delta}\right)(g\circ u_x)(t)$$
$$- \frac{C}{\delta}(g'\circ u_x)(t) + C\int_0^\ell x h^2(u_t)\mathrm{d}x, \quad \forall\ 0<\delta<1. \tag{3.11}$$

Proof. By direct computations and (1.1), we get

$$\begin{aligned} K'(t) =& \left(1 - \int_0^t g(s)\mathrm{d}s\right)\int_0^\ell x u_x \int_0^t g(t-s)\left(u_x(t)-u_x(s)\right)\mathrm{d}s\mathrm{d}x \\ &+ \int_0^\ell x\left(\int_0^t g(t-s)\left(u_x(t)-u_x(s)\right)\mathrm{d}s\right)^2 \mathrm{d}x - \int_0^\ell x u_t \int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ &- \left(\int_0^t g(s)\mathrm{d}s\right)\int_0^\ell x u_t^2\mathrm{d}x + \int_0^\ell x h(u_t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ &- b\int_0^\ell x|u|^{p-2}u\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{3.12}$$

By Young's inequality and Lemma 3.2, we have

$$\left(1 - \int_0^t g(s)\mathrm{d}s\right)\int_0^\ell x u_x \int_0^t g(t-s)\left(u_x(t)-u_x(s)\right)\mathrm{d}s\mathrm{d}x \le \delta\int_0^\ell x u_x^2\mathrm{d}x + \frac{C}{\delta}(g\circ u_x)(t), \tag{3.13}$$

$$\int_0^\ell x\left(\int_0^t g(t-s)\left(u_x(t)-u_x(s)\right)\mathrm{d}s\right)^2\mathrm{d}x \le C(g\circ u_x)(t), \tag{3.14}$$

$$-\int_0^\ell x u_t\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \le \delta\int_0^\ell x u_t^2\mathrm{d}x - \frac{C}{\delta}(g'\circ u_x)(t), \tag{3.15}$$

$$\int_0^\ell x h(u_t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \le C\int_0^\ell x h^2(u_t)\mathrm{d}x + C(g\circ u_x)(t), \tag{3.16}$$

As for the sixth term, using Lemma 2.2, (2.6) and (3.1), we get

$$\begin{aligned} &- b\int_0^\ell x|u|^{p-2}u\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ \le& b\delta\int_0^\ell x|u|^{2p-2}\mathrm{d}x + \frac{C}{2\delta}(g\circ u_x)(t) \le b\delta C_p\left(\int_0^\ell x u_x^2\mathrm{d}x\right)^{p-2}\left(\int_0^\ell x u_x^2\mathrm{d}x\right) + \frac{C}{2\delta}(g\circ u_x)(t) \\ \le& b\delta C_p\left[\frac{2pE(u_0,u_1)}{(p-2)l}\right]^{p-2}\left(\int_0^\ell x u_x^2\mathrm{d}x\right) \le \frac{\delta l^2}{bC_p}\int_0^\ell x u_x^2\mathrm{d}x + \frac{C}{2\delta}(g\circ u_x)(t). \end{aligned} \tag{3.17}$$

Combining (3.12)-(3.17), the assertion of the lemma is established.

Now select N_1, N_2 large so that (3.3) remains valid and $\frac{l}{4N_2\left(1+\frac{l^2}{bC_p}\right)} \le \frac{l}{2(1+C_*)}$. Set $g_0 = \int_0^{t_0} g(s)\mathrm{d}s$ for some fixed $t_0 > 0$. By combining (2.5), (3.2), (3.7) and (3.11), we take $\delta = \frac{l}{4N_2\left(1+\frac{l^2}{bC_p}\right)}$ and obtain, for all $t \ge t_0$,

$$\mathcal{L}'(t) \le -\frac{l}{4}\int_0^\ell x u_x^2\mathrm{d}x - \left(N_2 g_0 - \frac{l}{4} - 1\right)\int_0^\ell x u_t^2\mathrm{d}x + \left(\frac{4CN_2^2\left(1+\frac{l^2}{bC_p}\right)}{l} + C\right)(g\circ u_x)(t)$$
$$+ \left(\frac{1}{2}N_1 - \frac{4CN_2^2\left(1+\frac{l^2}{bC_p}\right)}{l}\right)(g'\circ u_x)(t) + (CN_2 + C)\int_0^\ell x h^2(u_t)\mathrm{d}x + b\int_0^\ell x|u|^p\mathrm{d}x.$$

At this point, since N_2 large enough, so we can have $k := N_2 g_0 - \frac{l}{4} - 1 > 0$, then N_1 large enough so that (3.6) remains valid and $\frac{1}{2}N_1 - \frac{4CN_2^2\left(1+\frac{l^2}{bC_p}\right)}{l} > 0$. Thus, using (H1), it turns out that

$$\mathcal{L}'(t) \le -\frac{l}{4}\int_0^\ell x u_x^2 \mathrm{d}x - k\int_0^\ell x u_t^2 \mathrm{d}x + C(g \circ u_x)(t) + C\int_0^\ell x h^2(u_t)\mathrm{d}x + b\int_0^\ell x|u|^p \mathrm{d}x,$$

which implies

$$E(t) \le -m\mathcal{L}'(t) + C(g \circ u_x)(t) + C\int_0^\ell x h^2(u_t)\mathrm{d}x, \quad \forall\ t \ge t_0. \tag{3.18}$$

Proof of Theorem 2.4. (Sketch) Define $\phi(t) = 1 + \int_1^t \frac{1}{h_0\left(\frac{1}{s}\right)}\mathrm{d}s$, $\forall\ t \ge 1$ and $\sigma(t) = \phi^{-1}\left(\int_0^t \xi(s)\mathrm{d}s\right)$, for $\forall\ t \ge t_1 \ge t_0$. Then continue as that of [13, Theorem 3.5] we can complete the proof.

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(Y. Sun) College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

(G. Li) College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

(W. J. Liu) College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

E-mail address: `wjliu@nuist.edu.cn`

Measuring fuzziness of generalized fuzzy rough sets induced by pseudo-operations

Zhan-hong Shi[1,2,*], Zeng-tai Gong[2,*]

1. College of Science, Gansu Agricultural University, Lanzhou 730070, P.R. China

2. College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, P.R. China

Abstract: Rough sets is a new mathematical tool to handle imprecision, vagueness and uncertainty in data analysis. But, in Pawlak's rough set model, equivalence relation is a key and primitive notion and this equivalence relation seems to be a very stringent condition that limited the application domain of the rough sets. Various fuzzy generalizations of rough approximations have been made over the years. In this paper, we consider pseudo-operation of the following form: $x \oplus y = g^{-1}(g(x) + g(y))$, where g is a positive strictly monotone generating function and g^{-1} is its pseudo-inverse. Using this type of pseudo-operation, the pseudo-generalized fuzzy rough sets are presented and some properties of the pseudo fuzzy rough approximation operators are investigated. Moreover, we define a measure of fuzziness based on pseudo-generalized fuzzy rough sets with the new pseudo-lower and pseudo-upper approximations.

Keywords: Fuzzy sets; Rough sets; Pseudo-operations; Approximation operators

1. Introduction

The theory of rough set was originally proposed by Pawlak [1] as a mathematical approach to handle imprecision, vagueness and uncertainty in data analysis. By using the concepts of lower and upper approximations in rough set theory, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. However, in Pawlak's rough set model, an equivalence relation is a key and primitive notion. This equivalence relation, however, seems to be a very stringent condition that may limit the application domain of the rough set model. Generalizations of rough set theory were considered by scholars in order to deal with complex practical problems [2-7].

There are at least two approaches for the development of definitions of lower and upper approximation operators, namely, the constructive and axiomatic approaches. In the constructive approach, some authors have extended equivalence relation to tolerance relations [8], similarity relations [9], ordinary binary relations [7,10], and others [11-13]. Meanwhile, some authors have relaxed the partition of universe to the covering and obtain the covering-based rough sets [4,14-20]. In addition, generalizations of rough sets to the fuzzy environment have also been made [2,5,21-26]. By introducing the lower and upper approximations in fuzzy set theory, Dubois and

$*$ Corresponding author. E-mail Address: szh780323@163.com (Z.H. Shi); zt-gong@163.com (Z.T. Gong)

Prade [27] formulated rough fuzzy sets and fuzzy rough sets, they constructed a pair of lower and upper approximation operators for fuzzy sets with respect to fuzzy similarity relation by using the t-norm Min and its dual conorm Max. By using a residual implication (for short, R-implication) to define the lower approximation operator, Morsi and Yakout [28] generalized the fuzzy rough sets in the sense of Dubois and Prade. Later, Radzikowska and Kerre [29] proposed a more general approach to the fuzzification of a rough set. This approach is based on a border implication $\mathcal{I}$ (not necessarily a R-implication) and a triangular norm $\mathcal{T}$. Recently, Mi et al. [30] presented the generalized fuzzy rough sets determined by a triangular norm, Ouyang et al. [31] discussed fuzzy rough sets based on tolerance relations.

In the axiomatic approaches, a set of axioms is used to characterize the approximations. Lin and Liu [32] proposed six axioms on a pair of abstract operators on the power set of universe in the framework of topological spaces. Under these axioms, there exists an equivalence relation such that the lower and upper approximations are the same as the abstract operators. The most important axiomatic studies for crisp rough sets were made by Yao [7,10,33]. Recently, the research of the axiomatic approach has also been extended to approximation operators in the fuzzy environment [28,30,34-37].

In some problems with uncertainty in the theory of probabilistic metric spaces, fuzzy logics and fuzzy measures, the pseudo-operations such as pseudo-additions and pseudo-multiplications are used [38-40]. Pseudo-analysis [38-47] has been applied in different fields, e.g., measure theory, integration, convolution, Laplace transform, optimization, nonlinear differential and difference equations, economics, game theory, etc. Interestingly, by using the Aczel's theorem [48], the pseudo-additions and pseudo-multiplications could be transferred into the corresponding results of reals such as the addition operator and multiplication operator. This can bring us the convenience of calculation.

We note that there are some literatures about pseudo integrals [7,8,10,25,35], but little literatures about rough set model based on pseudo-operations. The main purpose of this paper is to present a general framework for the study of fuzzy rough approximation operators based on pseudo-operations. By using the pseudo-operations, the pseudo-lower and pseudo-upper approximation operators are defined. Meanwhile, some properties of the proposed pseudo fuzzy rough approximation operators are investigated. Connections between the new and the existing fuzzy rough approximation operators are also discussed. Compared with the previous rough set models based on triangular norms [28-30,37], the pseudo-generalized fuzzy rough set proposed in this paper has its advantage to calculate its lower and upper approximations conveniently.

The remainder of this paper is organized as follows. In section 2, we recall some basic concepts of fuzzy sets, fuzzy relation, rough sets and pseudo-operations. In section 3, the pseudo-generalized fuzzy rough sets are presented. Some properties of the proposed pseudo fuzzy rough approximation operators are also investigated in this section. In section 4, the fuzziness of pseudo-generalized fuzzy rough sets is given. Section 5 presents conclusions.

2. Preliminaries

2.1 Fuzzy sets

Let U be a universe. Fuzzy set A is a mapping from U into the unit interval $[0, 1]$:

$$A : U \to [0, 1],$$

where for each $x \in U$, we call $A(x)$ the membership degree of x in A. If $U = \{x_1, x_2, \cdots, x_n\}$, then the fuzzy set A on U can be expressed by $\sum_{i=1}^{n} A(x_i)/x_i$. Additionally, the fuzzy power set, i.e., the set of all fuzzy sets in the universe U is denoted by $\mathcal{F}(U)$ [49].

For fuzzy sets $A, B \in \mathcal{F}(U)$,
$A \subseteq B \Leftrightarrow A(x) \leq B(x)$;
$(A \cap B)(x) = A(x) \wedge B(x) = \min\{A(x), B(x)\}$;
$(A \cup B)(x) = A(x) \vee B(x) = \max\{A(x), B(x)\}$;
$(\sim A)(x) = 1 - A(x)$, where $\sim A$ is the complement of A.

2.2 Fuzzy relation

Let U and W be two nonempty sets. The Cartesian product of U and W is denoted by $U \times W$. A fuzzy relation R from U to W is a fuzzy subset of $U \times W$, i.e., $R \in \mathcal{F}(U \times W)$, and $R(x, y)$ is called the degree of relation between x and y. In particular, if $U = W$, we call R a fuzzy relation on U. Usually, a fuzzy relation can be expressed by a fuzzy matrix.

2.3 Rough sets

In traditional Pawlak rough set theory, the pair (U, R) is called an approximation space (it is also called Pawlak approximation space), where U is a finite and non-empty set called the universe and R is an equivalence relation on U, i.e., R is reflexive, symmetrical and transitive. The relation R decomposes the set U into a disjoint class in such a way that two elements x and y are in the same class iff $(x, y) \in R$. Suppose R is an equivalence relation on U. With respect to R, we can define an equivalence class of an element x in U as follows:

$$[x]_R = \{y|(x, y) \in R\}.$$

The quotient set of U by the relation R is denoted by U/R, and

$$U/R = \{X_1, X_2, \cdots, X_m\}.$$

where X_i $(i = 1, 2, \cdots, m)$ is an equivalence class of R.

Given an arbitrary set $X \subseteq U$, it may not be possible to describe X precisely in the approximation space (U, R). One may characterize X by a pair of lower and upper approximations defined as follows:

$$\underline{R}X = \{x \in U|[x]_R \subseteq X\} = \cup\{Y \in U/R|Y \subseteq X\};$$
$$\overline{R}X = \{x \in U|[x]_R \cap X \neq \phi\} = \cup\{Y \in U/R|Y \cap X \neq \phi\}.$$

The pair $(\underline{R}X, \overline{R}X)$ is referred to as a rough set of X.

2.4 Pseudo-operations

Throughout this paper, we only consider the case of pseudo-addition and present the fuzzy generalized rough sets using pseudo-addition. For the case of pseudo-multiplication, the discussion can be given similarly.

Definition 2.1 An operation $\oplus : [0,\infty]^2 \to [0,\infty]$ is called a pseudo-addition if it satisfies the following axioms:

(1) Associativity: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in [0,\infty]$.

(2) Monotonicity: $a \oplus b \leq c \oplus d$ whenever $0 \leq a \leq c \leq \infty, 0 \leq b \leq d \leq \infty$.

(3) 0 is neutral element: $a \oplus 0 = 0 \otimes a = a$ for all $a \in [0,\infty]$.

(4) Continuity: for any sequences $(a_n)_{n\in N}$, $(b_n)_{n\in N}$ in $[0,\infty]^N$ such that $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$ it holds $\lim\limits_{n\to\infty} a_n \oplus b_n = a \oplus b$.

(5) Commutativity: $a \oplus b = b \oplus a$ for all $a, b \in [0,\infty]$.

Lemma 2.1 (Aczel's theorem) Let g be a positive strictly monotone function defined on $[a,b] \subseteq (-\infty,+\infty)$ such that $0 \in Ran(g)$. The generalized generated pseudo-addition $\oplus$ and the generalized generated pseudo-multiplication $\odot$ are given by

$$x \oplus y = g^{-1}(g(x) + g(y)),$$

$$x \odot y = g^{-1}(g(x)g(y)),$$

where g^{-1} is pseudo-inverse function for function g: $g^{-1}(y) = \sup\{x \in [a,b] | g(x) < y\}$ if g is a non-decreasing function and $g^{-1}(y) = \sup\{x \in [a,b] | g(x) > y\}$ if g is a non-increasing function.

Example 2.2 Suppose that $g(x) = 1 - x$ $(x \in [0,1])$, then its pseudo-inverse is

$$g^{-1}(x) = \begin{cases} 1 - x, & x \in [0,1], \\ 0, & x \in [1,+\infty). \end{cases}$$

And $x \oplus y = g^{-1}(g(x) + g(y)) = \max\{0, x + y - 1\}$, this is Lukasiewicz t-norm.

3. Construction of pseudo fuzzy rough approximation operators

Definition 3.1 Let U and W be two nonempty sets, R a fuzzy relation from U to W, then (U,W,R) is called a fuzzy approximation space. $g : [0,1] \to [0,+\infty)$ is a strictly decreasing function such that $g(1) = 0$ and $g(x) + g(y) \in Ran(g) \cup [g(0^+),+\infty)$ for all $(x,y) \in [0,1]^2$. Then for any $A \in \mathcal{F}(W)$, the pseudo-lower approximation $\underline{R}_{\oplus}(A)$ and the pseudo-upper approximation $\overline{R}_{\oplus}(A)$ of A are defined as follows, respectively:

$$\underline{R}_{\oplus}(A)(x) = \bigwedge_{y\in W} \{1 - R(x,y) \oplus (1 - A(y))\} = \bigwedge_{y\in W} \{1 - g^{-1}(g(R(x,y)) + g(1 - A(y)))\}, x \in U;$$

$$\overline{R}_{\oplus}(A)(x) = \bigvee_{y\in W} \{R(x,y) \oplus A(y)\} = \bigvee_{y\in W} \{g^{-1}(g(R(x,y)) + g(A(y)))\}, x \in U.$$

The pair $(\underline{R}_{\oplus}(A), \overline{R}_{\oplus}(A))$ is called a pseudo-generalized fuzzy rough set. $\underline{R}_{\oplus}$ and $\overline{R}_{\oplus}$ are referred to as the pseudo-lower and pseudo-upper fuzzy rough approximation operators, respectively.

Example 3.1 Suppose that (U,W,R) is a fuzzy approximation space, where U and W are two sets called object set and attribute set. Let $U = \{x_1, x_2, x_3\}$, $W = \{a_1, a_2, a_3, a_4\}$. $R \in \mathcal{F}(U \times W)$ is a fuzzy relation from U to W and R can be seen in Table 2:

For a fuzzy attribute set

$$A = 0.8/a_1 + 0.3/a_2 + 1/a_3 + 0.9/a_4 \in \mathcal{F}(W),$$

Table 1: A fuzzy approximation space

	a_1	a_2	a_3	a_4
x_1	1	0.4	0	0.1
x_2	0.3	0.9	0.7	0.6
x_3	0.9	0.2	1	0

if we take a strictly decreasing function as

$$g(x) = 1 - x \ (x \in [0, 1]),$$

then the pseudo-lower approximation $\underline{R}_{\oplus}(A)$ and the pseudo-upper approximation $\overline{R}_{\oplus}(A)$ of A can be computed as follows:

$\underline{R}_{\oplus}(A)(x_1) = \min\{1 - g^{-1}(0 + 0.8), 1 - g^{-1}(0.6 + 0.3), 1 - g^{-1}(1 + 1), 1 - g^{-1}(0.9 + 0.9)\} = 0.8;$
$\underline{R}_{\oplus}(A)(x_2) = \min\{1 - g^{-1}(0.7 + 0.8), 1 - g^{-1}(0.1 + 0.3), 1 - g^{-1}(0.3 + 1), 1 - g^{-1}(0.4 + 0.9)\} = 0.4;$
$\underline{R}_{\oplus}(A)(x_3) = \min\{1 - g^{-1}(0.1 + 0.8), 1 - g^{-1}(0.8 + 0.3), 1 - g^{-1}(0 + 1), 1 - g^{-1}(1 + 0.9)\} = 0.9;$
$\overline{R}_{\oplus}(A)(x_1) = \max\{g^{-1}(0 + 0.2), g^{-1}(0.6 + 0.7), g^{-1}(1 + 0), g^{-1}(0.9 + 0.1)\} = 0.8;$
$\overline{R}_{\oplus}(A)(x_2) = \max\{g^{-1}(0.7 + 0.2), g^{-1}(0.1 + 0.7), g^{-1}(0.3 + 0), g^{-1}(0.4 + 0.1)\} = 0.7;$
$\overline{R}_{\oplus}(A)(x_3) = \max\{g^{-1}(0.1 + 0.2), g^{-1}(0.8 + 0.7), g^{-1}(0 + 0), g^{-1}(1 + 0.1)\} = 1.$

That is,

$$\underline{R}_{\oplus}(A) = 0.8/x_1 + 0.4/x_2 + 0.9/x_3,$$
$$\overline{R}_{\oplus}(A) = 0.8/x_1 + 0.7/x_2 + 1/x_3.$$

Remark 3.1 If R is a crisp binary relation from U to W, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [36]. That is, for every $A \in \mathcal{F}(W),\ x \in U$,

$$\overline{R}_{\oplus}(A)(x) = \sup\{A(y)|y \in R_s(x)\},\quad \underline{R}_{\oplus}(A)(x) = \inf\{A(y)|y \in R_s(x)\},$$

where $R_s(x) = \{y \in W|(x, y) \in R\}$.

In fact,

$$\begin{aligned}
&\overline{R}_{\oplus}(A)(x) \\
&= \bigvee_{y \in W} \{g^{-1}(g(R(x, y)) + g(A(y)))\} \\
&= \sup\{g^{-1}(g(1) + g(A(y)))|y \in R_s(x)\} \bigvee \sup\{g^{-1}(g(0) + g(A(y)))|y \notin R_s(x)\} \\
&= \sup\{g^{-1}(g(1) + g(A(y)))|y \in R_s(x)\} \\
&= \sup\{g^{-1}(0 + g(A(y)))|y \in R_s(x)\} \\
&= \sup\{A(y)|y \in R_s(x)\}, \\
&\underline{R}_{\oplus}(A)(x) \\
&= \bigwedge_{y \in W} \{1 - g^{-1}(g(R(x, y)) + g(1 - A(y)))\} \\
&= \inf\{1 - g^{-1}(g(1) + g(1 - A(y)))|y \in R_s(x)\} \bigwedge \inf\{1 - g^{-1}(g(0) + g(1 - A(y)))|y \notin R_s(x)\} \\
&= \inf\{1 - g^{-1}(g(1) + g(1 - A(y)))|y \in R_s(x)\} \\
&= \inf\{1 - g^{-1}(0 + g(1 - A(y)))|y \in R_s(x)\} \\
&= \inf\{A(y)|y \in R_s(x)\}.
\end{aligned}$$

Remark 3.2 If R is a crisp binary relation on U and A is a crisp set on U, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [7]. That is, for any $A \in P(U),\ x \in U$,

$$\overline{R}_{\oplus}(A) = \{x \in U | R_s(x) \cap A \neq \phi\}, \ \ \underline{R}_{\oplus}(A) = \{x \in U | R_s(x) \subseteq A\}.$$

where $R_s(x) = \{y \in U | (x, y) \in R\}$.

In fact, by Remark 3.2, we know that if $A \in P(U)$ then for any $x \in U$,

$x \in \overline{R}_{\oplus}(A) \Leftrightarrow \overline{R}_{\oplus}(A)(x) = 1 \Leftrightarrow \exists\ y \in R_s(x)$ such that $A(y) = 1$, i.e., $y \in A \Leftrightarrow R_s(x) \cap A \neq \phi$,

$x \in \underline{R}_{\oplus}(A) \Leftrightarrow \underline{R}_{\oplus}(A)(x) = 1 \Leftrightarrow A(y) = 1$ for every $y \in R_s(x)$, i.e., $y \in A \Leftrightarrow R_s(x) \subseteq A$.

Remark 3.3 If R is a crisp equivalence relation on U and A is a fuzzy set on U, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [27]. That is, for every $A \in \mathcal{F}(U),\ x \in U$,

$$\overline{R}_{\oplus}(A)(x) = \sup\{A(y) | y \in [x]_R\}, \ \ \underline{R}_{\oplus}(A)(x) = \inf\{A(y) | y \in [x]_R\}.$$

In fact, if R is a crisp equivalence relation on U, then $R_s(x) = [x]_R$.

Remark 3.4 If R is a crisp equivalence relation on U and A is a crisp set on U, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [1]. That is, for any $A \in P(U),\ x \in U$,

$$\overline{R}_{\oplus}(A) = \{x \in U | [x]_R \cap A \neq \phi\}, \ \ \underline{R}_{\oplus}(A) = \{x \in U | [x]_R \subseteq A\}.$$

Example 3.2 Let $U = \{x_1, x_2, x_3\}$ be the universe of discourse, $R = \begin{pmatrix} 0.8 & 0.9 & 0.6 \\ 0.7 & 0.9 & 0.1 \\ 0.8 & 0.2 & 0.8 \end{pmatrix}$ be a fuzzy relation on U. Suppose that $A, B, C \in \mathcal{F}(U)$, and

$$A = 0.4/x_1 + 0.5/x_2 + 0.8/x_3;$$

$$B = 0.6/x_1 + 0.7/x_2 + 0.2/x_3;$$

$$C = 0.6/x_1 + 0.8/x_2 + 0.9/x_3.$$

Let $g : [0, 1] \to [0, +\infty)$ given by $g(x) = 1 - x$ be a generating function for pseudo-addition $\oplus$, then we can compute that

$$\underline{R}_{\oplus}(A) = 0.6/x_1 + 0.6/x_2 + 0.6/x_3;$$

$$\overline{R}_{\oplus}(A) = 0.4/x_1 + 0.4/x_2 + 0.6/x_3;$$

$$\underline{R}_{\oplus}(B) = 0.6/x_1 + 0.8/x_2 + 0.4/x_3;$$

$$\overline{R}_{\oplus}(B) = 0.6/x_1 + 0.6/x_2 + 0.4/x_3;$$

$$\underline{R}_{\oplus}(C) = 0.9/x_1 + 1/x_2 + 0.9/x_3;$$

$$\overline{R}_{\oplus}(C) = 0.7/x_1 + 0.7/x_2 + 0.7/x_3.$$

From computation above, we can find $A \subseteq C$ implies that $\underline{R}_{\oplus}(A) \subseteq \underline{R}_{\oplus}(C)$ and $\overline{R}_{\oplus}(A) \subseteq \overline{R}_{\oplus}(C)$. Furthermore,

$$A \cap B = 0.4/x_1 + 0.5/x_2 + 0.2/x_3,$$

$$A \cup B = 0.6/x_1 + 0.7/x_2 + 0.8/x_3.$$

And

$$\underline{R}_{\oplus}(A \cap B) = 0.6/x_1 + 0.6/x_2 + 0.4/x_3;$$
$$\overline{R}_{\oplus}(A \cap B) = 0.4/x_1 + 0.4/x_2 + 0.2/x_3;$$
$$\underline{R}_{\oplus}(A \cup B) = 0.8/x_1 + 0.8/x_2 + 0.8/x_3;$$
$$\overline{R}_{\oplus}(A \cup B) = 0.6/x_1 + 0.6/x_2 + 0.6/x_3.$$

Thus, we notice that

$$\underline{R}_{\oplus}(A \cap B) = \underline{R}_{\oplus}(A) \cap \underline{R}_{\oplus}(B),\ \overline{R}_{\oplus}(A \cup B) = \overline{R}_{\oplus}(A) \cup \overline{R}_{\oplus}(B);$$

$$\underline{R}_{\oplus}(A \cup B) \supseteq \underline{R}_{\oplus}(A) \cup \underline{R}_{\oplus}(B),\ \overline{R}_{\oplus}(A \cap B) \subseteq \overline{R}_{\oplus}(A) \cap \overline{R}_{\oplus}(B).$$

4. Measuring fuzziness of pseudo-generalized fuzzy rough sets

Let (U, W, R) be a fuzzy approximation space, where U and W are two nonempty sets, R is a fuzzy relation from U to W. For any $A \in \mathcal{F}(W)$, the pseudo-generalized fuzzy rough set of A is $(\underline{R}_{\oplus}(A), \overline{R}_{\oplus}(A))$. Thus in the fuzzy approximation space (U, W, R), A is approximated by two fuzzy sets, one called the pseudo-lower approximation of A, and another called the pseudo-upper approximation of A. In this section, we suppose that $U = W$ and give an approach to measuring the fuzziness of pseudo-generalized fuzzy rough sets.

Definition 4.1 Let U be a universe of discourse, R be a fuzzy relation on U. For any $x \in U$ and $A \in \mathcal{F}(U)$, the degree of rough membership of x in A is defined by

$$r(A)(x) = \frac{\sum_{y \in U}[R(x, y) \oplus A(y)]}{\sum_{y \in U} R(x, y)}.$$

From Definition 4.1, we note that the fuzzy set A and fuzzy relation R on U can induce a new fuzzy set $r(A)$ of U.

Theorem 4.1 For any fuzzy sets $A, B \in \mathcal{F}(U)$,

(1) if $A \subseteq B$, then $r(A) \subseteq r(B)$;

(2) $r(A \cap B) \subseteq r(A) \cap r(B)$, $r(A \cup B) \supseteq r(A) \cup r(B)$.

Proof

(1) Since for any $x \in U$, $A(x) \leq B(x)$. By Definition 4.1, we have

$$r(A)(x) = \frac{\sum_{y \in U}[R(x, y) \oplus A(y)]}{\sum_{y \in U} R(x, y)} \leq \frac{\sum_{y \in U}[R(x, y) \oplus B(y)]}{\sum_{y \in U} R(x, y)} = r(B)(x).$$

So $r(A) \subseteq r(B)$.

(2) For any $A, B \in \mathcal{F}(U)$, we have $A \cap B \subseteq A$ and $A \cap B \subseteq B$. It implies that

$$r(A \cap B) \subseteq r(A), \quad r(A \cap B) \subseteq r(B).$$

Thus, $r(A \cap B) \subseteq r(A) \cap r(B)$.

$r(A \cup B) \supseteq r(A) \cup r(B)$ can be proved in a similar way. $\square$

Definition 4.2 Let U be a universe of discourse, R be a fuzzy relation on U, $A \in \mathcal{F}(U)$. The fuzziness of pseudo-generalized fuzzy rough set $(\underline{R}_{\oplus}(A), \overline{R}_{\oplus}(A))$ is defined by

$$FR(A) = -\frac{1}{|U|}\sum_{x\in U} r(A)(x)\cdot \log_2 r(A)(x).$$

Example 4.1 (Continue the Example 3.2)

In Example 3.2, fuzzy relation $R = \begin{pmatrix} 0.8 & 0.9 & 0.6 \\ 0.7 & 0.9 & 0.1 \\ 0.8 & 0.2 & 0.8 \end{pmatrix}$, three fuzzy sets A, B, C are denoted as follows, respectively:

$$A = 0.4/x_1 + 0.5/x_2 + 0.8/x_3;$$

$$B = 0.6/x_1 + 0.7/x_2 + 0.2/x_3;$$

$$C = 0.6/x_1 + 0.8/x_2 + 0.9/x_3.$$

Meanwhile, $g(x) = 1 - x$ $(x \in [0,1])$. Thus, we can compute that

$$\begin{aligned} r(A)(x_1) &= \frac{g^{-1}(0.2+0.6) + g^{-1}(0.1+0.5) + g^{-1}(0.4+0.2)}{0.8+0.9+0.6} \\ &= \frac{0.2+0.4+0.4}{0.8+0.9+0.6} \\ &= 0.435. \end{aligned}$$

In a similar way, we get

$$r(A)(x_2) = \frac{0.1+0.4+0}{0.7+0.9+0.1} = 0.294,$$

$$r(A)(x_3) = \frac{0.2+0+0.6}{0.8+0.2+0.8} = 0.444.$$

That is,

$$r(A) = 0.435/x_1 + 0.294/x_2 + 0.444/x_3.$$

In addition, we can obtain that

$$r(B) = 0.435/x_1 + 0.529/x_2 + 0.222/x_3,$$

$$r(C) = 0.783/x_1 + 0.588/x_2 + 0.555/x_3,$$

$$r(A\cap B) = 0.261/x_1 + 0.294/x_2 + 0.111/x_3,$$

$$r(A\cup B) = 0.609/x_1 + 0.529/x_2 + 0.555/x_3.$$

From computation above, we note that
$A \subseteq C \Rightarrow r(A) \subseteq r(C)$, $r(A\cap B) \subseteq r(A)\cap r(B)$ and $r(A\cup B) \supseteq r(A)\cup r(B)$ hold.
Furthermore, we have

$$FR(A) = -\frac{1}{3}(0.435\times\log_2 0.435 + 0.294\times\log_2 0.294 + 0.444\times\log_2 0.444) \approx 0.521;$$

$$FR(B) = -\frac{1}{3}(0.435\times\log_2 0.435 + 0.529\times\log_2 0.529 + 0.222\times\log_2 0.222) \approx 0.497;$$

$$FR(C) = -\frac{1}{3}(0.783 \times \log_2 0.783 + 0.588 \times \log_2 0.588 + 0.555 \times \log_2 0.555) \approx 0.399;$$

$$FR(A \cap B) = -\frac{1}{3}(0.261 \times \log_2 0.261 + 0.294 \times \log_2 0.294 + 0.111 \times \log_2 0.111) \approx 0.459;$$

$$FR(A \cup B) = -\frac{1}{3}(0.609 \times \log_2 0.609 + 0.529 \times \log_2 0.529 + 0.555 \times \log_2 0.555) \approx 0.464.$$

From the results of Example 4.1, we note that $FR(A) \geq FR(C)$ whenever $A \subseteq C$, but for $A \cap B \subseteq A$, $FR(A \cap B) \leq FR(A)$.

It can be shown that for any $A, B \in \mathcal{F}(U)$, if $A \subseteq B$, $FR(A) \leq FR(B)$ or $FR(A) \geq FR(B)$ does not hold.

5. Conclusions

At present, there are many researchers about pseudo-analysis. Pseudo-analysis has been applied in different fields. It is interesting to combine pseudo-operations and rough set in order to expand the application domain of pseudo-analysis and rough set. In this paper, we presented a generalized fuzzy rough set model based on pseudo-operation, constructed pseudo fuzzy rough approximation operations. Some properties of the proposed generalized fuzzy rough approximation operators also investigated. At the same time, the fuzziness of pseudo-generalized fuzzy rough sets is given.

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Global analysis for delay virus infection model with multitarget cells

A. M. Elaiw[a,b] and M. A. Alghamdi[a]
[a]Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
[b]Department of Mathematics, Faculty of Science,
Al-Azhar University (Assiut Branch), Assiut, Egypt.
Email: a_m_elaiw@yahoo.com

Abstract

This paper investigates the qualitative behavior of viral infection model with multitarget cells in vivo. The infection rate is given by Crowley-Martin functional response. By assuming that the virus attack n classes of uninfected target cells, we study a viral infection model of dimension $2n+1$ with distributed delay. To describe the latent period for the contacted target cells with viruses to begin producing viruses, two types of distributed delay are incorporated into the model. The basic reproduction number R_0 of the model is defined which determines the dynamical behavior of the model. Utilizing Lyapunov functionals and LaSalle's invariance principle, we have proven that if $R_0 \leq 1$ then the uninfected steady state is globally asymptotically stable, and if $R_0 > 1$ then the infected steady state is globally asymptotically stable.
.

Keywords: Viral infection; Global stability; Delay; Crowley-Martin functional response.
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1 Introduction

Mathematical models have proven their importance in understanding the dynamical behaviors of various viruses such as human immunodeficiency virus (HIV), hepatitis B virus (HBV), hepatitis C virus (HCV), etc. [1]. The interatcion of the virus and target cells has been formulated as ordinary differential equations in several works (see e.g. [2], [3], [4], [12], [11], [5] and [6]). The basic mathematical model describing the dynamics of viral infection can be written in a general form as [6]:

$$\dot{x} = \lambda - dx - h(x,v), \tag{1}$$
$$\dot{y} = h(x,v) - \delta y, \tag{2}$$
$$\dot{v} = ky - rv, \tag{3}$$

where x, y and v represent the populations of the uninfected target cells, infected cells and free virus particles, respectively. The uninfected cells are generated from sources within the body at rate λ. The parameter d is the death rate constant of the uninfected target cells. Eq. (2) describes the population dynamics of the infected cells and shows that they die with rate constant δ. The virus particles are produced by the infected cells with rate constant k, and are cleared from plasma with rate constant r. The function $h(x,v)$ represents the incidence rate of infection and it has been considered in the viral infection models by different forms:

- Bilinear incidence rate [2], [3]: $h(x,v) = \beta xv$.
- Saturated incidence rate [30]: $h(x,v) = \frac{\beta xv}{1+bv}$.
- Holling type II functional response [34]: $h(x,v) = \frac{\beta xv}{1+ax}$.
- Beddington-DeAngelis infection rate [28]: $h(x,v) = \frac{\beta xv}{1+ax+bv}$.
- Crowley-Martin functional response [31], [32]: $h(x,v) = \frac{\beta xv}{(1+ax)(1+bv)}$,where $a, b \geq 0$ and β is the rate constant characterizing infections of the cells. The Crowley-Martin type of functional response was first introduced by Crowley and Martin [33].

Model (1)-(3) is based on the assumption that, once the virus contacts a target cell, the cell begins producing new virus particles. More realistic models incorporate the delay between the time of viral entry into the target cell and the time the production of new virus particles, modeled with discrete time delay or distributed time delay using functional differential equations. Many researchers have devoted their effort in developing various mathematical models of viral infections with discrete or distributed delays and studying their qualitative behaviors (see e.g. [8], [10], [9], [27], [29], [24], [26], [22], [21], [34]).

In the literature, most of the proposed mathematical models for viral infection assume that the virus has one class of target cells, (e.g. $CD4^+$ T cells in case of HIV or hepatic cells in case of HCV and HBV) (see e.g. [2], [3] and the book Nowak and May [1]). In [7], [25], [13], [15], [18], [19], and [16], some HIV models with two classes of target cells, $CD4^+$ T cells and macrophages have been proposed. The global stability of these models has been investigated in ([13], [15] and [16]). Because the interactions of some types of viruses in *vivo* is complex and is not known clearly, we would suppose that the virus may attack n classes of target cells where $n \geq 1$ [14], [17]. In [17], models with discrete-time delays and saturated incidence rate have been studied. Elaiw [14] studied a class of virus infection models with multitarget cells without time delay.

The purpose of this paper is to propose a viral infection model with multitarget cells and Crowley-Martin functional response and investigate its qualitative behavior. We incorporate distributed delay into the model which represents an intracellular latent period for the contacted uninfected target cells with virus to begin producing new virus particles. The global stability of this model is established using Lyapunov functionals and LaSalle's invariance principle. We prove that the global dynamics of this model is determined by the basic reproduction number R_0. If $R_0 \leq 1$, then the uninfected steady state is globally asymptotically stable (GAS) and if $R_0 > 1$, then the infected steady state exists and is GAS.

2 Model with distributed time delays

In this section we propose a virus dynamics model with multitarget cells and multiple distributed intracellular delays.

$$\dot{x}_i(t) = \lambda_i - d_i x_i - \frac{\beta_i x_i(t) v(t)}{(1 + a_i x_i(t))(1 + b_i v(t))}, \qquad i = 1, ..., n \tag{4}$$

$$\dot{y}_i(t) = \beta_i \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \frac{x_i(t-\tau) v(t-\tau)}{(1 + a_i x_i(t-\tau))(1 + b_i v(t-\tau))} d\tau - \delta_i y_i(t), \qquad i = 1, ..., n \tag{5}$$

$$\dot{v}(t) = \sum_{i=1}^{n} k_i \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} y_i(t-\tau) d\tau - r v(t), \tag{6}$$

where x_i and y_i represent the populations of the uninfected target cells and infected cells of class i, respectively, v is the population of the virus particles. To account for the time lag between viral contacting a target cell and the production of new virus particles, two distributed intracellular delays are introduced. It is assumed that the target cells of class i are contacted by the virus particles at time $t - \tau$ become infected cells at time t, where τ is a random variable with a probability distribution $f_i(\tau)$ over the interval $[0, \tau_i]$ and τ_i is limit superior of this delay. The factor $e^{-m_i \tau}$ accounts for the loss of target cells during delay period where m_i is positive constant. On the other hand, it is assumed that, a cell infected at time $t - \tau$ starts to yield new infectious virus at time t where τ is distributed according to a probability distribution $g_i(\tau)$ over the interval $[0, \mu_i]$ and μ_i is limit superior of this delay. The factor $e^{-n_i \tau}$ account for the cells loss during this delay period where n_i is positive constant. All the other parameters of the model have the same biological meaning as given in model (1)-(3).

The probability distribution functions $f_i(\tau) : [0, \tau_i] \to \mathbb{R}_+$ and $g_i(\tau) : [0, \mu_i] \to \mathbb{R}_+$ are integral functions with $\int_0^{\tau_i} f_i(\tau) d\tau = \int_0^{\mu_i} g_i(\tau) d\tau = 1, \; i = 1, ..., n$. Define $F_i = \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} d\tau$ and $G_i = \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} d\tau$, $m_i \geq 0$, $n_i \geq 0$. It is clear that $0 < F_i \leq 1$ and $0 < G_i \leq 1$, $i = 1, ..., n$.

The initial conditions for system (4)-(6) take the form

$$\begin{aligned} &x_j(\theta) = \varphi_j(\theta), \; y_j(\theta) = \varphi_{j+n}(\theta), \quad j = 1, ..., n, \; v(\theta) = \varphi_{2n+1}(\theta), \\ &\varphi_j(\theta) \geq 0, \; \theta \in [-\ell, 0), \; \varphi_j(0) > 0, \; j = 1, ..., 2n+1, \end{aligned} \tag{7}$$

where $\ell = \max\{\tau_1, ..., \tau_n, \mu_1, ..., \mu_n\}$, $(\varphi_1(\theta), \varphi_2(\theta), ..., \varphi_{2n+1}(\theta)) \in C$ and $C = C([-\ell, 0], \mathbb{R}_+^{2n+1})$ is the Banach space of continuous functions mapping the interval $[-\ell, 0]$ into $\mathbb{R}_+^{2n+1}$. By the fundamental theory of functional differential equations [20], system (4)-(6) has a unique solution satisfying initial conditions (7).

2.1 Non-negativity and boundedness of solutions

In the following, we establish the non-negativity and boundedness of solutions of (4)-(6) with initial conditions (7). Let $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ and $\mathbf{y} = (y_1, y_2, ..., y_n)^T$.

Proposition 2. Let $(\mathbf{x}(t), \mathbf{y}(t), v(t))$ be any solution of (4)-(6) satisfying the initial conditions (7), then $\mathbf{x}(t), \mathbf{y}(t)$ and $v(t)$ are all non-negative for $t \geq 0$ and ultimately bounded.

Proof. First, we prove that $x_i(t) > 0$, $i = 1, ..., n$, for all $t \geq 0$. Assume that $x_i(t)$ lose its non-negativity on some local existence interval $[0, \omega]$ for some constant ω and let $t_1 \in [0, \omega]$ be such that $x_i(t_1) = 0$. From Eq. (4) we have $\dot{x}_i(t_1) = \lambda_i > 0$. Hence $x_i(t) < 0$ for some $t \in (t_1 - \varepsilon, t_1)$, where $\varepsilon > 0$ is sufficiently small. This leads to a contradiction and hence $x_i(t) > 0$, for all $t \geq 0$. Further, from Eqs. (5) and (6) we have

$$y_i(t) = y_i(0)e^{-\delta_i t} + \beta_i \int_0^t e^{-\delta_i(t-\eta)} \int_0^{\tau_i} f_i(\tau)e^{-m_i\tau} \frac{x_i(\eta-\tau)v(\eta-\tau)}{(1+a_i x_i(\eta-\tau))(1+b_i v(\eta-\tau))} d\tau d\eta,$$

$$v(t) = v(0)e^{-rt} + \sum_{i=1}^n k_i \int_0^t e^{-r(t-\eta)} \int_0^{\mu_i} g_i(\tau)e^{-n_i\tau} y_i(\eta-\tau) d\tau d\eta,$$

confiming that $y_i(t) \geq 0$, $i = 1, ..., n$, and $v(t) \geq 0$ for all $t \in [0, \ell]$. By a recursive argument, we obtain $y_i(t) \geq 0$, $i = 1, ..., n$, and $v(t) \geq 0$ for all $t \geq 0$.

Now we show the boundedness of the solutions of (4)-(6). Eqs. (4) imply that $\limsup_{t\to\infty} x_i(t) \leq x_i^0$, where $x_i^0 = \lambda_i/d_i$, and thus $x_i(t)$ is ultimately bounded. If follows that $\int_0^{\tau_i} f_i(\tau)e^{-m_i\tau} x_i(t-\tau)d\tau \leq F_i x_i^0$. Let $X_i(t) = \int_0^{\tau_i} f_i(\tau)e^{-m_i\tau} x_i(t-\tau)d\tau + y_i(t)$, $i = 1, ..., n$, then

$$\begin{aligned}\dot{X}_i(t) &= \int_0^{\tau_i} f_i(\tau)e^{-m_i\tau}\left(\lambda_i - d_i x_i(t-\tau) - \frac{\beta_i x_i(t-\tau)v(t-\tau)}{(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))}\right) d\tau \\ &+ \int_0^{\tau_i} f_i(\tau)e^{-m_i\tau} \frac{\beta_i x_i(t-\tau)v(t-\tau)}{(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))} d\tau - \delta_i y_i(t) \leq F_i\lambda_i - \sigma_i X_i(t),\end{aligned}$$

where $\sigma_i = \min\{d_i, \delta_i\}$. Hence $\limsup_{t\to\infty} X_i(t) \leq L_i$, where $L_i = \lambda_i F_i/\sigma_i$. Since $\int_0^{\tau_i} f_i(\tau)e^{-m_i\tau} x_i(t-\tau)d\tau > 0$, we get $\limsup_{t\to\infty} y_i(t) \leq L_i$. On the other hand,

$$\dot{v}(t) \leq \sum_{i=1}^n k_i L_i \int_0^{\mu_i} g_i(\tau)e^{-n_i\tau} d\tau - rv = \sum_{i=1}^n k_i L_i G_i - rv,$$

then $\limsup_{t\to\infty} v(t) \leq L^*$, where $L^* = \sum_{i=1}^n \frac{k_i L_i G_i}{r}$. Therefore, $\mathbf{x}(t), \mathbf{y}(t)$ and $v(t)$ are ultimately bounded.□

2.2 Steady states

System (4)-(6) has an uninfected steady state $E_0 = (\mathbf{x}^0, \mathbf{y}^0, v^0)$, where $x_i^0 = \frac{\lambda_i}{d_i}$, $y_i^0 = 0$, $i = 1, ..., n$ and $v^0 = 0$. In addition to E_0, the system can has a positive infected steady state $E_1(\mathbf{x}^*, \mathbf{y}^*, v^*)$. The coordinates of the infected steady state, if they exist, satisfy the equalities:

$$\lambda_i = d_i x_i^* + \frac{\beta_i x_i^* v^*}{(1+a_i x_i^*)(1+b_i v^*)}, \qquad i = 1, ..., n, \tag{8}$$

$$\delta_i y_i^* = F_i \frac{\beta_i x_i^* v^*}{(1+a_i x_i^*)(1+b_i v^*)}, \qquad i = 1, ..., n, \tag{9}$$

$$rv^* = \sum_{i=1}^n G_i k_i y_i^*. \tag{10}$$

The basic reproduction number of system (4)-(6) is given by

$$R_0 = \sum_{i=1}^n R_i = \sum_{i=1}^n \frac{F_i G_i \beta_i k_i x_i^0}{\delta_i r(1+a_i x_i^0)}, \tag{11}$$

where R_i is the basic reproduction number for the dynamics of the interaction of the virus only with the target cells of class i.

Lemma 1. If $R_0 > 1$, then there exists a positive steady state E_1.

Proof. To compute the steady states of model (4)-(6), we let the right-hand sides of Eqs. (4)-(6) equal zero,

$$\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+a_i x_i)(1+b_i v)} = 0, \qquad i = 1, ..., n, \tag{12}$$

$$\frac{F_i \beta_i x_i v}{(1+a_i x_i)(1+b_i v)} - \delta_i y_i = 0, \qquad i = 1, ..., n, \tag{13}$$

$$\sum_{i=1}^{n} G_i k_i y_i - rv = 0. \tag{14}$$

Solving Eq. (12) with respect to x_i, we get x_i as a function of v as:

$$x_i^+ = \frac{a_i x_i^0(1+b_i v) - (1+\phi_i v) + \sqrt{\left[(1+\phi_i v) - a_i x_i^0(1+b_i v)\right]^2 + 4a_i x_i^0(1+b_i v)^2}}{2a_i(1+b_i v)}, \tag{15}$$

$$x_i^- = \frac{a_i x_i^0(1+b_i v) - (1+\phi_i v) - \sqrt{\left[(1+\phi_i v) - a_i x_i^0(1+b_i v)\right]^2 + 4a_i x_i^0(1+b_i v)^2}}{2a_i(1+b_i v)}, \tag{16}$$

where, $\phi_i = b_i + \frac{\beta_i}{d_i}$.

It is clear that if $v > 0$ then $x_i^+ > 0$ and $x_i^- < 0$. Let us choose $x_i = x_i^+$. From Eqs. (12)-(14) we have

$$\sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} (\lambda_i - d_i x_i) - rv = 0. \tag{17}$$

Since x_i is a function of v, then we can define a function $S_1(v)$ as:

$$S_1(v) = \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} (\lambda_i - d_i x_i) - rv = 0.$$

It is clear that when $v = 0$, then $x_i = x_i^0$ and $S_1(0) = 0$ and when $v = \overline{v} = \sum\limits_{i=1}^{n} \frac{F_i G_i k_i \lambda_i}{\delta_i r} > 0$, then substituting it in Eq. (15) we obtain $\overline{x}_i > 0$ and

$$S_1(\overline{v}) = -\sum_{i=1}^{n} \frac{k_i d_i F_i G_i}{\delta_i} \overline{x}_i < 0.$$

Since $S_1(v)$ is continuous for all $v \geq 0$, we have that

$$S_1'(0) = \sum_{i=1}^{n} \frac{k_i \beta_i x_i^0 F_i G_i}{\delta_i \left(1 + a_i x_i^0\right)} - r = r(R_0 - 1).$$

Therefore, if $R_0 > 1$, then $S_1'(0) > 0$. It follows that there exists $v^* \in (0, \overline{v})$ such that $S_1(v^*) = 0$. From Eq. (15), we obtain $x_i^* > 0$, $i = 1, ..., n$. Moreover, from Eq, (13) we get $y_i^* > 0$, $i = 1, ..., n$. $\square$

2.3 Global stability

In this section, we prove the global stability of the uninfected and infected steady states of system (4)-(6) employing the method of Lyapunov functional which is used in [23] for SIR epidemic model with distributed delay.

Next we shall use the following notation: $z = z(t)$, for any $z \in \{x_i, y_i, v,\ i = 1, ..., n\}$. We also define a function $H : (0, \infty) \to [0, \infty)$ as $H(z) = z - 1 - \ln z$. It is clear that $H(z) \geq 0$ for any $z > 0$ and H has the global minimum $H(1) = 0$.

Theorem 1. (i) If $R_0 \leq 1$, then E_0 is GAS.

(ii) If $R_0 > 1$, then E_1 is GAS.

Proof. (i) Define a Lyapunov functional W_1 as:

$$W_1 = \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[\frac{x_i^0}{1+a_i x_i^0} H\left(\frac{x_i}{x_i^0}\right) + \frac{1}{F_i} y_i + \frac{\delta_i}{F_i G_i} \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} \int_0^{\tau} y_i(t-\theta) d\theta d\tau \right.$$
$$\left. + \frac{\beta_i}{F_i} \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \int_0^{\tau} \frac{x_i(t-\theta) v(t-\theta)}{(1+a_i x_i(t-\theta))(1+b_i v(t-\theta))} d\theta d\tau \right] + v.$$

The time derivative of W_1 along the trajectories of (4)-(6) satisfies

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[\frac{1}{1+a_i x_i^0} \left(1 - \frac{x_i^0}{x_i}\right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+a_i x_i)(1+b_i v)}\right) \right. \\
&+ \frac{\beta_i}{F_i} \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \frac{x_i(t-\tau)v(t-\tau)}{(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))} d\tau - \frac{\delta_i}{F_i} y_i \\
&+ \frac{\beta_i}{F_i} \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \left(\frac{x_i v}{(1+a_i x_i)(1+b_i v)} - \frac{x_i(t-\tau)v(t-\tau)}{(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))} \right) d\tau \\
&\left. + \frac{\delta_i}{F_i G_i} \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} (y_i - y_i(t-\tau)) d\tau \right] + \sum_{i=1}^{n} k_i \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} y_i(t-\tau) d\tau - rv \\
&= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[\frac{\lambda_i}{1+a_i x_i^0} \left(2 - \frac{x_i}{x_i^0} - \frac{x_i^0}{x_i}\right) - \frac{\beta_i x_i v}{(1+a_i x_i^0)(1+a_i x_i)(1+b_i v)} \right. \\
&\left. + \frac{\beta_i x_i^0 v}{(1+a_i x_i^0)(1+a_i x_i)(1+b_i v)} + \frac{\beta_i x_i v}{(1+a_i x_i)(1+b_i v)} \right] - rv \\
&= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[\frac{-\lambda_i}{x_i x_i^0 (1+a_i x_i^0)} \left(x_i - x_i^0\right)^2 + \frac{\beta_i x_i^0 v}{(1+a_i x_i^0)(1+b_i v)} \right] - rv \\
&= -\sum_{i=1}^{n} \frac{k_i F_i G_i d_i \left(x_i - x_i^0\right)^2}{\delta_i x_i (1+a_i x_i^0)} + r \sum_{i=1}^{n} \frac{F_i G_i k_i \beta_i x_i^0 v}{\delta_i r (1+a_i x_i^0)(1+b_i v)} - rv \\
&= -\sum_{i=1}^{n} \left(\frac{k_i F_i G_i d_i \left(x_i - x_i^0\right)^2}{\delta_i x_i (1+a_i x_i^0)} + \frac{r b_i R_i v^2}{1+b_i v} \right) + (R_0 - 1) rv. \qquad (18)
\end{aligned}$$

If $R_0 \le 1$, then $\frac{dW_1}{dt} \le 0$ for all $x_i, v > 0$. By Theorem 5.3.1 in [20], the solutions of system (4)-(6) limit to M, the largest invariant subset of $\left\{ \frac{dW_1}{dt} = 0 \right\}$. Clearly, it follows from (18) that $\frac{dW_1}{dt} = 0$ if and only if $x_i = x_i^0$ and $v = 0$. Noting that M is invariant, for each element of M we have $v = 0$, then $\dot{v} = 0$. From Eq. (6) we drive that $0 = \dot{v} = \sum_{i=1}^{n} \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} k_i y_i(t-\tau) d\tau$. This yields $y_i = 0$ and hence $\frac{dW_1}{dt} = 0$ if and only if $x_i = x_i^0$, $y_i = 0$ and $v = 0$. From LaSalle's invariance principle, E_0 is GAS.

(ii) We construct the following Lyapunov functional

$$\begin{aligned}
W_2 &= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[x_i - x_i^* - \int_{x_i^*}^{x_i} \frac{x_i^*(1+a_i \eta)}{\eta(1+a_i x_i^*)} d\eta + \frac{1}{F_i} y_i^* H\left(\frac{y_i}{y_i^*}\right) \right. \\
&+ \frac{1}{F_i} \frac{\beta_i x_i^* v^*}{(1+a_i x_i^*)(1+b_i v^*)} \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \int_0^{\tau} H\left(\frac{x_i(t-\theta)v(t-\theta)(1+a_i x_i^*)(1+b_i v^*)}{x_i^* v^* (1+a_i x_i(t-\theta))(1+b_i v(t-\theta))} \right) d\theta d\tau \\
&\left. + \frac{\delta_i y_i^*}{F_i G_i} \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} \int_0^{\tau} H\left(\frac{y_i(t-\theta)}{y_i^*}\right) d\theta d\tau \right] + v^* H\left(\frac{v}{v^*}\right).
\end{aligned}$$

Differentiating with respect to time yields

$$\begin{aligned}
\frac{dW_2}{dt} &= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[\left(1 - \frac{x_i^*(1+a_i x_i)}{x_i(1+a_i x_i^*)}\right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+a_i x_i)(1+b_i v)}\right) \right. \\
&+ \frac{1}{F_i} \left(1 - \frac{y_i^*}{y_i}\right) \left(\beta_i \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \frac{x_i(t-\tau)v(t-\tau)}{(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))} d\tau - \delta_i y_i \right) \\
&+ \frac{\beta_i}{F_i} \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \left\{ \frac{x_i v}{(1+a_i x_i)(1+b_i v)} - \frac{x_i(t-\tau)v(t-\tau)}{(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))} \right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{x_i^* v^*}{(1+a_i x_i^*)(1+b_i v^*)} \ln\left(\frac{x_i(t-\tau)v(t-\tau)(1+a_i x_i)(1+b_i v)}{x_i v(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))}\right)\Bigg\} d\tau \\
&+\frac{\delta_i}{F_i G_i}\int_0^{\mu_i} g_i(\tau)e^{-n_i\tau}\left(y_i - y_i(t-\tau) + y_i^* \ln\left(\frac{y_i(t-\tau)}{y_i}\right)\right)d\tau\Bigg] \\
&+\left(1-\frac{v^*}{v}\right)\left(\sum_{i=1}^n k_i \int_0^{\mu_i} g_i(\tau)e^{-n_i\tau} y_i(t-\tau)d\tau - rv\right). \qquad (19)
\end{aligned}
$$

Collecting terms of (19) we obtain

$$
\begin{aligned}
\frac{dW_2}{dt} &= \sum_{i=1}^n \frac{k_i F_i G_i}{\delta_i}\left[\left(1-\frac{x_i^*(1+a_i x_i)}{x_i(1+a_i x_i^*)}\right)(\lambda_i - d_i x_i)\right. \\
&+\frac{\beta_i v x_i^*}{(1+a_i x_i^*)(1+b_i v)} - \frac{\beta_i}{F_i}\int_0^{\tau_i} f_i(\tau)e^{-m_i\tau}\frac{y_i^* x_i(t-\tau)v(t-\tau)}{y_i(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))}d\tau + \frac{\delta_i}{F_i}y_i^* \\
&+\frac{1}{F_i}\frac{\beta_i x_i^* v^*}{(1+a_i x_i^*)(1+b_i v^*)}\int_0^{\tau_i} f_i(\tau)e^{-m_i\tau}\ln\left(\frac{x_i(t-\tau)v(t-\tau)(1+a_i x_i)(1+b_i v)}{x_i v(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))}\right)d\tau \\
&\left.+\frac{\delta_i y_i^*}{F_i G_i}\int_0^{\mu_i} g_i(\tau)e^{-n_i\tau}\ln\left(\frac{y_i(t-\tau)}{y_i}\right)d\tau\right] - rv - \frac{v^*}{v}\sum_{i=1}^n k_i\int_0^{\mu_i} g_i(\tau)e^{-n_i\tau}y_i(t-\tau)d\tau + rv^*.
\end{aligned}
$$

Using the infected steady state conditions (8)-(10), we obtain

$$
\begin{aligned}
\frac{dW_2}{dt} &= \sum_{i=1}^n \frac{k_i F_i G_i}{\delta_i}\left[\left(1-\frac{x_i^*(1+a_i x_i)}{x_i(1+a_i x_i^*)}\right)(d_i x_i^* - d_i x_i) - \frac{\delta_i}{F_i}y_i^*\frac{x_i^*(1+a_i x_i)}{x_i(1+a_i x_i^*)} + \frac{\delta_i}{F_i}y_i^*\frac{v(1+b_i v^*)}{v^*(1+b_i v)}\right. \\
&-\frac{\delta_i}{F_i^2}y_i^*\int_0^{\tau_i} f_i(\tau)e^{-m_i\tau}\frac{x_i(t-\tau)v(t-\tau)y_i^*(1+a_i x_i^*)(1+b_i v^*)}{x_i^* v^* y_i\,(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))}d\tau \\
&+3\frac{\delta_i}{F_i}y_i^* + \frac{\delta_i}{F_i^2}y_i^*\int_0^{\tau_i} f_i(\tau)e^{-m_i\tau}\ln\left(\frac{x_i(t-\tau)v(t-\tau)(1+a_i x_i)(1+b_i v)}{x_i v(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))}\right)d\tau \\
&\left.+\frac{\delta_i y_i^*}{F_i G_i}\int_0^{\mu_i} g_i(\tau)e^{-n_i\tau}\ln\left(\frac{y_i(t-\tau)}{y_i}\right)d\tau - \frac{\delta_i}{F_i}y_i^*\frac{v}{v^*} - \frac{\delta_i}{F_i G_i}y_i^*\int_0^{\mu_i} g_i(\tau)e^{-n_i\tau}\frac{v^* y_i(t-\tau)}{v y_i^*}d\tau\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^n \frac{k_i F_i G_i}{\delta_i}\left[\left(1-\frac{x_i^*(1+a_i x_i)}{x_i(1+a_i x_i^*)}\right)(d_i x_i^* - d_i x_i) + \frac{\delta_i}{F_i}y_i^*\left(-1-\frac{v}{v^*}+\frac{v(1+b_i v^*)}{v^*(1+b_i v)}+\frac{1+b_i v}{1+b_i v^*}\right)\right. \\
&-\frac{\delta_i}{F_i}y_i^* H\left(\frac{x_i^*(1+a_i x_i)}{x_i(1+a_i x_i^*)}\right) - \frac{\delta_i}{F_i}y_i^* H\left(\frac{1+b_i v}{1+b_i v^*}\right) \\
&-\frac{\delta_i}{F_i^2}y_i^*\int_0^{\tau_i} f_i(\tau)e^{-m_i\tau}H\left(\frac{x_i(t-\tau)v(t-\tau)y_i^*(1+a_i x_i^*)(1+b_i v^*)}{x_i^* v^* y_i\,(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))}\right)d\tau \\
&\left.-\frac{\delta_i}{F_i G_i}y_i^*\int_0^{\mu_i} g_i(\tau)e^{-n_i\tau}H\left(\frac{v^* y_i(t-\tau)}{v y_i^*}\right)d\tau\right]
\end{aligned}
$$

$$= -\sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[\frac{d_i (x_i - x_i^*)^2}{x_i (1 + a_i x_i^*)} + \frac{\delta_i y_i^* b_i (v - v^*)^2}{F_i v^* (1 + b_i v)(1 + b_i v^*)} + \frac{\delta_i}{F_i} y_i^* H\left(\frac{x_i^*(1 + a_i x_i)}{x_i(1 + a_i x_i^*)}\right) \right.$$
$$+ \frac{\delta_i}{F_i} y_i^* H\left(\frac{1 + b_i v}{1 + b_i v^*}\right) + \frac{\delta_i}{F_i^2} y_i^* \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} H\left(\frac{x_i(t-\tau) v(t-\tau) y_i^* (1 + a_i x_i^*)(1 + b_i v^*)}{x_i^* v^* y_i (1 + a_i x_i(t-\tau))(1 + b_i v(t-\tau))}\right) d\tau$$
$$\left. + \frac{\delta_i}{F_i G_i} y_i^* \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} H\left(\frac{v^* y_i(t-\tau)}{v y_i^*}\right) d\tau \right].$$

It is easy to see that if $x_i^*, y_i^*, v^* > 0$, $i = 1, ..., n$, then $\frac{dW_2}{dt} \le 0$. By Theorem 5.3.1 in [20], the solutions of system (4)-(6) limit to M, the largest invariant subset of $\left\{\frac{dW_2}{dt} = 0\right\}$. It can be seen that $\frac{dW_2}{dt} = 0$ if and only if $x_i = x_i^*$, $v = v^*$, and $H = 0$ i.e.

$$\frac{x_i(t-\tau) v(t-\tau) y_i^* (1 + a_i x_i^*)(1 + b_i v^*)}{x_i^* v^* y_i (1 + a_i x_i(t-\tau))(1 + b_i v(t-\tau))} = \frac{v^* y_i(t-\tau)}{v y_i^*} = 1 \text{ for all } \tau \in [0, \ell]. \tag{20}$$

If $v = v^*$, then from Eq. (20) we have $y_i = y_i^*$, and hence $\frac{dW_2}{dt}$ equal to zero at E_1. LaSalle's invariance principle implies global stability of E_1.□

3 Conclusion

In this paper, we have investigated mathematical model of virus dynamics with distributed delay. We have assumed that the virus attack n classes of target cells. The infection rate is given by Crowley-Martin functional response. By defining the delay-dependent basic reproduction number R_0, we have discussed the existence of the steady states. The global stability of the uninfected and infected steady states of the model has been established using suitable Lyapunov functionals and LaSalle's invariant principle. We have proven that, if $R_0 < 1$, then the uninfected steady state is GAS and if $R_0 > 1$, then infected steady state is GAS.

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The parameter reduction of soft sets and its algorithm *

Zhaowen Li† Ninghua Gao‡

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Abstract: Soft set theory is a new mathematical tool to deal with uncertain problems. In this paper, we prove the fact that there exists a one-to-one correspondence between "the set of all soft sets" and "the set of all 2-value information systems". Base on this fact, we investigate the parameter reduction of soft sets by means of the knowledge reduction in rough set theory and give an algorithm. Parameters of soft sets are classified and the core of soft sets are obtained.

Keywords: Soft sets; Rough sets; Information systems; One-to-one correspondences; Parameter reductions; Cores.

1 Introduction

In 1999, Molodtsov [6] proposed soft set theory as a new mathematical tool for dealing with uncertainties which is free from the difficulties affecting existing method. As reported in [6, 7], a wide range of applications of soft sets have been developed in many different fields, including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory.

Presently, works on theory of soft sets are progressing rapidly. Maji et al. [8, 9] further studied the theory of soft sets, used this theory to solve some decision making problems. Jiang et al. [4] extended soft sets with description logics. Ge et al. [3] discussed relationships between soft sets and topological spaces.

Rough set theory was initiated by [10] for dealing with vagueness and granularity in information systems. This theory handles the approximation of an arbitrary subset of a universe by two definable or observable subsets called lower and upper approximations. It has been successfully applied to machine learning,

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†Corresponding Author, School of Science, Guangxi University for Nationalities, Nanning, Guangxi 530006, P.R.China. lizhaowen8846@163.com

‡School of Science, Guangxi University for Nationalities, Nanning, Guangxi 530006, P.R.China. gaoninghua1987513@126.com

intelligent information systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert information systems and many other fields (see [11]).

Soft set itself has classification ability. The parameter reduction of soft sets means reducing the number of parameters for a soft set to the minimum without distorting its original classification ability. Thus, the parameter reduction of soft sets is a very important problem in soft set theory. Maji et al. [9] introduce parameter reduction of soft sets. Unfortunately some errors in [9] were pointed out by Chen et al. [2]. They present a new definition of parameterization reduction in soft sets. In [5], Kong et al. pointed out some odd situations which may occur when method of reduction of parameters in case of soft sets given in [2] is applied. So they introduced the concept of reduction of normal parameters.

In [1], it has been seen that there is a very close relationship between soft sets and rough sets. The purpose of this paper is to investigate further the parameter reduction of soft sets with the help of rough set theory. We prove the fact that there exists a one-to-one correspondence between "the set of all soft sets" and "the set of all 2-value information systems". Base on this fact, we can do consider the parameter reduction of soft sets by means of the knowledge reduction in rough set theory.

2 Preliminaries

2.1 Soft sets

Definition 2.1 ([6]). *Let U be an initial universe and let A be a set of parameters. A pair (f, A) is called a soft set over U, if f is a mapping given by $f : A \to 2^U$ where 2^U is the power set of U. We denote (f, A) by f_A.*

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $f(e)$ may be considered as the set of e-approximate elements of the soft set f_A.

Example 2.2. *Let $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a universe consisting of five stores. Let $A = \{a_1, a_2, a_3, a_4,$
$a_5, a_6, a_7\}$ be is a set of status of stores where a_1, a_2, a_3, a_4, a_5, a_6 and a_7 represent respectively the parameters "high empowerment of sales personnel", "medium empowerment of sales personnel", "low empowerment of sales personnel", "good perceived quality of merchandise", "average perceived quality of merchandise", "high traffic location" and "low traffic location", respectively. We define f_A as follows*

$$f(a_1) = \{h_1\},\ \ f(a_2) = \{h_2, h_3, h_5\},\ \ f(a_3) = \{h_4\},\ \ f(a_4) = \{h_1, h_2, h_3\},$$
$$f(a_5) = \{h_4, h_5\},\ \ f(a_6) = \{h_1, h_2, h_3\},\ \ f(a_7) = \{h_4, h_5\}.$$

Soft sets f_A can be described as the following Table 1. If $h_i \in f(a_j)$, then $h_{ij} = 1$; otherwise $h_{ij} = 0$, where h_{ij} are the entries in Table 1.

Table 1: Tabular representation of the soft set f_A

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
h_1	1	0	0	1	0	1	0
h_2	0	1	0	1	0	1	0
h_3	0	1	0	1	0	1	0
h_4	0	0	1	0	1	0	1
h_5	0	1	0	0	1	0	1

Definition 2.3. *Let f_A be a soft set over U. f_A is called non-trivial, if for any $a \in A$, $f(a) \neq \emptyset$ and $f(a) \neq U$.*

In this paper, we only consider non-trivial soft sets.

2.2 Information systems

Definition 2.4 ([11, 12])**.** *Let U be a finite set of objects and let A be a finite set of attributes. The pair (U, A, V, g) is called an information system (a knowledge representation system), if g is an information function from $U \times A$ to $V = \bigcup\limits_{a \in A} V_a$ where every $V_a = \{g(x, a) : a \in A$ and $x \in U\}$ is the values of the attribute a.*

Definition 2.5. *An information system (U, A, V, g) is called 2-value, if $V = \{0, 1\}$.*

Example 2.6. *Let $U = \{h_1, h_2, h_3, h_4\}$ be a universe consisting of four patients, and let $A = \{a_1, a_2, a_3\}$ be a set of attributes where a_1, a_2 and a_3 represent respectively the attributes " headache", " muscle pain" and " fever".*

Now, we consider an information system (U, A, V, g), which describes the " symptoms of patients". For instance, "$g(h_1, a_1) = yes$" means "h_1 suffers from headache" and its functional value is yes; "$g(h_3, a_2) = no$ "means "h_3 has no muscle pain" and its functional value is no; "$g(h_3, a_3) = no$" means "h_3 doesn't have a fever" and its functional value is no.

We define
$g(h_1, a_1) =$*yes*, $g(h_1, a_2) =$*yes*, $g(h_1, a_3) =$*no;*
$g(h_2, a_1) =$*yes*, $g(h_2, a_2) =$*yes*, $g(h_2, a_3) =$*yes;*
$g(h_3, a_1) =$*yes*, $g(h_3, a_2) =$*yes*, $g(h_3, a_3) =$*no;*
$g(h_4, a_1) =$*no*, $g(h_4, a_2) =$*yes*, $g(h_4, a_3) =$*no.*

Let h_{ij} be the entries. If $g(h_i, a_j) =$yes, then $h_{ij} = 1$; if $g(h_i, a_j) =$no, then $h_{ij} = 0$. A 2-value information system (U, A, V, g) can be described as the following Table 2.

In Table 2, $V_{a_1} = \{0, 1\}$, $V_{a_2} = \{0, 1\}$, $V_{a_3} = \{0, 1\}$, $V = \bigcup_{a \in A} V_a = \{0, 1\}$.

Let (U, A, V, g) be an information system and let $P \subseteq A$. We denote

$$ind(P) = \{(x, y) \in U \times U : g(x, a) = g(y, a) \ for \ any \ a \in P\}.$$

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Table 2: The 2-value information system (U, A, V, g)

	a_1	a_2	a_3
h_1	1	1	0
h_2	1	1	1
h_3	1	1	0
h_4	0	1	0

Obviously, $ind(P)$ is an equivalence relation on U, which is called the equivalence relation induced by P. Sometimes, we replace respectively $ind(P)$ and $U/ind(P)$ by $\mathbf{P}$ and $U/\mathbf{P}$ where

$$U/ind(P) = \{[x]_{ind(P)} : x \in U\}.$$

Specially, we replace $ind(\{a\})$ by $\mathbf{a}$ for $a \in A$.

Theorem 2.7. *Let $S = f_A$ be a soft set over U and let $I_S = (U, A, V, g_s)$ be a 2-value information system induced by S. Then for any $a \in A$, $U/\mathbf{a} = \{f(a), U - f(a)\}$.*

Proof. Since

$$\mathbf{a} = \{(x, y) \in U \times U : g_s(x, a) = g_s(y, a)\},$$

$$g_s(x, a) = g_s(y, a) = 1 \text{ or } g_s(x, a) = g_s(y, a) = 0.$$

This implies that $\{x, y\} \subset f(a)$ *or* $\{x, y\} \subset U - f(a)$. Thus $U/\mathbf{a} = \{f(a), U - f(a)\}$. $\square$

2.3 The relationship between soft sets and information systems

Definition 2.8. *Let $S = f_A$ be a soft set over U. Then $I_S = (U, A, V, g_s)$ is called a 2-value information system induced by S where $g_s : U \times A \to V$.*

For any $x \in U$ and $a \in A$,

$$g_s(x, a) = \begin{cases} 1, & x \in f(a), \\ 0, & x \notin f(a). \end{cases}$$

Definition 2.9. *Let $I = (U, A, V, g)$ be a 2-value information system. Then $S_I = (f_I, A)$ is called a soft set over U induced by I where $f_I : A \to 2^U$ and for any $x \in U$ and $a \in A$, $f_I(a) = \{x \in U : g(x, a) = 1\}$.*

Lemma 2.10. *Let $S = f_A$ be a soft set over U, let $I_S = (U, A, V, g_s)$ be a 2-value information system induced by S over U and let $S_{I_S} = (f_{I_S}, A)$ be a soft set over U induced by I_S. Then $S = S_{I_S}$.*

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Proof. By Definition 2.9, for any $a \in A$, $f_{I_S}(a) = \{x \in U : g_s(x,a) = 1\}$.

By Definition 2.8, for any $x \in U$ and $a \in A$,

$$g_s(x,a) = \begin{cases} 1, & x \in f(a), \\ 0, & x \notin f(a). \end{cases}$$

This implies that $g_s(x,a) = 1 \Leftrightarrow x \in f(a)$. So, for $\forall x \in U, a \in A$, $f(a) = f_{I_S}(a)$. Hence $f_A = (f_{I_S}, A)$. This implies that $S = S_{I_S}$. □

Lemma 2.11. *Let $I = (U, A, V, g)$ be a 2-value information system, Let $S_I = (f_{I,A})$ be a soft set over U induced by I and let $I_{S_I} = (U, A, V, g_{s_I})$ be a 2-value information system induced by S_I. Then $I = I_{S_I}$.*

Proof. By Definition 2.8, for any $x \in U$ and $a \in A$,

$$g_{s_I}(x,a) = \begin{cases} 1, & x \in f_I(a), \\ 0, & x \notin f_I(a). \end{cases}$$

For any $x \in U$ and $a \in A$, by Definition 2.9, $f_I(a) = \{x \in U : g(x,a) = 1\}$. Since $I = (U, A, V, g)$ is a 2-value information system, $g(x,a) = 0$ for $x \notin f_I(a)$, This implies that

$$g(x,a) = \begin{cases} 1, & x \in f_I(a), \\ 0, & x \notin f_I(a). \end{cases}$$

So for any $x \in U$ and $a \in A$, $g_{s_I}(x,a) = g(x,a)$. Hence $g_{s_I} = g$. This implies that that $I = I_{S_I}$. □

Theorem 2.12. *Let*

$$\Sigma = \{S : \ S = f_A \ is \ a \ soft \ set \ over \ U\}$$

and

$$\Gamma = \{I : I = (U, A, V, g) isa2 - valueinformationsystem\}.$$

Then there exists a one-to-one correspondence between Σ and Γ.

Proof. Two mappings $F : \Sigma \to \Gamma$ and $G : \Gamma \to \Sigma$ are defined as follows:

$$F(S) = I_S \ for \ \forall S \in \Sigma; \quad G(I) = S_I \ for \ \forall I \in \Gamma.$$

By Lemma 2.9, $G \circ F = i_\Sigma$ where $G \circ F$ is the composition of F and G, and i_Σ is the identity mapping on Σ.

By Lemma 2.10, $F \circ G = i_\Gamma$ where $G \circ F$ is the composition of G and F, and i_Γ is the identity mapping on Γ.

Hence F and G are both a one-to-one correspondence. This prove that there exists a one-to-one correspondence between Σ and Γ. □

3 The parameter reduction of soft sets

Soft sets and rough sets are two different concepts to deal with uncertainty. Both of these concepts help in decision-making problems. Soft set itself has classification ability. The parameter reduction of soft sets means reducing the number of parameters for a soft set to the minimum without distorting its original classification ability. Specific approach is first classifying the parameter according to the importance of parameters and then finding the minimum set of parameters (ie., the core for a soft set) without distorting the original classification ability of soft sets.

Reduction of parameters of soft sets plays a vital role in decision-making problems. Reduction of parameters can save expensive tests and time.

Since there exists a one-to-one correspondence between "the set of all soft sets" and "the set of all 2-value information systems" (see Theorem 2.12), we can do the parameter reduction of soft sets with the help of the knowledge reduction in rough set theory.

Definition 3.1. *Let f_A be a soft set over U.*

(1) *$A^* \subseteq A$ is called a parameter reduction of f_A (brief. a f_A-parameter reduction), if $ind(A) = ind(A^*)$ and $ind(A) \neq ind(B)$ for any $B \subsetneqq A^*$.*

(2) *The intersection set of all f_A-parameter reductions is called the core of f_A. We denote it by core(f_A).*

In this paper, we denote the set of all f_A-parameter reductions by $pr(f_A)$.

Proposition 3.2. *Let f_A be a soft set over U. Then $pr(f_A) \neq \emptyset$.*

Proof. (1) If $ind(A) \neq ind(A - \{a\})$ for any $a \in A$, then A itself is a f_A-parameter reduction.

(2) If $ind(A) = ind(A - \{a\})$ for some $a \in A$, then we consider $B_1 = A - \{a\}$. If $ind(A) \neq ind(B_1 - \{b_1\})$ for any $b_1 \in B_1$, B_1 is a f_A-parameter reduction. Otherwise, we consider $B_1 - \{b_1\}$ again and repeat the above mentioned process. Since A is a finite set, we can find a f_A-parameter reduction.

Thus, $pr(f_A) \neq \emptyset$. □

Definition 3.3. *Let f_A be a soft set over U and let $pr(f_A) = \{C_i : 1 \leq i \leq n\}$. Then*

(1) *$a \in A$ is called core, if $a \in \bigcap_{i=1}^{n} C_i = core(f_A)$.*

(2) *$a \in A$ is called relative indispensable, if $a \in \bigcup_{i=1}^{n} C_i - core(f_A)$.*

(3) *$a \in A$ is called absolutely dispensable, if $a \in A - \bigcup_{i=1}^{n} C_i$.*

(4) *$a \in A$ is called dispensable, if $a \in A - core(f_A)$.*

Obviously, $a \in A$ is dispensable if and only if a is relative indispensable or absolutely dispensable.

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Definition 3.4. *Let $\mathcal{A}, \mathcal{B} \subset 2^U$. $\mathcal{A}$ is called a refinement of $\mathcal{B}$, if for any $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $A \subseteq B$. We denote it by $\mathcal{A} \leq \mathcal{B}$.*

Lemma 3.5. *Let R and ρ be two equivalence relations on U. If $R \subseteq \rho$, then $U/R \leq U/\rho$.*

Proof. Suppose that $A \in X/R$. Since R is an equivalence relation on X, there exists $x \in X$, such that $A = [x]_R$.

Suppose that $y \in [x]_R$. Then xRy. This implies that $(x, y) \in R$. Since $R \subseteq \rho$, $(x, y) \in \rho$. This implies that $y \in [x]_\rho$. Then $[x]_R \subseteq [x]_\rho$.

Pick $B = [x]_\rho$. Then $A \subseteq B$ and so $X/R \leq X/\rho$. $\square$

The following Theorem 3.6 and Corollary 3.7 give the parameter reduction of soft sets.

Theorem 3.6. *Let f_A be a soft set over U. Then*
(1) $|pr(f_A)| = 1$ if and only if $core(f_A) \in pr(f_A)$.
(2) $a \in core(f_A)$ if and only if $U/ind(A) \neq U/ind(A - \{a\})$.
(3) $a \in A$ is dispensable if and only if $U/ind(A) = U/ind(A - \{a\})$.

Proof. (1) Sufficiency. Let $core(f_A) \in pr(f_A)$. Note that $pr(f_A) = \{C_i : 1 \leq i \leq n\}$. We only need to prove $n = 1$.

1) Suppose $n = 2$. Then there are only two different f_A-parameter reductions C_1 and C_2.

a) If $C_1 \subsetneq C_2$. Since $C_2 \in pr(f_A)$, $ind(A) \neq ind(C_1)$. Then $C_1 \notin pr(f_A)$. This is a contradiction.

b) If $C_2 \subsetneq C_1$. We can similarly prove that this implies a contradiction.

c) If $C_1 \nsubseteq C_2$ and $C_2 \nsubseteq C_1$. Obviously, $core(f_A) = C_1 \cap C_2$ and $core(f_A) \subsetneq C_1$. Since $C_1 \in pr(f_A)$, $ind(A) \neq ind(core(f_A))$. Then $core(f_A) \notin pr(f_A)$. This is also a contradiction.

2) Suppose $n \geq 3$. This is similar to the proof of 1).

Thus $|pr(f_A)| = 1$.

Necessity. This is obvious.

(2) Sufficiency. Suppose that $U/ind(A) \neq U/ind(A - \{a\})$. We claim that $a \in C_i$ for any $1 \leq i \leq n$. Otherwise. $a \notin C_{i_0}$ for some C_{i_0}. This implies that $U/ind(A) = U/ind(C_{i_0})$. Since $ind(C_{i_0}) \supseteq ind(A - \{a\}) \supseteq ind(A)$, by Lemma 3.5, $U/ind(C_{i_0}) \geq U/ind(A - \{a\}) \geq U/ind(A)$. So $U/ind(A) = U/ind(A - \{a\})$, a contradiction.

This implies that $a \in core(f_A)$.

Necessity. Suppose that $U/ind(A) = U/ind(A - \{a\})$. Since $pr(f_A) \neq \emptyset$, there exists $B_1' \subseteq A - \{a\}$ such that $B_1' \in pr(f_A)$. So $a \notin core(f_A)$. This is a contradiction.

Thus $U/ind(A) \neq U/ind(A - \{a\})$.

(3) Sufficiency. Suppose that $U/ind(A) = U/ind(A - \{a\})$. Since $A - \{a\}$ is a finite set, there exists $B_2 \subseteq A - \{a\}$ such that $B_2 \in pr(f_A)$. So $a \notin core(f_A)$. This implies that $a \in A - core(f_A)$.

Thus a is a dispensable parameter.

Necessity. Suppose that $U/ind(A) \neq U/ind(A-\{a\})$. Similar to the proof of (2), we have $a \in core(f_A)$. Then $a \notin A - core(f_A)$. Note that a is a dispensable parameter. Then $a \in A - core(f_A)$. This implies a contradiction.

Thus $U/ind(A) = U/ind(A - \{a\})$. □

Corollary 3.7. $core(f_A) = \{a \in A : U/ind(A) \neq U/ind(A - \{a\})\}$.

4 Algorithms

Algorithms 4.1. *Let f_A be a soft set over U. The algorithm of parameter reduction is shown as follows:*

Input: A soft set f_A.

Output: $pr(f_A)$ and $core(f_A)$.

Step 1. Calculate $U/ind(A)$ and $U/ind(A - \{a\})$ for any $a \in A$;

Step 2. If $U/ind(A) \neq U/ind(A - \{a\})$ for any $a \in A$, then $pr(f_A) = \{A\}$ and $core(f_A) = A$;

Step 3. If $U/ind(A) = U/ind(A - \{a\})$ for some $a \in A$, then we consider $B_1 = A - \{a\}$. If $U/ind(A) \neq U/ind(B_1 - \{b_1\})$ for any $b_1 \in B_1$, then $B_1 \in pr(f_A)$; Otherwise, we consider $B_1 - \{b_1\}$ again;

Step 4. Output $pr(f_A)$ and $core(f_A)$.

Example 4.2. *Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$ and let f_A be a soft set over U, defined as follows*

$$f(a_1) = \{h_1, h_2, h_5\}, \quad f(a_2) = \emptyset, \; f(a_3) = \{h_3\}, \; f(a_4) = \{h_3, h_4\}.$$

By Theorem 2.7, we have $U/\mathbf{a_1} = \{f(a_1), U - f(a_1)\} = \{\{h_1, h_2, h_5\}, \{h_3, h_4\}\}$,

$U/\mathbf{a_2} = \{f(a_2), U - f(a_2)\} = \{\{h_1, h_2, h_3, h_4, h_5\}\}$,

$U/\mathbf{a_3} = \{f(a_3), U - f(a_3)\} = \{\{h_3\}, \{h_1, h_2, h_4, h_5\}\}$,

$U/\mathbf{a_4} = \{f(a_4), U - f(a_4)\} = \{\{h_3, h_4\}, \{h_1, h_2, h_5\}\}$. *And*

$U/\mathbf{A} = \{\{h_1, h_2, h_5\}, \{h_3\}, \{h_4\}\}$. $U/ind(A-\{a_1\}) = \{\{h_1, h_2, h_5\}, \{h_3\}, \{h_4\}\} = U/ind(A)$.

$U/ind(A - \{a_2\}) = \{\{h_1, h_2, h_5\}, \{h_3\}, \{h_4\}\} = U/ind(A)$.

$U/ind(A - \{a_3\}) = \{\{h_1, h_2, h_5\}, \{h_3, h_4\}\} \neq U/ind(A)$.

$U/ind(A - \{a_4\}) = \{\{h_1, h_2, h_5\}, \{h_3\}, \{h_4\}\} = U/ind(A)$.

This implies that

$U/ind(\{a_2, a_3, a_4\}) = U/ind(\{a_1, a_3, a_4\}) = U/ind(\{a_1, a_2, a_3\}) = U/ind(A)$.

Since $U/ind(\{a_2, a_3, a_4\}) = U/ind(\{a_3, a_4\})$, $U/ind(\{a_3, a_4\}) \neq U/ind(\{a_3\})$ *and* $U/ind(\{a_3, a_4\}) \neq U/ind(\{a_4\})$, $\{a_3, a_4\}$ *is a f_A-parameter reduction.*

Since $U/ind(\{a_1, a_3, a_4\}) = U/ind(\{a_1, a_3\})$, $U/ind(\{a_1, a_3\}) \neq U/ind(\{a_1\})$ *and* $U/ind(\{a_1, a_3\}) \neq U/ind(\{a_4\})$, $\{a_1, a_3\}$ *also is a f_A-parameter reduction.*

Obviously,

$$pr(f_A) = \{\{a_3, a_4\}, \{a_1, a_3\}\},$$
$$core(f_A) = \{a_3, a_4\} \cap \{a_1, a_3\} = \{a_3\}.$$

Example 4.3. *In Example 4.2, we have*
(1) a_3 *is core.* (2) a_1 *and* a_4 *are relative indispensable.*
(3) a_2 *is absolutely dispensable.* (4) a_1*,* a_2 *and* a_4 *are dispensable.*

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FUNCTIONAL INEQUALITIES ASSOCIATED WITH BI-CAUCHY ADDITIVE FUNCTIONAL EQUATIONS

GANG LU, CHOONKIL PARK, AND DONG YUN SHIN*

ABSTRACT. In this paper, we prove the Hyers-Ulam stability for the following functional inequalities:

$$\|f(x_1,y_1)+f(x_2,y_2)+f(x_3,y_3)\| \le \|f(x_1+x_2+x_3,y_1+y_2+y_3)\|, \tag{1}$$

$$\|f(x_1,y_1)+f(x_2,y_2)+f(x_3,y_3)\| \le \left\|2f\left(\frac{x_1+x_2+x_3}{2},\frac{y_1+y_2+y_3}{2}\right)\right\|, \tag{2}$$

$$\|f(x_1,y_1)+f(x_2,y_2)+2f(x_3,y_3)\| \le \left\|2f\left(\frac{x_1+x_2}{2}+x_3,\frac{y_1+y_2}{2}+y_3\right)\right\| \tag{3}$$

in Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms: Let $(G_1,*)$ be a group and let $(G_2,\diamond,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta(\epsilon)>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality

$$d(h(x*y),h(x)\diamond h(y))<\delta$$

for all $x,y\in G_1$, then there is a homomorphism $H:G_1\to G_2$ with

$$d(h(x),H(x))<\epsilon$$

for all $x\in G_1$? If the answer is affirmative, we would say that the question of homomorphism $H(x*y)=H(x)\diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equation is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f:X\to Y$ satisfies

$$\|f(x+y)-f(x)-f(y)\|\le\epsilon$$

for all $x,y\in X$ and some $\epsilon\ge0$. Then there exists a unique additive mapping $T:X\to Y$ such that

$$\|f(x)-T(x)\|\le\epsilon$$

for all $x\in X$.

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f:X\to Y$ to be a mapping such that $f(tx)$ is continuous in $t\in\mathcal{R}$ for each fixed $x\in X$. Th.M. Rassias

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*The corresponding author.

[3] introduced the following inequality, that we call *Cauchy-Rassias inequality* : Assume that there exist constants $\lambda \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \lambda(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Th.M. Rassias [3] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\lambda}{2 - 2^p}\|x\|^p$$

for all $x \in X$. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [4] generalized the Rassias' result.

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation.* In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping.* A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replace by an Abelian group. In [7], Czerwik proved the Hyers-Ulam stability of the quadratic functional equation. Borelli and Forti [8] generalized the stability result. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found in [9]–[28].

In this paper, let X be a vector space and Y a Banach space. A mapping $f : X \to Y$ is called a Cauchy additive mapping if f satisfies the functional equation $f(x+y) = f(x) + f(y)$. For a given mapping $f : X \times X \to Y$, we define

$$f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_2, y_2) \tag{1.1}$$

for all $(x_1, y_1), (x_2, y_2) \in X \times X$. A mapping $f : X \times X \to Y$ is called a bi-Cauchy mapping if f satisfies the functional equation (1.1). We investigate the functional inequalities (1), (2) and (3) and prove the Hyers-Ulam stability of the functional inequalities (1), (2) and (3).

2. Hyers-Ulam stability of the functional inequality (1)

Proposition 2.1. *Let $f : X \times X \to Y$ be a mapping such that*

$$\|f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3)\| \leq \|f(x_1 + x_2 + x_3, y_1 + y_2 + y_3)\| \tag{2.1}$$

for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times X$. Then the mapping $f : X \to Y$ is bi-Cauchy additive.

Proof. Letting $(x_1, y_1) = (x_2, y_2) = (x_3, y_3) = (0, 0)$ in (2.1), we have

$$\|3f(0,0)\| \leq \|f(0,0)\|$$

and so $f(0, 0) = 0$.

Letting $x_1 = x, x_2 = -x, x_3 = 0, y_1 = y, y_2 = -y, y_3 = 0$ in (2.1), we get

$$\|f(x, y) + f(-x, -y)\| \leq 0$$

and so $f(x, y) = -f(-x, -y)$ for all $(x, y) \in X \times X$.

Next, we show that f is a bi-Cauchy additive mapping.

$$\begin{aligned}&\|f(x_1,y_1)+f(x_2,y_2)-f(x_1+x_2,y_1+y_2)\|\\&=\|f(x_1,y_2)+f(x_2,y_2)+f(-x_1-x_2,-y_1-y_2)\|\\&\le\|f(0,0)\|=0\end{aligned}$$

and so $f(x_1+x_2,y_1+y_2)=f(x_1,y_1)+f(x_2,y_2)$ for all $(x_1,y_1),(x_2,y_2)\in X\times X$, as desired. □

Theorem 2.2. *Assume that a mapping $f:X\times X\to Y$ satisfies the inequality*

$$\begin{aligned}&\|f(x_1,y_1)+f(x_2,y_2)+f(x_3,y_3)\|\\&\le\|f\left(x_1+x_2+x_3,y_1+y_2+y_3\right)\|+\phi((x_1,y_1),(x_2,y_2),(x_3,y_3)),\end{aligned}\tag{2.2}$$

where $\phi:(X\times X)^3\to[0,\infty)$ satisfies

$$\widetilde{\phi}((x_1,y_1),(x_2,y_2),(x_3,y_3)):=\sum_{j=1}^{\infty}2^j\phi\left(\left(\frac{x_1}{2^j},\frac{y_1}{2^j}\right),\left(\frac{x_2}{2^j},\frac{y_2}{2^j}\right),\left(\frac{x_3}{2^j},\frac{y_3}{2^j}\right)\right)<\infty\tag{2.3}$$

for all $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in X\times X$. Then there exists a unique bi-Cauchy additive mapping $A:X\times X\to Y$ such that

$$\|A(x,y)-f(x,y)\|\le\widetilde{\phi}\left(\left(\frac{x}{2},\frac{y}{2}\right),\left(\frac{x}{2},\frac{y}{2}\right),(-x,-y)\right)+\widetilde{\phi}((x,y),(-x,-y),(0,0))\tag{2.4}$$

for all $(x,y)\in X\times X$.

Proof. Letting $x_1=x_2=x_3=0$ and $y_1=y_2=y_3=0$ in (2.2), we get $f(0,0)=0$.

Letting $x_1=x_2=x,y_1=y_2=y$ and $x_3=-2x,y_3=-2y$ in (2.2), we get

$$\|2f(x,y)+f(-2x,-2y)\|\le\phi((x,y),(x,y),(-2x,-2y))$$

for all $(x,y)\in X\times X$.

Letting $x_1=2x,x_2=-2x,x_3=0$ and $y_1=2y,y_2=-2y,y_3=0$ in (2.2), we obtain

$$\|f(2x,2y)+f(-2x,-2y)\|\le\phi((2x,0),(-2x,-2y),(0,0))$$

for all $(x,y)\in X\times X$.

Thus

$$\begin{aligned}&\left\|f(x,y)-2f\left(\frac{1}{2}x,\frac{1}{2}y\right)\right\|\\&\le\left[\phi\left(\left(\frac{1}{2}x,\frac{1}{2}y\right),\left(\frac{1}{2}x,\frac{1}{2}y\right),(-x,-y)\right)+\phi((x,y),(-x,-y),(0,0))\right]\end{aligned}$$

and so

$$\begin{aligned}&\left\|2^lf\left(\frac{x}{2^l},\frac{y}{2^l}\right)-2^mf\left(\frac{x}{2^m},\frac{y}{2^m}\right)\right\|\\&\le\sum_{j=l}^{m-1}2^j\left[\phi\left(\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}}\right),\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}}\right),\left(-\frac{x}{2^j},-\frac{y}{2^j}\right)\right)\right.\\&\left.+\phi\left(\left(\frac{x}{2^j},\frac{y}{2^j}\right),\left(-\frac{x}{2^j},-\frac{y}{2^j}\right),(0,0)\right)\right]\end{aligned}\tag{2.5}$$

for all nonnegative integers m and l with $m>l$ and all $(x,y)\in X\times X$. It follows from (2.3) and (2.5) that the sequence $\{2^kf(\frac{x}{2^k},\frac{y}{2^k})\}$ is a Cauchy sequence for all $(x,y)\in X\times X$. Since Y is complete, the sequence $\{2^k f(\frac{x}{2^k}, \frac{y}{2^k})\}$ converges. So we can define the mapping $A : X \times X \to Y$ by

$$A(x,y) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}, \frac{y}{2^k}\right)$$

for all $(x,y) \in X \times X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$, we get (2.4).

Now, we show that $A(x,y)$ is a bi-Cauchy additive mapping.

It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\|A(x,y) + A(-x,-y)\| \\
&= \lim_{k\to\infty} 2^k \left\| f\left(\frac{x}{2^k}, \frac{y}{2^k}\right) + f\left(\frac{-x}{2^k}, \frac{-y}{2^k}\right) + f(0,0) \right\| \\
&\le \lim_{k\to\infty} 2^k \left[\left\| f\left(\frac{x}{2^k} + \frac{-x}{2^k} + 0, \frac{y}{2^k} + \frac{-y}{2^k} + 0\right) \right\| + \phi\left(\left(\frac{x}{2^k}, \frac{y}{2^k}\right), \left(\frac{-x}{2^k}, \frac{-y}{2^k}\right), (0,0)\right) \right] \\
&= 0
\end{aligned}$$

and so $A(x,y) = -A(-x,-y)$ for any $(x,y) \in X \times X$.

$$\begin{aligned}
&\|A(x_1,y_1) + A(x_2,y_2) - A(x_1+x_2, y_1+y_2)\| \\
&= \|A(x_1,y_1) + A(x_2,y_2) + A(-x_1-x_2, -y_1-y_2)\| \\
&= \lim_{k\to\infty} 2^k \left\| f\left(\frac{x_1}{2^k}, \frac{y_1}{2^k}\right) + f\left(\frac{x_2}{2^k}, \frac{y_2}{2^k}\right) + f\left(\frac{-x_1-x_2}{2^k}, \frac{-y_1-y_2}{2^k}\right) \right\| \\
&\le \lim_{k\to\infty} 2^k \left[\left\| f\left(\frac{x_1}{2^k} + \frac{x_2}{2^k} + \frac{-x_1-x_2}{2^k}, \frac{y_1}{2^k} + \frac{y_2}{2^k} + \frac{-y_1-y_2}{2^k}\right) \right\| \right. \\
&\left. + \phi\left(\left(\frac{x_1}{2^k}, \frac{y_1}{2^k}\right), \left(\frac{x_2}{2^k}, \frac{y_2}{2^k}\right), \left(\frac{-x_1-x_2}{2^k}, \frac{-y_1-y_2}{2^k}\right)\right) \right] \\
&= 0
\end{aligned}$$

for all $(x_1,y_1), (x_2,y_2) \in X \times X$. Thus the mapping $A : X \times X \to$ is bi-Cauchy additive.

Next, we prove the uniqueness of A. Suppose that $T : X \times X \to Y$ is another additive mapping satisfying (2.4). We may obtain

$$\begin{aligned}
\|A(x,y) - T(x,y)\| &= \lim_{k\to\infty} 2^k \left\| A\left(\frac{x}{2^k}, \frac{y}{2^k}\right) - T\left(\frac{x}{2^k}, \frac{y}{2^k}\right) \right\| \\
&\le \lim_{k\to\infty} 2^k \left\| A\left(\frac{x}{2^k}, \frac{y}{2^k}\right) - f\left(\frac{x}{2^k}, \frac{y}{2^k}\right) \right\| \\
&+ \lim_{k\to\infty} 2^k \left\| T\left(\frac{x}{2^k}, \frac{y}{2^k}\right) - f\left(\frac{x}{2^k}, \frac{y}{2^k}\right) \right\| \\
&\le \lim_{k\to\infty} 2 \left[\widetilde{\phi}\left(\left(\frac{x}{2^{k+1}}, \frac{y}{2^{k+1}}\right), \left(\frac{x}{2^{k+1}}, \frac{y}{2^{k+1}}\right), \left(-\frac{x}{2^k}, -\frac{y}{2^k}\right)\right) \right. \\
&\left. + \widetilde{\phi}\left(\left(\frac{x}{2^k}, \frac{y}{2^k}\right), \left(-\frac{x}{2^k}, -\frac{y}{2^k}\right), (0,0)\right) \right] \\
&= 0
\end{aligned}$$

for all $(x,y) \in X \times X$. Thus we can conclude that $A(x,y) = T(x,y)$ for all $(x,y) \in X \times X$. This complete the proof. □

3. Hyers-Ulam stability of the functional inequality (2)

Proposition 3.1. *Let $f : X \times X \to Y$ be a mapping such that*

$$\|f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3)\| \leq \left\|2f\left(\frac{x_1 + x_2 + x_3}{2}, \frac{y_1 + y_2 + y_3}{2}\right)\right\| \tag{3.1}$$

for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times X$. Then the mapping $f : X \times X \to Y$ is bi-Cauchy additive.

Proof. Letting $x_1 = x_2 = x_3 = 0, y_1 = y_2 = y_3 = 0$ in (3.1), we get

$$\|3f(0, 0)\| \leq \|2f(0, 0)\|.$$

So $f(0, 0) = 0$.

Letting $x_1 = x, y_1 = y, x_2 = -x, y_2 = -y$ and $x_3 = y_3 = 0$ in (3.1), we get

$$\|f(x, y) + f(-x, -y) + f(0, 0)\| \leq \|2f(0, 0)\| = 0$$

for all $(x, y) \in X \times X$. So $f(-x, -y) = -f(x, y)$ for all $(x, y) \in X \times X$.

Letting $x_3 = -x_1 - x_2, y_3 = -y_1 - y_2$ in (3.1), we obtain

$$\begin{aligned}&\|f(x_1, y_1) + f(x_2, y_2) - f(x_1 + x_2, y_1 + y_2)\| \\ &= \|f(x_1, y_1) + f(x_2, y_2) + f(-x_1 - x_2, -y_1 - y_2)\| \leq \|2f(0, 0)\| = 0\end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in X \times X$. Thus

$$f(x_1, y_1) + f(x_2, y_2) = f(x_1 + x_2, y_1 + y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in X \times X$, as desired. □

Theorem 3.2. *Assume that a mapping $f : X \times X \to Y$ satisfies the inequality*

$$\begin{aligned}\|f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3)\| &\leq \left\|2f\left(\frac{x_1 + x_2 + x_3}{2}, \frac{y_1 + y_2 + y_3}{2}\right)\right\| \\ &+ \phi((x_1, y_1), (x_2, y_2), (x_3, y_3))\end{aligned} \tag{3.2}$$

where $\phi : (X \times X)^3 \to [0, \infty)$ satisfies

$$\widetilde{\phi}((x_1, y_1), (x_2, y_2), (x_3, y_3)) := \sum_{j=1}^{\infty} 2^j \phi\left(\left(\frac{x_1}{2^j}, \frac{y_1}{2^j}\right), \left(\frac{x_2}{2^j}, \frac{y_2}{2^j}\right), \left(\frac{x_3}{2^j}, \frac{y_3}{2^j}\right)\right) < \infty$$

for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times X$. Then there exists a unique bi-Cauchy additive mapping $A : X \times X \to Y$ such that

$$\|A(x, y) - f(x, y)\| \leq \widetilde{\phi}\left(\left(\frac{x}{2}, \frac{y}{2}\right), \left(\frac{x}{2}, \frac{y}{2}\right), (-x, -y)\right) + \widetilde{\phi}((x, y), (-x, -y), (0, 0))$$

for all $(x, y) \in X \times X$.

Proof. Letting $x_1 = x_2 = x_3 = 0$ and $y_1 = y_2 = y_3 = 0$ in (3.2), we get $f(0, 0) = 0$.

Letting $x_1 = x_2 = x, y_1 = y_2 = y$ and $x_3 = -2x, y_3 = -2y$ in (3.2), we get

$$\|2f(x, y) + f(-2x, -2y)\| \leq \phi((x, y), (x, y), (-2x, -2y))$$

for all $(x, y) \in X \times X$.

Letting $x_1 = 2x, x_2 = -2x, x_3 = 0$ and $y_1 = 2y, y_2 = -2y, y_3 = 0$ in (3.2), we obtain

$$\|f(2x, 2y) + f(-2x, -2y)\| \leq \phi((2x, 0), (-2x, -2y), (0, 0))$$

for all $(x, y) \in X \times X$.

Thus we get

$$\begin{aligned}&\left\|f(x,y)-2f\left(\frac{1}{2}x,\frac{1}{2}y\right)\right\|\\&\le\left[\phi\left(\left(\frac{1}{2}x,\frac{1}{2}y\right),\left(\frac{1}{2}x,\frac{1}{2}y\right),(-x,-y)\right)+\phi((x,y),(-x,-y),(0,0))\right]\end{aligned}$$

for any $(x,y)\in X\times X$.

The rest of the proof is the same as in the proof of Theorem 2.2. □

4. Hyers-Ulam stability of the functional inequality (3)

Proposition 4.1. *Let $f:X\times X\to Y$ be a mapping such that*

$$\|f(x_1,y_1)+f(x_2,y_2)+2f(x_3,y_3)\|\le\left\|2f\left(\frac{x_1+x_2}{2}+x_3,\frac{y_1+y_2}{2}+y_3\right)\right\| \tag{4.1}$$

for all $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in X\times X$. Then the mapping $f:X\times X\to Y$ is bi-Cauchy additive.

Proof. Letting $x_1=x_2=x_3=0,y_1=y_2=y_3=0$ in (4.1), we get

$$\|4f(0,0)\|\le\|2f(0,0)\|.$$

So $f(0,0)=0$.

Letting $x_1=x,x_2=-x,x_3=0$ and $y_1=y,y_2=-y,y_3=0$ in (4.1), we obtain

$$\|f(x,y)+f(-x,-y)+2f(0,0)\|\le\|2f(0,0)\|=0$$

for all $(x,y)\in X\times X$. Hence $f(-x,-y)=-f(x,y)$ for all $(x,y)\in X\times X$.

Letting $x_1=-2x,x_2=0,x_3=x$ and $y_1=-2y,y_2=0,y_3=y$ in (4.1), we get

$$\|f(-2x,-2y)+f(0,0)+2f(x,y)\|\le\|2f(0,0)\|=0$$

for all $(x,y)\in X\times X$. Hence $f(2x,2y)=2f(x,y)$ for all $(x,y)\in X\times X$.

Replacing $x_3=-\frac{x_1+x_2}{2}$ and $y_3=-\frac{y_1+y_2}{2}$ in (4.1), we have

$$\begin{aligned}&\|f(x_1,y_1)+f(x_2,y_2)-f(x_1+x_2,y_1+y_2)\|\\&=\left\|f(x_1,y_1)+f(x_2,y_2)+2f\left(-\frac{x_1+x_2}{2},-\frac{y_1+y_2}{2}\right)\right\|\\&\le\|2f(0,0)\|=0\end{aligned}$$

for all $(x_1,y_1),(x_2,y_2)\in X\times X$. Thus

$$f(x_1,y_1)+f(x_2,y_2)=f(x_1+x_2,y_1+y_2)$$

for all $(x_1,y_1),(x_2,y_2)\in X\times X$, as desired. □

Theorem 4.2. *Assume that a mapping $f:X\times X\to Y$ satisfies the inequality*

$$\begin{aligned}\|f(x_1,y_1)+f(x_2,y_2)+2f(x_3,y_3)\le&\left\|2f\left(\frac{x_1+x_2}{2}+x_3,\frac{y_1+y_2}{2}+y_3\right)\right\|\\&+\phi((x_1,y_1),(x_2,y_2),(x_3,y_3))\end{aligned} \tag{4.2}$$

where $\phi:(X\times X)^3\to[0,\infty)$ satisfies

$$\widetilde{\phi}((x_1,y_1),(x_2,y_2),(x_3,y_3)):=\sum_{j=1}^{\infty}2^j\phi\left(\left(\frac{x_1}{2^j},\frac{y_1}{2^j}\right),\left(\frac{x_2}{2^j},\frac{y_2}{2^j}\right),\left(\frac{x_3}{2^j},\frac{y_3}{2^j}\right)\right)<\infty \tag{4.3}$$

for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times X$. *Then there exists a unique bi-Cauchy additive mapping* $A : X \times X \to Y$ *such that*

$$\begin{aligned}&\|A(x,y) - f(x,y)\| \\ &\leq \widetilde{\phi}\left((x,y),(0,0),\left(-\frac{1}{2}x,-\frac{1}{2}y\right)\right) + \widetilde{\phi}\left(\left(\frac{1}{2}x,\frac{1}{2}y\right),\left(\frac{1}{2}x,\frac{1}{2}y\right),\left(-\frac{1}{2}x,-\frac{1}{2}y\right)\right)\end{aligned}$$

for all $(x,y) \in X \times X$.

Proof. It follows from (4.3) that $\phi((0,0),(0,0),(0,0)) = 0$. Letting $x_1 = x_2 = x_3 = 0$ and $y_1 = y_2 = y_3 = 0$ in (4.2), we get $\|4f(0,0)\| \leq \|2f(0,0)\| + \phi((0,0),(0,0),(0,0)) = \|2f(0,0)\|$. So $f(0,0) = 0$.

Letting $x_1 = 2x, x_2 = 0, x_3 = -x$ and $y_1 = 2y, y_2 = 0, y_3 = -y$ in (4.2), we get

$$\|f(2x,2y) + f(0,0) + 2f(-x,-y)\| \leq \phi((2x,2y),(0,0),(-x,-y))$$

for all $(x,y) \in X \times X$.

Letting $x_1 = x, x_2 = x, x_3 = -x$ and $y_1 = y, y_2 = y, y_3 = -y$ in (4.2), we get

$$\|2f(x,y) + 2f(-x,-y)\| \leq \phi((x,y),(x,y),(-x,-y))$$

for any $(x,y) \in X \times X$.

Thus we get

$$\|f(2x,2y) - 2f(x,y)\| \leq \phi((2x,2y),(0,0),(-x,-y)) + \phi((x,y),(x,y),(-x,-y))$$

for all $(x,y) \in X \times X$. So

$$\begin{aligned}&\left\|f(x,y) - 2f\left(\frac{x}{2},\frac{y}{2}\right)\right\| \\ &\leq \phi\left((x,y),(0,0),\left(-\frac{1}{2}x,-\frac{1}{2}y\right)\right) + \phi\left(\left(\frac{1}{2}x,\frac{1}{2}y\right),\left(\frac{1}{2}x,\frac{1}{2}y\right),\left(-\frac{1}{2}x,-\frac{1}{2}y\right)\right)\end{aligned}$$

for all $(x,y) \in X \times X$.

The rest of the proof is the same as in the proof of Theorem 2.2. □

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Gang Lu
Department of Mathematics, ShenYang University of Technology, School of Science, Shenyang 110178, P.R. China
E-mail address: `lvgang1234@hanmail.net`

Choonkil Park
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea
E-mail address: `baak@hanyang.ac.kr`

Dong Yun Shin
Department of Mathematics, University of Seoul, Seoul 130-743, Korea
E-mail address: `dyshin@uos.ac.kr`

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Robust CVaR-based portfolio optimization under a genal affine data perturbation uncertainty set *

Zhifeng Dai[1,2] Fenghua Wen[2] †

[1]College of Mathematics and Computational Science, Changsha University of Science and Technology, Changsha 410114, China

[2]College of business, Central South University , Changsha 410004, China.

Abstract: Under a genal affine data perturbation uncertainty set, we propose a computationally tractable robust optimization method for minimizing the CVaR of a portfolio. Using L_1 norm, the robust counterpart problem can be a linear programming problem. Moreover, it is less conservative than the Quaranta and Zaffaroni's method which is under box uncertainty set. We present some numerical experiments with real market data to illustrate the behavior of robust optimization model.
Keywords: Conditional value at risk(CVaR), robust optimization, line programming(LP), second-order cone programming(SOCP).

1. Introduction

Portfolio selection optimization is one of the best known and most widely used methods in financial optimization. The mean-variance (MV) portfolio selection model, proposed by Markowitz [1], provides a fundamental basis for portfolio selection in both theoretical and practical applications.

Since the middle of 1990s, Value-at-Risk (VaR, [2]), a new measure of downside risk, has become popular in financial risk management. It has even been recommended as a standard on banking supervision by the Basel Committee. However, VaR has been criticized for its theoretical deficiency[3]. Conditional value at risk(CVaR), defined as the mean of the tail distribution exceeding VaR, has attracted much attention in recent years. CVaR is known to have nice properties such as the coherence [3, 4] and the consistency with the second-order stochastic dominance [5]. Also, Rockafellar and Uryasev [6, 7] show that the minimization of CVaR results in a tractable optimization problem.

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†The corresponding author: mailtowfh@126.com

However, as pointed out by Black and Litterman [8], in the classical mean-variance model, the portfolio decision is very sensitive to the mean and the covariance matrix, especially to the mean. Chopra and Ziemba [9] showed that small changes in the input parameters can result in large changes in the optimal portfolio allocation. Thus, the modeling risk arises due to the uncertainty of the underlying probability distribution. Being aware of the importance of robustness, researchers have paid increasing attention to the robust version of portfolio selection problems, for example, (Lobo and Boyd [14], Goldfarb and Iyengar [15], El Ghaoui, Oks and Oustry [16], Zhu and Fukushima [17]).

By utilizing the Soyster's approach [18], under the assumption that the expected returns lie in a box uncertainty set, Quaranta and Zaffaroni [10] considered robust optimization of conditional value at risk and portfolio selection problem. Although they succeeded in obtaining a linear robust copy of the bicriteria minimization model proposed by Rockafellar and Uryasev, the associated consequences are that the resulting robust portfolios can be too conservative. Under the assumption that the expected returns lie in an ellipsoidal uncertainty set, An and Luo (2010) [11] considered robust optimization of conditional value at risk and portfolio selection problem. They showed that the robust optimization problem can be reformulated as a second order cone programming (SOCP), however, a practical drawback of such an approach, is that it leads to nonlinear, although convex, models, which are more demanding computationally than the earlier linear models by Quaranta and Zaffaroni [10].

In robust portfolio selection problems, one try to find portfolios with the worst-case return under a given uncertainty set, in which asset returns can be realized. A too large uncertainty set will lead to a too conservative robust portfolio. However, if the given uncertainty set is not large enough, the realized returns of resulting portfolios will be outside of the uncertainty set when an extreme event such as market crash or a large shock of asset returns occurs. Motivated by the works in [20], under an affine data perturbation uncertainty set, we provide a computationally tractable robust optimization method for minimizing the CVaR of a portfolio which is less conservative than box uncertainty set. specifically the robust optimization problem retains its original structure,i.e., the robust counterpart problem is still a linear programming problem.

The rest of this paper is organized as follows: In the next section, we introduce the concept of CVaR and the mean-CVaR portfolio optimization model. In Section 3, we review the main ideas behind the robust optimization methodology, and present the computationally tractable robust optimization method for minimizing the CVaR of a portfolio. In Section 4, we report some numerical results to test the proposed methods.

2. Conditional value-at-risk measure

Conditional VaR (CVaR) is a popular example of such a coherent risk measure and is discussed in Rockafellar and Uryasev [6, 7]. The CVaR measure can be written as

$$CVaR_\beta(\boldsymbol{x}) = (1-\beta)^{-1}\int_{f(\boldsymbol{x},\boldsymbol{y})\geq VaR_\beta(\boldsymbol{x})} f(\boldsymbol{x},\boldsymbol{y})p(\boldsymbol{y})d\boldsymbol{y} \tag{2.1}$$

where

$$VaR_\beta(\boldsymbol{x}) = \min\{\alpha\in R : \Psi(\boldsymbol{x},\alpha)\geq\beta\} \tag{2.2}$$

and

$$\Psi(\boldsymbol{x},\alpha) = \int_{f(\boldsymbol{x},\boldsymbol{y})\leq\alpha} p(\boldsymbol{y})d\boldsymbol{y} \tag{2.3}$$

is the probability of $f(\boldsymbol{x},\boldsymbol{y})$ not exceeding a threshold α.

In practice, the probability density function $p(\boldsymbol{y})$ is often not available, or is very difficult to estimate. Instead, we might have T different scenarios $Y = (\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_T)$ that are sampled from the probability distribution or that have been obtained from computer simulations. Evaluating the auxiliary function $\tilde{F}_\beta(\boldsymbol{x},\alpha)$ using the scenarios Y, we have

$$\tilde{F}_\beta(\boldsymbol{x},\alpha) = \alpha + (1-\beta)^{-1}\sum_{j=1}^{T}\pi_j[f(\boldsymbol{x},\boldsymbol{y}_{[j]})-\alpha]^+ \tag{2.4}$$

where $\boldsymbol{y}_{[j]}$ denotes the jth sample (the subscript $[j]$ is used to distinguish a vector from a scalar) generated by simple random sampling with respect to $\mathbf{y}$ according to its density function $p(.)$, and T denotes the number of samples. where π_j are probabilities of scenarios $\boldsymbol{y}_{[j]}$. If π_j is equal to T^{-1} for all j, then (2.4) reduces to

$$\tilde{F}_\alpha(\boldsymbol{x},\alpha) = \alpha + \frac{1}{T(1-\beta)}\sum_{j=1}^{T}[f(\boldsymbol{x},\boldsymbol{y}_{[j]})-\alpha]^+ \tag{2.5}$$

Obviously, $\tilde{F}_\alpha(\boldsymbol{x},\alpha)$ is convex and piecewise linear with respect to α. Further, $\tilde{F}_\alpha(\boldsymbol{x},\alpha)$ is convex for $(\mathbf{x},\alpha)$, if $f(\boldsymbol{x},\boldsymbol{y})$ is convex (see Theorem 2 in [6]).

Replacing $[f(\boldsymbol{x},\boldsymbol{y}_{[j]})-\alpha]^+$ by the auxiliary variables z_j along with appropriate constraints, we obtain the equivalent optimization problem

$$\begin{array}{lll} \min\limits_{\boldsymbol{x},\alpha} & \alpha + \dfrac{1}{T(1-\beta)}\displaystyle\sum_{i=1}^{T} z_j & \\ s.t. & z_j \geq f(\boldsymbol{x},\boldsymbol{y}_{[j]})-\alpha, & j=1,\ldots,T \\ & z_j\geq 0 & \end{array} \tag{2.6}$$

Generally, the loss and return functions of portfolio allocation are chosen by:

$$f(\boldsymbol{x},\boldsymbol{y}) = -\boldsymbol{x}^T\boldsymbol{y}, \quad R_p(\boldsymbol{x}) = E_p[\boldsymbol{x}^T\boldsymbol{y}] = \boldsymbol{x}^T E_p[\boldsymbol{y}] = \boldsymbol{x}^T\mathbf{r} \tag{2.7}$$

in which $\mathbf{y}$ is the vector of the assets' return, $\boldsymbol{x}^T\boldsymbol{r}$ is the mean return of the portfolio.

2.1. portfolio optimization with CVaR measure

Portfolio optimization tries to find an optimal trade-off between the risk and the return according to the investor's preference. Thus, the portfolio selection problem using CVaR as a risk measure can be represented as

$$\min_{\boldsymbol{x} \in \mathcal{X}} \quad CVaR_\beta(\boldsymbol{x})$$

where $\mathcal{X}$ denotes the constraint on the portfolio position, which usually includes the budget constraint and no short sales constraint

$$\boldsymbol{x}^T \mathbf{1} = 1, \qquad \mathbf{x} \geq 0. \tag{2.8}$$

Let μ be the worst-case minimum mean return required by the investor. From (2.7), this can be represented as

$$\min \quad \boldsymbol{x}^T \boldsymbol{r} \geq \mu \tag{2.9}$$

Hence, adding an auxiliary variable $\theta \in R$, the mean-CVaR Portfolio optimization can be be written as the following linear program

$$\begin{aligned}
\min \quad & \theta \\
s.t. \quad & \alpha + \frac{1}{T(1-\beta)} \sum_{j=1}^{T} z_j \leq \theta && (2.10) \\
& z_j \geq -\boldsymbol{x}^T \boldsymbol{y}_{[j]} - \alpha, \qquad j = 1, \ldots, T && (2.11) \\
& z_j \geq 0 && (2.12) \\
& \boldsymbol{x}^T \boldsymbol{r} \geq \mu && (2.13) \\
& \boldsymbol{x}^T \mathbf{1} = 1, \qquad \mathbf{x} \geq 0. && (2.14)
\end{aligned}$$

The expected return $\boldsymbol{r}$ in constraint (2.9) is assumed to be exactly known. In fact, the expected assets' return $\boldsymbol{r}$ is uncertain. On the other hand, it is difficult to estimate the mean return vector, and the solution to the problem is sensitive to the mean return vector. One way to address this issue is to consider a robust version of the portfolio problem. More specifically, we propose the following robust version of the constraint on the expected portfolio return

$$\min \quad \boldsymbol{x}^T \boldsymbol{r} \geq \mu_w. \tag{2.15}$$

where μ_w denotes the worst-case required expected return specified by the investor. In next section, we will investigate a tractable robust formulation of the constraint on the expected return which belongs to a genal affine data perturbation uncertainty set.

3. Robust portfolio optimization

Robust optimization is emerging as a leading methodology to address optimization problems under uncertainty. In this section, we will discuss different robust optimization methods which can be applied to deal with the uncertain minimal constraint (2.15).

3.1. Genal Affine Data Perturbation uncertainty

In this paper, We consider instead the following uncertainty set, suggested by Chen et al. (2007) [20] in the context of stochastic programming applications:

$$V_\Omega = \Big\{ \boldsymbol{r} : \boldsymbol{r} = \boldsymbol{r}_0 + \sum_{j=1}^{N} \Delta \boldsymbol{r}_j z_j, \boldsymbol{z} \in A_\Omega(\boldsymbol{z}) \Big\}, \tag{3.1}$$

where

$$A_\Omega(\boldsymbol{z}) = \Big\{ \boldsymbol{z} : \exists \boldsymbol{v}, \boldsymbol{w} \in R_+^N, \boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}, \|\mathbf{P}^{-1}\boldsymbol{v} + \mathbf{Q}^{-1}\boldsymbol{w}\| \le \Omega \Big\}, \tag{3.2}$$

and $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_N)$, $\mathbf{Q} = \mathrm{diag}(q_1, \ldots, q_N)$. The parameters $p_j > 0$ and $q_j > 0$ are the "forward"and the "backward" deviations of random variable $z_j, j = 1, \ldots, N$, respectively.

For the stochastic linear constraint (2.15), the **worst-case** convex support of the uncertain parameter can be specified as follows,

$$W = \Big\{ \boldsymbol{r} : \exists z \in R^N, \boldsymbol{r} = \boldsymbol{r}^0 + \sum_{j=1}^{N} \Delta \boldsymbol{r}^j \tilde{z}_j, -\underline{\boldsymbol{z}} \le \boldsymbol{z} \le \overline{\boldsymbol{z}} \Big\} \tag{3.3}$$

Therefore, under affine data perturbation, the worse-case uncertainty set is a parallelotope in which the feasible solution is characterized by Soyster [18], which, of course, is a very conservative.

To derive a less conservative approximation, we need to choose the budget of uncertainty, Ω, appropriately. The natural uncertainty set to consider is the intersection of a norm uncertainty set, V_Ω and the worst-case support set, W as follows.

$$S_\Omega = \Big\{ \boldsymbol{r} : \exists z \in R^N, \boldsymbol{r} = \boldsymbol{r}^0 + \sum_{j=1}^{N} \Delta \boldsymbol{r}^j \tilde{z}_j, \boldsymbol{z} \in A_\Omega(\boldsymbol{z}), -\underline{\boldsymbol{z}} \le \boldsymbol{z} \le \overline{\boldsymbol{z}} \Big\} \tag{3.4}$$

As the budget of uncertainty Ω increases, the norm uncertainty set, V_Ω expands radially from the point $\boldsymbol{r}^0$ until it engulfs the set W. In this case, the uncertainty set $S_\Omega = W$. Hence, for any choice of Ω, the uncertainty set S_Ω is always less conservative than the worst-case uncertainty set W. We call the uncertainty S_Ω as genal affine data perturbation uncertainty.

We will show an equivalent formulation of the corresponding robust counterpart of (2.15) under the generalized uncertainty set, S_Ω. The dual norm $\|u\|^*$ is defined as

$$\|u\|^* = \max_{\{\|\boldsymbol{x}\| \le 1\}} \boldsymbol{u}'\boldsymbol{x} \tag{3.5}$$

Theorem 3.1 *The robust counterpart of (2.15) in which $U_\Omega = S_\Omega$ is equivalent to*

$$\begin{cases} \exists \boldsymbol{u}, \lambda, \boldsymbol{s} \in R^N, h \in R \\ -\boldsymbol{r}_0' \boldsymbol{x} + \Omega h + \lambda' \overline{z} + s' \underline{z} \le -\mu \\ \|u\|^* \le h, \\ u_j \ge -p_j(\Delta r_j' \boldsymbol{x} + \lambda_j - s_j), \forall j = \{1, \dots, N\} \\ u_j \ge q_j(\Delta r_j' \boldsymbol{x} + \lambda_j - s_j), \forall j = \{1, \dots, N\} \\ \boldsymbol{u}, \lambda, \boldsymbol{s} \ge 0. \end{cases} \tag{3.6}$$

Proof: From (2.15), we have

$$max \quad -\mathbf{x}^T \mathbf{r} \le -\mu \tag{3.7}$$

Under the condition $U_\Omega = S_\Omega$, the robust counterpart of (3.7) is as follows,

$$-\boldsymbol{r}_0^T \boldsymbol{x} + \max_{\{z \in C\}} z' \boldsymbol{y} \le -\mu \tag{3.8}$$

where

$$C = \left\{ (\boldsymbol{v}, \boldsymbol{w}) : \|\mathbf{P}^{-1} \boldsymbol{v} + \mathbf{Q}^{-1} \boldsymbol{w}\| \le \Omega, -\underline{\boldsymbol{z}} \le \boldsymbol{z} \le \overline{\boldsymbol{z}}, \boldsymbol{v}, \boldsymbol{w} \ge 0 \right\}$$

and $y_j = -\Delta r^{j'} \boldsymbol{x}$. Since C is a compact convex set with nonempty interior, we can use strong duality to obtain the equivalent representation. Observe that

$$\begin{aligned} & \max_{\{(\boldsymbol{v},\boldsymbol{w}):\|\mathbf{P}^{-1}\boldsymbol{v}+\mathbf{Q}^{-1}\boldsymbol{w}\|\le\Omega,-\underline{\boldsymbol{z}}\le\boldsymbol{z}\le\overline{\boldsymbol{z}},\boldsymbol{v},\boldsymbol{w}\ge0\}} (\boldsymbol{v}-\boldsymbol{w})'\boldsymbol{y} \\ = & \min_{\mathbf{r},\ \mathbf{s}\ge0} \left\{ \max_{\{(v,w):\|\mathbf{P}^{-1}v+\mathbf{Q}^{-1}w\|\le\Omega,v,w\ge0\}} (v-w)'y + \mathbf{r}'(\overline{z}-v+w) + \mathbf{s}'(\underline{z}+v-w) \right\} \\ = & \min_{\mathbf{r},\ \mathbf{s}\ge0} \left\{ \max_{\{(v,w):\|\mathbf{P}^{-1}v+\mathbf{Q}^{-1}w\|\le\Omega,v,w\ge0\}} (y-r+s)'v - (y-r+s)'w + \mathbf{r}'\overline{z} + \mathbf{s}'\underline{z} \right\} \\ = & \min_{\mathbf{r},\ \mathbf{s}\ge0} \left\{ \max_{\{(v,w):\|v+w\|\le\Omega,v,w\ge0\}} \mathbf{P}(y-r+s)'v - \mathbf{Q}(y-r+s)'w + \mathbf{r}'\overline{z} + \mathbf{s}'\underline{z} \right\} \\ = & \min_{\mathbf{r},\ \mathbf{s}\ge0} \Omega\|\mathbf{u}\|^* + \mathbf{r}'\overline{\boldsymbol{z}} + \mathbf{s}'\underline{\boldsymbol{z}} \end{aligned}$$

where

$$\begin{aligned} u_j &= max\{p_j(y_j - r_j + s_j), -q_j(y_j - r_j + s_j)\} \\ &= max\{-p_j(\Delta r^{j'} \boldsymbol{x} + r_j - s_j), q_j(\Delta r^{j'} \boldsymbol{x} + r_j - s_j)\}. \end{aligned}$$

Hence the robust counterpart is the same as

$$-\boldsymbol{r}_0^T \boldsymbol{x} + \Omega\|\mathbf{u}\|^* + \mathbf{r}'\overline{z} + \mathbf{s}'\underline{z} \le -\mu \tag{3.9}$$

Adding an auxiliary variable $h \in R$, we can easily obtain the equivalent formulation of (3.9), that is (3.6).

The complete formulation and complexity class of the robust counterpart depends on the representation of the dual norm constraint, $\|u\|^* \leq h$. In this paper, we select the l_1 norm. So the $\|u\|^* \leq h$ is equivalent to

$$u_j \leq h, \forall j \in N. \tag{3.10}$$

By (2.10)-(2.14) and (3.10), the robust portfolio selection problem can be written as the following linear programming problem with variables $(\mathbf{x}, \mathbf{z}, \mathbf{u}, \lambda, \mathbf{s}, \mathbf{v}, \theta, \alpha, h) \in R^n \times R^T \times R^N \times R^N \times R^N \times R^N \times R \times R \times R$:

$$\begin{array}{ll} \min & \theta \\ s.t. & \alpha + \dfrac{1}{T(1-\beta)} \displaystyle\sum_{i=1}^{J} z_j \leq \theta \\ & z_j \geq -\mathbf{x}^T \mathbf{y}_{[j]} - \alpha, \qquad j = 1, \ldots, T \\ & \mathbf{x}^T \mathbf{1} = 1, \qquad \mathbf{x} \geq 0 \\ & -\boldsymbol{r}_0' \boldsymbol{x} + \Omega h + \lambda' \overline{z} + s' \underline{z} \leq -\mu \\ & u_j \leq h, \forall j \in N \\ & u_j \geq -p_j(\Delta r_j' \boldsymbol{x} + \lambda_j - s_j), \forall j = \{1, \ldots, N\} \\ & u_j \geq q_j(\Delta r_j' \boldsymbol{x} + \lambda_j - s_j), \forall j = \{1, \ldots, N\} \\ & \mathbf{z}, \mathbf{u}, \lambda, \mathbf{s} \geq 0, v \in R_+^N, p \in R_+ \end{array} \tag{3.11}$$

4. Empirical Results

In this section, we apply the robust portfolio optimization methods discussed in the previous sections to real market data and compare the behavior of the solutions obtained by the robust optimization technique.

In all tables and figure, the methods have the following meanings:

- "CVaR" stands for the initial CVaR method in [7].
- "BCVAR" stands for the robust mean-CVaR Portfolio optimization under box uncertainty set in [10].
- "ECVaR" stands for the robust mean-CVaR Portfolio optimization under ellipsoidal uncertainty set in [11].
- "ACVaR" stands for the robust mean-CVaR Portfolio optimization (3.11) under a genal affine data perturbation uncertainty set.

We utilize MatLab7.0 for solving models CVaR, BCVAR, and ACVaR which are linear programming problems. The model ECVaR is an SOCP and solved by SeDuMi1.02 [21].

We consider a portfolio of 10 small cap stocks from 5 different industry categories of the S&P 600 index(Table 2), and use historical returns from May, 1998 to June, 2006. There are a total of 2,000 observations for each stock.

Table 2: List of Stocks and Corresponding Industries

Industry discretionary	Company name (ticker)
Consumer discretionary	Aztar Corp. (AZR), Hancock Fabrics Inc. (HKF)
Financials	Downey S & L Assn. (DSL), HARB
Industrials	AAR Corp. (AIR), CDI Corp. (CDI)
Information technology	FEI Company (FEIC), Exar Corp. (EXAR)
Healthcare	BioLase Technology (BLTI), BDR

In our first experiment, using the data presented above, we generated the classical and robust efficient frontiers. The parameters for all optimization models are set as follows:

- For the CVaR formulation, mean return r is given by the sample mean.
- For the BCVaR formulation, we assume that mean return r^0 is given by the sample mean, and that $\overline{r}_i$ is determined by the standard deviation of the stock i' sample return.
- According to the ECVaR formulation,we assume that mean return r^0 is given by the sample mean. For simplicity, the scaling matrix of the ellipsoid P is assumed to be a diagonal matrix ρI, where ρ is a nonnegative parameter.
- For the ACVaR formulation, we assume that mean return r^0 is given by the sample mean, and that Δr_i is determined by the standard deviation of the stock i' sample return, and assume they are also stochastically independent (N=10). We set $\Omega = 0.8$, $\overline{z} = \underline{z} = 1$ and $p_j = 1.5, q_j = 2$ in our Numerical experiments.

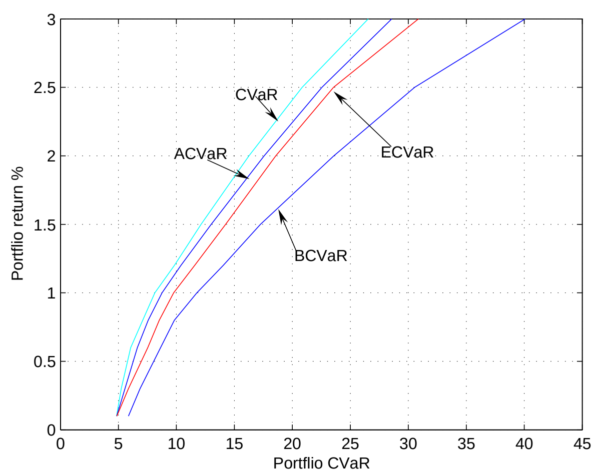

Figure 1-portfolio efficient frontiers for the different optimization formulations with $\beta = 1\%$.

As shown in Figure 1, it is apparent that ACVaR outperforms both ECVaR and BCVaR in terms of realized CVaR. As expected, CVaR is dominated by ACVaR. We also can see that the robust optimal portfolios are somewhat conservative in comparison to that of the CVaR model. But in the next experiment, we will see the robust optimal portfolios can result in more stable portfolio returns.

In our second experiment, we study the cumulative portfolio wealth if a portfolio manager employs a simple buy-and-hold strategy. The entire data sequence is divided into investment periods of length $T = 200$ days. In all there are $p = 10$ time periods. For each period p, first, we consider moving windows of $n = 10$ days and compute the parameters for all optimization models as experiment 1.

Once all the parameters are set, the portfolio $x^p_{CVaR}, x^p_{BCVaR}, x^p_{ECVaR}$, x^p_{ACVaR} for period p can be computed by solving the portfolio selection model C-VaR, BCVaR, ECVaR, and ACVaR respectively. The portfolio x^p_{CVaR}, x^p_{BCVaR}, x^p_{ECVaR}, x^p_{ACVaR} are held constant for the period p and then rebalanced to the portfolio $x^{p+1}_{CVaR}, x^{p+1}_{BCVaR}$, $x^{p+1}_{ECVaR}, x^{p+1}_{ACVaR}$ for period $p + 1$.

Let $W^p_{CVaR}, W^p_{BCVaR}, W^p_{ECVaR}, W^p_{ACVaR}$ denote the wealth at the end of period p of an investor with initial wealth $w_0 = 1$. Because these strategies require a block of data of length $T = 200$ to estimate all of parameters, the first investment period $p = 1$ starts from the time instant $T + 1$. Therefore, 10 time periods of length "$T = 200$" only have 9 investment periods.

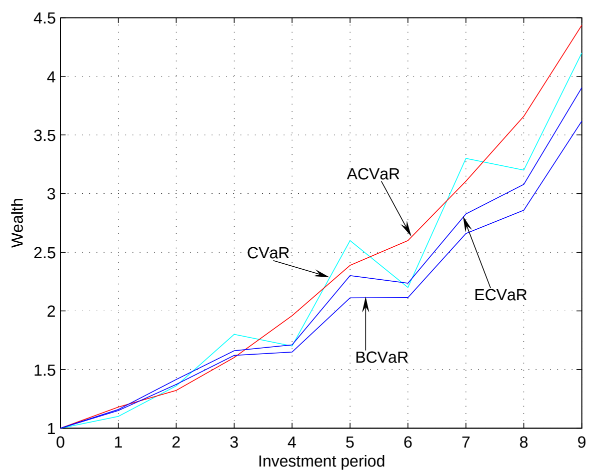

Figure 2-The wealth resulting from the four strategies with window n = 10 at each investment period.

It is clear that the wealth generated by the ACVaR model is much better than other models at the end of investment period. But in Figure 2 the wealth generated by the ACVaR model is a litter lower than other models at early investment periods. Therefore, it is not guaranteed that the ACVaR model always has an advantage of other three models. On the other hand, the optimal portfolio allocation based on the ACVaR approach tends to result in stable returns, whereas, for example, the behavior of the optimal portfolio obtained with the CVaR approach is erratic.

5. Conclusion

under a genal affine data perturbation uncertainty set, we propose a computationally tractable robust optimization method for minimizing the CVaR of a portfolio. The remarkable characteristic of the new method is that, using L_1 norm, the robust optimization model retains the complexity of original portfolio optimization problem, i.e., the robust counterpart problem is still a linear programming problem. This fact has important theoretical and practical implications. Since the computational complexity of an LP is simplest in all of program problems, it follows that robust portfolio optimization is able to provide protection against parameter fluctuations at light computational cost. Moreover, the LP problem is maybe the best known and the most frequently solved optimization problem in the real world. The numerical experiments presented in this paper suggest that the behavior of portfolios can be improved by using the robust CVaR model under a genal affine data perturbation uncertainty set. And the robustness is achieved at relatively high performance and low cost.

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RANDOM DERIVATIONS ON RANDOM NORMED ALGEBRAS

JUNG RYE LEE, CHOONKIL PARK, AND DONG YUN SHIN*

Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of random derivations in random normed algebras associated with the Cauchy functional equation.

1. Introduction

Fuzzy set theory is a powerful tool set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, e.g., population dynamics [4], chaos control [13], computer programming [15], etc. Recently, the fuzzy topology has proved to be a very useful tool to deal with such situations where the use of classical theories breaks down.

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [7, 25, 26, 31, 32]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1. ([31]) A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$ for all $a \in [0, 1]$;
(d) $T(a, b) \le T(c, d)$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t-norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz t-norm). Recall (see [16, 17]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0, 1]$, then $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \ge 2$. $T_{i=n}^\infty x_i$ is defined as

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*Corresponding author.

$T_{i=1}^{\infty} x_{n+i-1}$. It is known ([17]) that for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i-1} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty.$$

Definition 1.2. ([32]) A *random normed space* (briefly, RN-space) is a triple (X, μ, T_M), where X is a vector space and μ is a mapping from X into D^+ such that the following conditions hold:
(RN_1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN_2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \neq 0$;
(RN_3) $\mu_{x+y}(t + s) \geq T_M(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s > 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$. This space is called the induced random normed space.

Definition 1.3. Let (X, μ, T) be an RN-space.
(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.
(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.
(3) An RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4. ([31]) *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Definition 1.5. A *random normed algebra* is a random normed space with algebraic structure such that (RN_4) $\mu_{xy}(ts) \geq \mu_x(t)\mu_y(s)$ for all $x, y \in X$ and all $t, s > 0$.

Example 1.6. *Every normed algebra $(X, \|.\|)$ defines a random normed algebra (X, μ, T_M), where*

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$. This space is called the induced random normed algebra.

Definition 1.7. Let (X, μ, T_M) be a random normed algebra. An $\mathbb{R}$-linear mapping $f : X \to X$ is called a *random derivation* if $f(xy) = f(x)y + xf(y)$ for all $x, y \in X$.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
We recall a fundamental result in fixed point theory.

Theorem 1.8. ([6, 9]) *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

The stability problem of functional equations originated from a question of Ulam [33] concerning the stability of group homomorphisms. Hyers [18] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [30] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [30] has provided a lot of influence in the development of what we call *Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 3, 5, 8, 10, 11, 19, 21, 22, 23, 29]).

In 1996, G. Isac and Th.M. Rassias [20] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [25, 27, 28]).

The Hyers-Ulam stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [24, 26].

Using the fixed point method, we prove the Hyers-Ulam stability of random derivations in random normed algebras, associated with the Cauchy functional equation

$$f(x+y) = f(x) + f(y).$$

Throughout this paper, assume that (X, μ, T_M) is a complete random normed algebra.

2. Hyers-Ulam stability of random derivations in random normed algebras

Using the fixed point method, we prove the Hyers-Ulam stability of random derivations associated with the Cauchy functional equation.

Theorem 2.1. *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists a constant* $0 < L < \frac{1}{2}$ *with*

$$\varphi(x, y) \leq \frac{L}{2} \varphi(2x, 2y)$$

for all $x, y \in X$. *Let* $f : X \to X$ *be a mapping satisfying*

$$\mu_{f(rx+ry)-rf(x)-rf(y)}(t) \geq \frac{t}{t+\varphi(x,y)}, \tag{2.1}$$

$$\mu_{f(xy)-f(x)y-xf(y)}(t) \geq \frac{t}{t+\varphi(x,y)} \tag{2.2}$$

for all $r \in \mathbb{R}$, *all* $x, y \in X$ *and all* $t > 0$. *Then*

$$D(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for each $x \in X$ and defines a random derivation $D : X \to X$ such that

$$\mu_{f(x)-D(x)}(t) \geq \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x)} \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ and $r = 1$ in (2.1), we get

$$\mu_{f(2x)-2f(x)}(t) \geq \frac{t}{t + \varphi(x,x)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$. So

$$\mu_{f(x)-2f\left(\frac{x}{2}\right)}(t) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{2t}{2t + L\varphi(x,x)} \tag{2.5}$$

for all $x \in X$ and all $t > 0$.

Consider the set

$$S := \{g : X \to X\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\nu \in \mathbb{R}_+ : \mu_{g(x)-h(x)}(\nu t) \geq \frac{t}{t+\varphi(x,x)}, \ \forall x \in X, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see the proof of [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} \mu_{Jg(x)-Jh(x)}(L\varepsilon t) &= \mu_{2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right)}(L\varepsilon t) = \mu_{g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right)}\left(\frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,x)} = \frac{t}{t + \varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.5) that

$$\mu_{f(x)-2f\left(\frac{x}{2}\right)}\left(\frac{L}{2}t\right) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.8, there exists a mapping $D : X \to X$ satisfying the following:

(1) D is a fixed point of J, i.e.,

$$D\left(\frac{x}{2}\right) = \frac{1}{2}D(x) \tag{2.6}$$

for all $x \in X$. The mapping D is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that D is a unique mapping satisfying (2.6) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{f(x)-D(x)}(\nu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f, D) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = D(x)$$

for all $x \in X$;

(3) $d(f, D) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, D) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (2.3) holds.

By (2.1),

$$\mu_{2^n f\left(\frac{x}{2^n}+\frac{y}{2^n}\right)-2^n f\left(\frac{x}{2^n}\right)-2^n f\left(\frac{y}{2^n}\right)}\left(2^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\mu_{2^n f\left(\frac{x}{2^n}+\frac{y}{2^n}\right)-2^n f\left(\frac{x}{2^n}\right)-2^n f\left(\frac{y}{2^n}\right)}\left(t\right) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi\left(x, y\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\mu_{D(x+y)-D(x)-D(y)}\left(t\right) = 1$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $D : X \to X$ is Cauchy additive.

Let $y = 0$ in (2.1). By (2.1),

$$\mu_{2^n f\left(\frac{rx}{2^n}\right)-2^n r f\left(\frac{x}{2^n}\right)}\left(2^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, 0\right)}$$

for all $r \in \mathbb{R}$, all $x \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\mu_{2^n f\left(\frac{rx}{2^n}\right)-2^n r f\left(\frac{x}{2^n}\right)}\left(t\right) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi\left(x, 0\right)}$$

for all $r \in \mathbb{R}$, all $x \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,0)} = 1$ for all $x \in X$ and all $t > 0$,

$$\mu_{H(rx)-rH(x)}(t) = 1$$

for all $r \in \mathbb{R}$, all $x \in X$ and all $t > 0$. Thus the additive mapping $D : X \to X$ is $\mathbb{R}$-linear.

By (2.2),

$$\mu_{4^n f\left(\frac{x}{2^n}\cdot\frac{y}{2^n}\right)-2^n f\left(\frac{x}{2^n}\right)\cdot y - x\cdot 2^n f\left(\frac{y}{2^n}\right)}(4^n t) \geq \frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\mu_{4^n f\left(\frac{x}{2^n}\cdot\frac{y}{2^n}\right)-2^n f\left(\frac{x}{2^n}\right)\cdot y - x\cdot 2^n f\left(\frac{y}{2^n}\right)}(t) \geq \frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{2^n}\varphi(x,y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{2^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\mu_{D(xy)-D(x)y-xD(y)}(t) = 1$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $D : X \to X$ satisfies $D(xy) = D(x)y + xD(y)$ for all $x, y \in X$.

Therefore, there exists a unique random derivation $D : X \to X$ satisfying (2.3). $\square$

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists a constant $0 < L < 1$ with*

$$\varphi(x,y) \leq 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to X$ be a mapping satisfying (2.1) and (2.2). Then

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

exists for each $x \in X$ and defines a random derivation $D : X \to X$ such that

$$\mu_{f(x)-D(x)}(t) \geq \frac{(2-2L)t}{(2-2L)t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

It follows from (2.4) that

$$\mu_{f(x)-\frac{1}{2}f(2x)}\left(\frac{1}{2}t\right) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq \frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

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Jung Rye Lee
Department of Mathematics, Daejin University, Kyeonggi 487-711, South Korea
E-mail address: jrlee@daejin.ac.kr

Choonkil Park
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, South Korea
E-mail address: baak@hanyang.ac.kr

Dong Yun Shin
Department of Mathematics, University of Seoul, Seoul 130-743, South Korea
E-mail address: dyshin@uos.ac.kr

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A note on the q-extension of second kind Euler numbers and polynomials

C. S. Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper, by using the p-adic integral on $\mathbb{Z}_p$, we construct a new type of the q-extension of the second kind Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$. From these numbers and polynomials, we establish some interesting identities and relations. By using the q-extension of the second kind Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$, the q-Euler zeta function and Hurwitz-type q-Euler zeta functions are defined.

Key words : the second kind Euler numbers and polynomials, the q-extension of the second kind Euler numbers and polynomials

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80

1. Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$.Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \text{ cf. } [1, 2, 3, 4, 5, 6]\ .$$

For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

Kim[1, 2] defined the p-adic integral on $\mathbb{Z}_p$ as follows:

$$I_1(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0 \le x < p^N} g(x)(-1)^x. \tag{1.1}$$

From (1.1), we obtain

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} g(l), \text{ (see [1-3])}. \tag{1.2}$$

where $g_n(x) = g(x+n)$.

First, we introduce the second kind Euler numbers E_n and polynomials $E_n(x)$(see [4]). The second kind Euler numbers E_n are defined by the generating function:

$$F(t) = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.3}$$

We introduce the second kind Euler polynomials $E_n(x)$ as follows:

$$F(x,t) = \frac{2e^t}{e^{2t}+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.4}$$

2. q-extension of the second kind Euler numbers and polynomials

In this section, we introduce the q-extension of the second kind Euler Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ and investigate their properties. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

For $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, q-extension of the second kind Euler numbers $E_{n,q}$ are defined by

$$E_{n,q} = \int_{\mathbb{Z}_p} q^x [2x+1]_q^n d\mu_{-1}(x). \tag{2.1}$$

By using p-adic integral on $\mathbb{Z}_p$, we obtain,

$$\begin{aligned}\int_{\mathbb{Z}_p} q^x [2x+1]_q^n d\mu_{-1}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} q^x [2x+1]_q^n (-1)^x \\ &= 2\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l \frac{1}{1+q^{2l+1}} \\ &= 2\sum_{m=0}^{\infty} (-1)^m q^m [2m+1]_q^n.\end{aligned} \tag{2.2}$$

By (2.1), we have the following theorem.

Theorem 1. For $h \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, we have

$$\begin{aligned}E_{n,q} &= 2\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l q^l \frac{1}{1+q^{2l+1}} \\ &= 2\sum_{m=0}^{\infty} (-1)^m q^m [2m+1]_q^n.\end{aligned}$$

We set

$$F_q(t) = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}.$$

By using above equation and (2.2), we have

$$\begin{aligned}F_q(t) = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!} &= 2\sum_{n=0}^{\infty}\left(\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^l \frac{1}{1+q^{2l+1}}\right)\frac{t^n}{n!} \\ &= 2\sum_{m=0}^{\infty} (-1)^m q^m e^{[2m+1]_q t}.\end{aligned} \tag{2.3}$$

Thus, q-extension of the second kind Euler numbers, $E_{n,q}$ are defined by means of the generating function

$$F_q(t) = 2\sum_{m=0}^{\infty} (-1)^m q^m e^{[2m+1]_q t}. \tag{2.4}$$

By using (2.1), we have

$$\sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} q^x [2x+1]_q^n d\mu_{-1}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} q^x e^{[2x+1]_q t} d\mu_{-1}(x). \tag{2.5}$$

By (2.3), (2.5), we have

$$\int_{\mathbb{Z}_p} q^x e^{[2x+1]_q t} d\mu_{-1}(x) = 2\sum_{m=0}^{\infty}(-1)^m q^m e^{[2m+1]_q t}.$$

Next, we introduce q-extension of the second kind Euler polynomials $E_{n,q}(x)$. The q-extension of the second kind Euler polynomials $E_{n,q}(x)$ are defined by

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} q^y [x+2y+1]_q^n d\mu_{-1}(y). \tag{2.6}$$

By using p-adic integral, we obtain

$$E_{n,q}(x) = 2\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{(x+1)l}\frac{1}{1+q^{2l+1}}. \tag{2.7}$$

We set

$$F_q(t,x) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{2.8}$$

By using (2.7) and (2.8), we obtain

$$F_q(t,x) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!} = 2\sum_{m=0}^{\infty}(-1)^m q^m e^{[2m+1+x]_q t}. \tag{2.9}$$

Since $[x+2y+1]_q = [x]_q + q^x[2y+1]_q$, we easily see that

$$\begin{aligned} E_{n,q}(x) &= \int_{\mathbb{Z}_p} q^y [x+2y+1]_q^n d\mu_{-1}(y) \\ &= \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l} q^{xl} E_{l,q} \\ &= 2\sum_{m=0}^{\infty}(-1)^m q^m [x+2m+1]_q^n, \end{aligned} \tag{2.10}$$

with the usual convention of replacing $(E_q)^n$ by $E_{n,q}$.

By (1.3), (1.4), (2.3), and (2.10), we have the following remark.

Remark 1. Note that

$$\begin{aligned} &(1)\ E_{n,q}(0) = E_{n,q}, \\ &(2)\ \text{If } q \to 1, \text{ then } E_{n,q}(x) = E_n(x), \quad E_{n,q} = E_n, \\ &(3)\ \text{If } q \to 1, \text{ then } F_q(x,t) = F(x,t), \quad F_q(t) = F(t). \end{aligned}$$

By (2.7), we obtain the following theorem.

Theorem 2(Property of complement).

$$E_{n,q^{-1}}(-x) = (-1)^n q^{n+1} E_{n,q}(x)$$

By (2.7), we have the following distribution relation:

Theorem 3. For any positive integer m(=odd), we have

$$E_{n,q}(x) = [m]_q^n \sum_{a=0}^{m-1}(-1)^a q^a E_{n,q^m}\left(\frac{2a+x+1-m}{m}\right), n \in \mathbb{Z}_+.$$

By (1.2), (2.1), and (2.6), we easily see that

$$q^n E_{m,q}(2n) + (-1)^{n-1} E_{m,q} = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l [2l+1]_q^m.$$

Hence, we obtain the following theorem.

Theorem 4. Let $m \in \mathbb{Z}_+$. If $n \equiv 0 \pmod 2$, then

$$q^n E_{m,q}(2n) - E_{m,q} = 2 \sum_{l=0}^{n-1} (-1)^{l+1} q^l [2l+1]_q^m.$$

If $n \equiv 1 \pmod 2$, then

$$q^n E_{m,q}(2n) + E_{m,q} = 2 \sum_{l=0}^{n-1} (-1)^{l} q^l [2l+1]_q^m.$$

From (1.2), we note that

$$\begin{aligned} 2e^t &= q \int_{\mathbb{Z}_p} e^{[2x+3]_q t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} e^{[2x+1]_q t} d\mu_{-1}(x) \\ &= \sum_{n=0}^{\infty} \left(q \int_{\mathbb{Z}_p} [2x+3]_q^n d\mu_{-1}(x) + \int_{\mathbb{Z}_p} [2x+1]_q^n d\mu_{-1}(x) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(qE_{n,q}(2) + E_{n,q} \right) \frac{t^n}{n!}. \end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 5. For $n \in \mathbb{Z}_+$, we have

$$qE_{n,q}(2) + E_{n,q} = 2.$$

By Theorem 5 and (2.10), we have the following corollary.

Corollary 6. For $n \in \mathbb{Z}_+$, we have

$$q(q^2 E_q + [2]_q)^n + E_{n,q} = 2,$$

with the usual convention of replacing $(E_q)^n$ by $E_{n,q}$.

3. The analogue of the Euler zeta function

By using q-extension of second kind Euler numbers and polynomials, q-Euler zeta function and Hurwitz q-Euler zeta functions are defined. These functions interpolate the q-extension of second kind Euler numbers $E_{n,q}$, and polynomials $E_{n,q}(x)$, respectively. Let q be a complex number with $|q| < 1$. From (2.4), we note that

$$\begin{aligned} \left. \frac{d^k}{dt^k} F_q(t) \right|_{t=0} &= 2 \sum_{m=0}^{\infty} (-1)^n q^m [2m+1]_q^k \\ &= E_{k,q}, (k \in \mathbb{N}). \end{aligned}$$

By using the above equation, we are now ready to define q-Euler zeta functions.

Definition 7. Let $s \in \mathbb{C}$ with $Res > 1$.

$$\zeta_q(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[2n+1]_q^s}. \tag{3.1}$$

Note that $\zeta_q(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $q \to 1$, then $\zeta_q(s) = \zeta(s)$ which is the Euler zeta functions(see [6]). Relation between $\zeta_q(s)$ and $E_{k,q}$ is given by the following theorem.

Theorem 8. For $k \in \mathbb{N}$, we have

$$\zeta_q(-k) = E_{k,q}.$$

Observe that $\zeta_q(s)$ function interpolates $E_{k,q}$ numbers at non-negative integers. By using (2.9), we note that

$$\left.\frac{d^k}{dt^k} F_q(t,x)\right|_{t=0} = 2\sum_{m=0}^{\infty} (-1)^m q^m [2x+1+m]_q^k \tag{3.2}$$

and

$$\left(\frac{d}{dt}\right)^k \left.\left(\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}\right)\right|_{t=0} = E_{k,q}(x), \text{ for } k \in \mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), we are now ready to define the Hurwitz q-Euler zeta functions.

Definition 9. Let $s \in \mathbb{C}$ with $Res > 1$.

$$\zeta_q(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n q^n}{[n+2x+1]_q^s}. \tag{3.4}$$

Note that $\zeta_q(s,x)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $q \to 1$, then $\zeta_q(s,x) = \zeta(s,x)$ which is the Hurwitz Euler zeta functions(see [6]). Relation between $\zeta_q(s,x)$ and $E_{k,q}(x)$ is given by the following theorem.

Theorem 10. For $k \in \mathbb{N}$, we have

$$\zeta_q(-k,x) = E_{k,q}(x).$$

Observe that $\zeta_q(-k,x)$ function interpolates $E_{k,q}(x)$ numbers at non-negative integers.

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SOME PROPERTIES OF BAZILEVIC FUNCTIONS RELATED WITH CONIC DOMAINS

KHALIDA INAYAT NOOR, MOHSAN RAZA* AND KAMRAN YOUSAF

Abstract. The aim of this paper is to study the Bazilevic functions associated with conic domains. Some properties of analytic functions related with Bazilevic functions by using the concept of convolution are examined. We investigate some results concerned with integral preserving property and radius problems which generalize the already proved results.

1. Introduction

Let A be the class of analytic functions

$$F(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

defined in the open unit disc $E = \{z : |z| < 1\}$. For any two analytic functions f and g with

$$f(z) = \sum_{n=0}^{\infty} b_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} c_n z^n, \ z \in E,$$

the convolution (Hadamard product) is given by

$$(f * g)(z) = \sum_{n=0}^{\infty} b_n c_n z^n, \ z \in E.$$

A function $f \in A$ is starlike univalent function of order ρ, if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \rho, \quad 0 \le \rho < 1, \ z \in E.$$

This class of functions is denoted by $S^*(\rho)$. Kanas and Wisnowska [7] studied $k - UCV$, the class of k-uniformly convex and $k - ST$, the corresponding class of k-starlike functions. A function $f \in A$ is said to be in the class $k - UCV$ of k-uniformly convex function, if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \ge k \left|\frac{zf''(z)}{f'(z)}\right|, \quad k \ge 0, \quad z \in E. \tag{1.2}$$

Similarly a function $f \in A$ is said to be in the class denoted by $k - ST$, if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \ge k \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad k \ge 0, \quad z \in E. \tag{1.3}$$

* Corresponding author

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Geometric interpretation. The function $f \in k-UCV$ and $f \in k-ST$, if and only if $\frac{zf''(z)}{f'(z)}+1$ and $\frac{zf'(z)}{f(z)}$, respectively, take all values in the conic domain Ω_k, which is included in the right half plane such that

$$\Omega_k = \{u+iv : u > k\sqrt{(u-1)^2+v^2}\},$$

with $p(z) = \frac{zf''(z)}{f'(z)}+1$ or $p(z) = \frac{zf'(z)}{f(z)}$ and considering the functions which map E onto the conic domain Ω_k such that $1 \in \Omega_k$, we may rewrite the conditions (1.2) or (1.3) in the form

$$p(z) \prec q_k(z).$$

The domain $\Omega_{k,\rho}$ is such that

$$\Omega_{k,\rho} = (1-\rho)\,\Omega_k + \rho, \quad 0 \leq \rho < 1.$$

The function $q_{k,\rho}$ plays the role of extremal for these classes and is given by

$$q_{k,\rho}(z) = \begin{cases} \frac{1+(1-\rho)z}{1-z}, \quad k=0, \\ 1+\frac{2\gamma(1-\rho)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad k=1, \\ 1+\frac{2(1-\rho)}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right], \quad 0<k<1, \\ 1+\frac{(1-\rho)}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right)+\frac{(1-\rho)}{k^2-1}, \quad k>1, \end{cases} \tag{1.4}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}},\ t \in (0,1),\ z \in E$ and t is chosen such that $k = cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, with $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. By virtue of (1.4) and the properties of the domains $\Omega_{k,\rho}$, we have $p \prec q_{k,\rho}$ implies

$$\operatorname{Re} p(z) > \operatorname{Re} q_{k,\rho}(z) > \frac{k+\rho}{k+1}.$$

A function p, analytic in E with $p(0)=1$, is said to be in the class $k-P(\rho) \subset P$, if it is subordinate to $q_{k,\rho}$ in E. That is $p \in k-P(\rho)$, if and only if $p \prec q_{k,\rho}$, where $q_{k,\rho}$ is given by (1.4) and $p(E) \subset q_{k,\rho}(E)$.

It is noted that $0-P(0) = P$, the class of analytic functions with positive real part and $p \in 0-P(\rho) = P(\rho)$ implies that $Rep(z) > \rho,\ z \in E$.

Recently Noor [11] has extended the class $k-P(\rho)$ and defined the following subclass of caratheodory class P.

Definition 1.1. Let p be analytic in E with $p(0)=1$. Then $p \in k-P_m(\rho)$, if and only if

$$p(z) = \left(\frac{m}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)p_2(z), \quad p_1(z),\ p_2(z) \in k-P(\rho),$$

for $m \geq 2,\ 0 \leq \rho < 1,\ k \in [0,\infty),\ z \in E$. We note that $k-P_2(\rho) = k-P(\rho)$ and $0-P_m(0) = P_m$, the well-known class defined in [12].

Definition 1.2. [11] Let $f \in A$. Then $f \in k - UR_m(\rho),\ 0 \leq \rho < 1,\ k \in [0, \infty)$ and $m \geq 2$, if and only if

$$\frac{zf'(z)}{f(z)} \in k - P_m(\rho),\ z \in E.$$

Noor called $k - UR_m(\rho)$, the class of functions of k-uniform bounded boundary rotation m with order ρ. It can easily be seen that $0 - UR_m(0) = R_m$, the class of functions of bounded boundary rotation. It is also noted that $1 - UR_2(0) = UST$, the class of unifromly starlike functions.

Now using the concepts of class $k - P_m$ and the class of uniformly starlike functions, we define the following:

Definition 1.3. Let $F \in A,\ \alpha, \beta \in \mathbb{R},\ \alpha > 0,\ f \in UST$. Then $F \in k - UB_m(\alpha, \beta)$, if and only if

$$\left\{ \frac{zF'(z)\, F^{\alpha + i\beta - 1}(z)}{z^{i\beta} f^{\alpha}(z)} \right\} \in k - P_m, \quad z \in E. \tag{1.5}$$

Remark 1.4. From (1.5) it can easily be seen that $F \in k - UB_m(\alpha, \beta)$ can be represented by the following integral representation

$$F(z) = \left[(\alpha + i\beta) \int_0^z h(t) f^{\alpha}(t) t^{i\beta - 1} dt \right]^{\frac{1}{\alpha + i\beta}} \quad h \in k - P_m,\ \ f \in UST,\ z \in E. \tag{1.6}$$

We note that, with $m = 2,\ k = 0$, the class $k - UB_m(\alpha, \beta)$ reduces to the class of Bazilevic functions introduced in [3], where he showed that a Bazilevic function is univalent in E and has the integral representation given by (1.6).

For recent work of the above mentioned classes, we refer [1,2,6,9,13].

We need the following lemmas which will be used in our main results.

2. Preliminary Results

Lemma 2.1. *Let $g \in UST$. Then $z\left(\frac{g(z)}{z}\right)^{\alpha}$, where $\alpha > 0$ also belongs to UST in E.*

Proof. Let

$$G_1(z) = z\left(\frac{g(z)}{z}\right)^{\alpha}.$$

Taking logarithmic differentiation of both sides we have

$$\begin{aligned} \frac{zG_1'(z)}{G_1(z)} &= \alpha \frac{zg'(z)}{g(z)} + (1 - \alpha) \\ &= \alpha h_0(z) + (1 - \alpha). \end{aligned}$$

Since $h_0 \in 1 - P$, $p_0(z) = 1 \in 1 - P$ and $1 - P$ is convex set, see [11], therefore $G_1(z) \in 1 - P$. ∎

Remark 2.2. This result can easily be extended to the class $k - UR_m$ using the fact that $k - P_m$ is convex set, see [11].

Lemma 2.3. *[15] Let $f \in C$ and $g \in S^*$. Then for any analytic function F with $F(0) = 1$ in E*

$$\frac{f * Fg}{f * g}(E) \subset \overline{co}F(E),$$

where $\overline{co}F(E)$ denotes the convex hull of $F(E)$ (the smallest convex set which contains $F(E)$).

Lemma 2.4. *[10] Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\psi(u, v)$ be a complex valued function satisfying the conditions:*
(**i**) *$\psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$,*
(**ii**) *$(1, 0) \in D$ and $Re\psi(1, 0) > 0$,*
(**iii**) *$Re\psi(iu_2, v_1) \leq 0$, whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.*
If $h(z) = 1 + c_1 z + \cdots$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $Re\psi(h(z), zh'(z)) > 0$ for $z \in E$, then $Reh(z) > 0$ in E.

3. Main Results

Theorem 3.1. *Let $\alpha \in \mathbb{R}, \alpha > 0$ and $c \in \mathbb{C}$, $\operatorname{Re} c \geq 0$ and let $f \in UST$. Then*

$$g(z) = \left[(c+1)\, z^{-c} \int_0^z t^{c-1} f^{\alpha}(t)dt\right]^{\frac{1}{\alpha}} \in UST, \ z \in E. \tag{3.1}$$

Proof. We can write (3.1) by using convolution as

$$g(z) = z\left[\left(\frac{f(z)}{z}\right)^{\alpha} * \frac{\phi_{\alpha+c}(z)}{z}\right]^{\frac{1}{\alpha}}, \tag{3.2}$$

where $\phi_{\alpha+c}(z) = \sum_{n=1}^{\infty} \frac{\alpha+c}{\alpha+c+n-1} z^n$ is convex in E, see [14]. Now from (3.2), we get

$$\frac{zg'(z)}{g(z)} = \frac{\phi_{\alpha+c}(z) * z\left(\frac{f(z)}{z}\right)^{\alpha} \frac{zf'(z)}{f(z)}}{\phi_{\alpha+c}(z) * z\left(\frac{f(z)}{z}\right)^{\alpha}}.$$

Since, by Lemma 2.1, $z\left(\frac{f(z)}{z}\right)^{\alpha} \in UST \subset S^*\ \frac{1}{2}) \subset S^*$, $\phi_{\alpha+c}(z)$ is convex, it follows from Lemma 2.3 that

$$\frac{\phi_{\alpha+c}(z) * z\left(\frac{f(z)}{z}\right)^{\alpha} H(z)}{\phi_{\alpha+c}(z) * z\left(\frac{f(z)}{z}\right)^{\alpha}}(E) \subset \overline{co}H(E), \quad H(z) = \frac{zf'(z)}{f(z)}.$$

This proves that $\frac{zg'(z)}{g(z)} \in 1 - P$ and thus $g \in UST$. ∎

Theorem 3.2. *Let $F \in k - UB_m(\alpha, \beta)$, $\phi \in C$. Then*

$$G(z) = z\left[\left(\frac{F(z)}{z}\right)^{\alpha+i\beta} * \frac{\phi(z)}{z}\right]^{\frac{1}{\alpha+i\beta}} \in k - UB_m(\alpha, \beta) \tag{3.3}$$

in E.

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Proof. Since $F \in k - UB_m(\alpha, \beta)$, there exists $f \in UST$ such that

$$\left\{\frac{zF'(z)\, F^{\alpha+i\beta-1}(z)}{z^{i\beta} f^{\alpha}(z)}\right\} \in k - P_m,\ z \in E. \tag{3.4}$$

Define

$$g(z) = z\left[\left(\frac{f(z)}{z}\right)^{\alpha} * \frac{\phi(z)}{z}\right]^{\frac{1}{\alpha}}. \tag{3.5}$$

Then $g \in UST$ by Theorem 3.1. Therefore from (3.3), (3.4) and (3.5), it follows that

$$\begin{aligned} \frac{zG'(z)G^{\alpha+i\beta-1}(z)}{z^{i\beta}g^{\alpha}(z)} &= \frac{\phi(z) * z\left(\frac{f(z)}{z}\right)^{\alpha} \frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)}}{\phi(z) * z\left(\frac{f(z)}{z}\right)^{\alpha}} \\ &= \frac{\phi(z) * f_1(z)H_0(z)}{\phi(z) * f_1(z)}, \end{aligned}$$

where $H_0 = \frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)} \in k - P_m$, $f_1(z) = z\left(\frac{f(z)}{z}\right)^{\alpha} \in S^*$. Now using Lemma 2.3, we obtain the desired result. ∎

Applications of Theorem 2.3

The class $k-UB_m(\alpha, \beta)$ is invariant under the following integral representation

$$F_1(z) = \left[(\alpha + i\beta + c)z^{-c}\int_0^z t^{c-1}F^{\alpha+i\beta}(t)dt\right]^{\frac{1}{\alpha+i\beta}},$$

where $\operatorname{Re} c \geq 0$ and $F(z) \in k - UB_m(\alpha, \beta)$. In fact we can write

$$F_1(z) = z\left[\left(\frac{F(z)}{z}\right)^{\alpha+i\beta} * \frac{\phi_{\alpha+i\beta+c}(z)}{z}\right]^{\frac{1}{\alpha+i\beta}},$$

and since $\phi_{\alpha+i\beta+c}$ is convex in E, the result is immediate from Theorem 3.2.

We note the following special cases.

(i) For $\beta = 0,\ \alpha = 1$, we have

$$F_1(z) = (1 + c)z^{-c}\int_0^z t^{c-1}F(t)dt, \quad \operatorname{Re} c \geq 0, \tag{3.6}$$

and this is the generalized Bernadi integral operator [4]. When $k = 0, m = 2$, we have the result for the class K of close-to-convex functions [4].

(ii) In (3.6), by taking $c = 1$, we obtain Libera operator [8] and $c = 0$ leads us to the well-known Alexander operator.

(iii) When $\beta = c = 0$ implies that

$$[F_1(z)]^{\alpha} = \alpha\int_0^z \left(\frac{F(t)}{t}\right)^{\alpha} dt, \quad \alpha > 0, \tag{3.7}$$

a generalized form of Alexander operator.

With $m = 2,\ k = 0$, we note that the class of Bazilevic functions of type α (see, [16]) is invariant under the integral operator defined in (3.6).

Class $B_m(\alpha, \beta, \gamma)$

We now deal the case $k = 0$, when $F \in A,\ \alpha,\ \beta \in \mathbb{R},\ \alpha > 0,\ f \in S^*$. Then $F \in B_m(\alpha, \beta, \gamma)$, if and only if

$$\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)} \in P_m(\gamma),\ 0 \le \gamma < 1.$$

Theorem 3.3. *Let $F \in B_m(\alpha, \beta, \gamma)$ and set*

$$G(z) = \left[(\alpha + i\beta + c)z^{-c}\int_0^z t^{c-1}F^{\alpha+i\beta}(t)dt\right]^{\frac{1}{\alpha+i\beta}},\ c \ge 0. \tag{3.8}$$

Then $G \in B_m(\alpha, \beta, \gamma_1)$, where

$$\gamma_1 = \frac{2\gamma C_1 + c + \alpha h_1}{2C_1 + c + \alpha h_1},\ h_1 = \operatorname{Re}\frac{zg'(z)}{g(z)},\ C_1 = \left|\alpha\frac{zg'(z)}{g(z)} + c + i\beta\right|^2,$$

and g is integral representation of $f \in S^*$ and is starlike.

Proof. Since $F \in B_m(\alpha, \beta, \gamma)$ so there exists $f \in S^*$ such that

$$\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)} \in P_m(\gamma),\ z \in E. \tag{3.9}$$

Set

$$\frac{zG'(z)G^{\alpha+i\beta-1}(z)}{z^{i\beta}g^{\alpha}(z)} = h(z), \tag{3.10}$$

where

$$g(z) = \left[\frac{\alpha + i\beta + c}{z^{c+i\beta}}\int_0^z t^{c+i\beta-1}f^{\alpha}(t)dt\right]^{\frac{1}{\alpha}} \in S^* \tag{3.11}$$

by Theorem 3.1. Since $g \in S^*$ we set $\frac{zg'(z)}{g(z)} = h_0(z) = h_1 + ih_2,\ h_0 \in P$ in E. Now from (3.8) – (3.11) , we obtain after some computations

$$\left\{h(z) + \frac{zh'(z)}{\alpha h_0(z) + c + i\beta}\right\} \in P_m(\gamma). \tag{3.12}$$

Writing

$$h(z) = (1 - \gamma_1)p(z) + \gamma_1.$$

It follows from (3.12) that for $i = 1, 2$,

$$\left\{(1 - \gamma_1)\, p_i(z) + \frac{(1 - \gamma_1)\, zp_i'(z)}{\alpha h_0(z) + c + i\beta} + \gamma_1 - \gamma\right\} \in P,\ z \in E.$$

We construct the functional $\psi(u, v)$ by taking $u = p_i(z),\ v = zp_i'(z)$ as follows:

$$(u, v) = (1 - \gamma_1)u + \frac{(1 - \gamma_1)v}{\alpha h_0(z) + c + i\beta} + (\gamma_1 - \gamma).$$

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The first two conditions of Lemma 2.4 can easily be verified. We check condition (iii) as below:

$$\begin{aligned}
\operatorname{Re}\psi(iu_2, v_1) &= (\gamma_1-\gamma)+\frac{(1-\gamma_1)(c+\alpha h_1)v}{|\alpha h_0(z)+c+i\beta|^2} \\
&\le (\gamma_1-\gamma)-\frac{(1-\gamma_1)(c+\alpha h_1)(1+u_2^2)}{2\,|\alpha h_0(z)+c+i\beta|^2} \\
&= \frac{2(\gamma_1-\gamma)C_1-(1-\gamma_1)(c+\alpha h_1)(1+u_2^2)}{2C_1} \\
&= \frac{2(\gamma_1-\gamma)C_1-(1-\gamma_1)(c+\alpha h_1)+(\gamma_1-1)(c+\alpha h_1)u_2^2}{2C_1} \\
&= \frac{A+Bu_2^2}{2C_1}, \qquad (3.13)
\end{aligned}$$

where $A=2(\gamma_1-\gamma)C_1-(1-\gamma_1)(c+\alpha h_1)$ and $B=(\gamma_1-1)(c+\alpha h_1)$. Since $C_1=|\alpha h_0(z)+c+i\beta|^2>0$ and the right hand side of (3.13) is less than or equal to zero, if $A\le 0$ and $B\le 0$. Now from $B\le 0$, we have $\gamma_1<1$ and from $A\le 0$, we obtain the value of γ_1 given by

$$\gamma_1=\frac{2\gamma C_1+c+\alpha h_1}{2C_1+c+\alpha h_1}.$$

Thus all the conditions of Lemma 2.4 are satisfied and $p_i\in P$ which mean $h_i\in P(\gamma_1)$ and hence $h\in P_m(\gamma_1)$. This proves our result. ∎

Theorem 3.4. *Let $G\in B_2(\alpha,\beta,0)$, where*

$$G(z)=\left[(\alpha+i\beta+c)z^{-c}\int_0^z t^{c-1}F^{\alpha+i\beta}(t)dt\right]^{\frac{1}{\alpha+i\beta}}, \quad c\ge 0.$$

Then $F\in B_2(\alpha,\beta,0)$, for $|z|<r_0$ and

$$r_0=\begin{cases} \frac{-(\alpha+1)+\sqrt{c^2+2\alpha+1}}{c-\alpha}, & c>\alpha, \\ \frac{1}{2}, & c=\alpha=1. \end{cases} \qquad (3.14)$$

Proof. Since $G\in B_2(\alpha,\beta,0)$, so there exists $g\in S^*$ such that

$$\frac{zG'(z)G^{\alpha+i\beta-1}(z)}{z^{i\beta}g^{\alpha}(z)}=h(z),$$

where

$$g(z)=\left[\frac{\alpha+i\beta+c}{z^{c+i\beta}}\int_0^z t^{c+i\beta-1}f^{\alpha}(t)dt\right]^{\frac{1}{\alpha}}\in S^*$$

by Theorem 3.1. Since $g\in S^*$ we set $\frac{zg'(z)}{g(z)}=h_0(z)\in P$ in E. Now from (3.8), (3.10) and (3.11), we obtain after some computation

$$\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)}=h(z)+\frac{zh'(z)}{\alpha h_0(z)+c+i\beta}.$$

This implies that

$$\begin{aligned}\operatorname{Re}\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)} &= \operatorname{Re}\left\{h(z)+\frac{zh'(z)}{\alpha h_0(z)+c+i\beta}\right\}\\ &\geq \operatorname{Re}h(z)-\left|\frac{zh'(z)}{\alpha h_0(z)+c+i\beta}\right|.\end{aligned}$$

Using the well-known distortion results for class P, we have

$$\operatorname{Re}\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)} \geq \operatorname{Re}h(z)\left\{1-\frac{2r}{1-r^2}\left|\frac{1}{\alpha h_0(z)+c+i\beta}\right|\right\}.$$

Since $h_0 \in P$, we have

$$\begin{aligned}|\alpha h_0(z)+c+i\beta| &\geq \operatorname{Re}\{\alpha h_0(z)+c+i\beta\}\\ &\geq \alpha\left(\frac{1-r}{1+r}\right)+c\\ &= \frac{\alpha(1-r)+c(1+r)}{1+r}.\end{aligned}$$

It follows easily that

$$\begin{aligned}\operatorname{Re}\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)} &\geq \operatorname{Re}h(z)\left\{1-\frac{2r}{(1-r)\{\alpha(1-r)+c(1+r)\}}\right\}\\ &= \operatorname{Re}h(z)\left\{\frac{(\alpha+c)-2(\alpha+1)r+(\alpha-c)r^2}{(1-r)\{\alpha(1-r)+c(1+r)\}}\right\}.\end{aligned}$$

Hence $F \in B_2(\alpha,\beta,0)$, for $|z|<r_0$, where r_0 is given in (3.14). This completes the proof. ∎

For $\alpha = 1$ and $\beta = 0$, we have the result proved by Bernardi [5] for Bernardi operator.

Corollary 3.5. *Let $G \in K$, where*

$$G(z) = (1+c)\,z^{-c}\int_0^z t^{c-1}F(t)dt.$$

Then $F \in K$, for $|z| < r_0$ and

$$r_0 = \begin{cases}\frac{-2+\sqrt{c^2+3}}{c-1}, & c>1,\\ \frac{1}{2}, & c=1.\end{cases}$$

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KHALIDA INAYAT NOOR
Department of Mathematics, COMSATS Institute of Information Technology, Islamabad Pakistan.
E-mail address: khalidanoor@hotmail.com

MOHSAN RAZA
Department of Mathematics, GC University Faisalabad, Pakistan.
E-mail address: mohsan976@yahoo.com

KAMRAN YOUSAF
Department of Mathematics, COMSATS Institute of Information Technology, Islamabad Pakistan.
E-mail address: kamranmaths@gmail.com.

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Some characterizations in some Möbius invariant spaces

A. El-Sayed Ahmed
Sohag University, Faculty of Science, Department of Mathematics,
Sohag 82524, Egypt and
Taif University, Faculty of Science, Mathematics Department
Box 888 El-Hawiyah, El-Taif 5700, Saudi Arabia and
e-mail: ahsayed80@hotmail.com
A. Kamal
Majmaah University, Faculty of Science and Humanities,
Mathematics Department
Box 66 Ghat 11914, Saudi Arabia
e-mail: alaa_ mohamed1@yahoo.com
Aydah Ahmadi
King Khalid University, Faculty for girls, Math.Dept, KSA

Abstract. We give two characterizations of the Möbius invariant $Q_K(p,q)$ spaces, one in terms of a double integral and the other in terms of the mean oscillation in the Bergman metric. Both characterizations avoid the use of derivatives.

1 Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane $\mathbb{C}$. The Green's function in the unit disk Δ with singularity at $a \in \Delta$ is given by $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. For $0 < r < 1$, let $\Delta(a,r) = \{z \in \Delta : |\varphi_a(z)| < r\}$ be the pseudo-hyperbolic disk with the center $a \in \Delta$ and radius r. Through this paper, we assume that $K : [0,\infty) \to [0,\infty)$ is a right continuous and nondecreasing function. For $0 < p < \infty$ and $-2 < q < \infty$, we say that a function f analytic in Δ belongs to the space $Q_K(p,q)$ if

$$\|f\|^p_{K,p,q} = \sup_{a\in\Delta} \int_\Delta \left|f'(z)\right|^p \left(1-|z|^2\right)^q K(1-|\varphi_a(z)|^2)dA(z) < \infty,$$

where $dA(z)$ is the Euclidean area element on Δ. It is clear that $Q_K(p,q)$ is a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{K,p,q}$ where $p \geq 1$. If $q+2 = p$, $Q_K(p,q)$ is Möbius invariant, i.e., $\|f \circ \varphi_a\| = \|f\|_{K,p,q}$ for all $a \in \Delta$. Now we consider some special cases. If $p = 2$, and $q = 0$, we obtain that $Q_K(p,q) = Q_K$ (cf. [4, 9]). If $K(t) = t^s$, then $Q_K(p,q) = F(p,q,s)$ (cf. [11]) that $F(p,q,s)$ is contained in $\frac{q+2}{p}-$ Bloch space.
The space $Q_{K,0}(p,q)$ consists of analytic function f in Δ with the property that

$$\lim_{|a|\to 1^-} \int_\Delta |f'(z)|^p(1-|z|^2)^q K(1-|\varphi_a(z)|^2)dA(z) = 0.$$

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1

It can be checked that $Q_{K,0}(p,q)$ is a closed subspace in $Q_K(p,q)$. The following identity is easily verified:

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} = (1-|z|^2)|\varphi_a'(z)|.$$

For $a \in \Delta$, the substitution $z = \varphi_a(w)$ results in the Jacobian change in measure given by $dA(w) = |\varphi_a(z)|^2 dA(z)$. For a Lebesgue integrable or a non-negative Lebesgue measurable function h on Δ we thus have the following change-of-variable formula:

$$\int_{\Delta(0,r)} h(\varphi_a(w))dA(w) = \int_{\Delta(a,r)} h(z)\left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^2 dA(z)\ .$$

Note that $\varphi_a(\varphi_a(z)) = z$ and thus $\varphi_a^{-1}(z) = \varphi_a(z)$. For $a, z \in \Delta$ and $0 < r < 1$, the pseudo-hyperbolic disk $\Delta(a,r)$ is defined by $\Delta(a,r) = \{z \in \Delta : |\varphi_a(z)| < r\}$. We will also need to use the so-called Berezin transform. More specifically, for any function $f \in L^1(\Delta, dA)$, we define a function Bf by

$$Bf(z) = \int_\Delta \frac{(1-|z|^2)^2}{|1-zw|^4} f(w)dA(w), \quad z \in \Delta$$

we call Bf the Berezin transform of f. By a change of variables, we can also write

$$Bf(z) = \int_\Delta f \circ \varphi_z(w)dA(w), \quad z \in \Delta$$

see [1, 2, 3, 6] and [12] for basic properties of the Berezin transform.
If the function K is only defined on $(0,1]$, then we extend it to $(0,\infty)$ by setting $K(t) = K(1)$ for $t > 1$. We can then define on auxiliary function as follows:

$$\varphi_K(s) = \sup_{0<t\le 1} \frac{K(st)}{K(t)}, \ \ 0 < s < \infty.$$

We further assume that K is continuous and nondecreasing on $(0,1]$ This ensures that the function φ_K is continuous and nondecreasing on $(0,\infty)$.
The following estimate is the key to the main results of this paper.

Lemma 1.1 *[10] Let K be any nonnegative and Lebesgue measurable function on $(0,\infty)$ and $f(z) = K(1-|z|^2)$. If*

$$\int_0^\infty \frac{\varphi_K(x)}{(1+x)^3}dx < \infty, \tag{1}$$

then there exists a positive constant C such that $Bf(z) \le Cf(z)$ for all $z \in \Delta$.

Hereafter, C stands for absolute constants, which may indicate different constants from one occurrence to the next.

2 A double integral characterization in $Q_K(p,q)$ spaces

In this section we characterize the space $Q_K(p,q)$ in terms of a double integral that does not involve the use of derivatives. We begin with the following estimate of Bloch type integrals.

Theorem 2.1 *Suppose that $K(t) \approx t^n K(t);\ 0 < t < 1,\ n \ge 0$. There exists a constant $C > 0$ (independent of K) such that*

$$\int_\Delta \left|f'(z)\right|^p \left(1-|z|^2\right)^{p-2} K(1-|z|^2)dA(z) \le CI(f)$$

for all analytic functions f in Δ, where

$$I(f) = \int_\Delta \int_\Delta \frac{\left|f(z)-f(w)\right|^p}{\left|1-z\bar{w}\right|^4}\left(1-|z|^2\right)^{p-2} K(1-|z|^2)d(z)dA(w).$$

Proof. We write the double integral $I(f)$ as an iterated integral

$$I(f) = \int_{\Delta} \frac{K(1-|z|^2)}{(1-|z|^2)^{4-p}} dA(z) \int_{\Delta} \frac{\left(1-|z|^2\right)^2}{\left|1-z\bar{w}\right|^4} \left|f(z)-f(w)\right|^p dA(w)$$

Making a change of variables in the inner integral, we obtain

$$I(f) = \int_{\Delta} \frac{K(1-|z|^2)}{(1-|z|^2)^{4-p}} dA(z) \int_{\Delta} \left|f(\varphi_z(w)) - f(z)\right|^p dA(w). \tag{2}$$

It is well known that

$$\int_{\Delta} |g(w)-g(0)|^p dA(w) \sim \int_{\Delta} |g'(w)|^p (1-|w|^2)^p dA(w), \tag{3}$$

for analytic functions g in Δ. Applying (3) to the inner integral in (2) with the function $g(w) = f(\varphi_z(w))$, we deduce that

$$I(f) \sim \int_{\Delta} \frac{K(1-|z|^2)}{(1-|z|^2)^{4-p}} dA(z) \int_{\Delta} \left|(f \circ \varphi_z)'(w)\right|^p \left(1-|w|^2\right)^p dA(w).$$

Therefore, by the chain rule and a change of variables, we get

$$I(f) \quad \sim \int_{\Delta} \left(1-|z|^2\right)^{p-2} K(1-|z|^2) dA(z) \int_{\Delta} \left|f'(w)\right|^p \frac{(1-|w|^2)^p}{|1-z\bar{w}|^4} dA(w). \tag{4}$$

Fix any positive radius R. Then there exists a constant $C > 0$ such that

$$I(f) \geq C \int_{\Delta} \left(1-|z|^2\right)^{p-2} K(1-|z|^2) dA(z) \int_{\Delta(z,R)} \left|f'(w)\right|^p \frac{(1-|w|^2)^p}{|1-z\bar{w}|^4} dA(w).$$

It is well known that (see e.g [8])

$$\frac{(1-|w|^2)}{|1-z\bar{w}|^2} \sim \frac{1}{(1-|z|^2)} \sim \frac{1}{\sqrt{|\Delta(z,R)|}}.$$

for $w \in \Delta(z,R)$. It is follows that there exists a positive constant C such that

$$I(f) \geq C \int_{\Delta} (1-|z|^2)^{p-2} K(1-|z|^2) dA(z) \frac{1}{|\Delta(z,R)|^{\frac{p}{2}}} \int_{\Delta(z,R)} \left|f'(w)\right|^p dA(w).$$

Then,

$$I(f) \geq C \int_{\Delta} \left|f'(z)\right|^p (1-|z|^2)^{p-2} K(1-|z|^2) dA(z).$$

This complete the proof of the theorem.

Theorem 2.2 *Let $p > 2$. If the function K satisfies condition (1) and suppose that $K(t) \approx t^n K(t)$; $0 < t < 1$, $n \geq 0$. Then there exists a constant $C > 0$ such that*

$$\int_{\Delta} \left|f'(z)\right|^p \left(1-|z|^2\right)^{p-2} K(1-|z|^2) dA(z) \geq C I(f)$$

for all analytic functions f in Δ, where $I(f)$ is as given in Lemma 2.1.

Proof. By Fubini's theorem, we can rewrite (4) as

$$\begin{aligned} I(f) &\sim \int_{\Delta} \left|f'(w)\right|^p (1-|w|^2)^{p-2} dA(w) \int_{\Delta} (1-|z|^2)^{p-2} \frac{\left(1-|w|^2\right)^2}{|1-z\bar{w}|^4} K(1-|z|^2) dA(z). \\ &\sim \int_{\Delta} \left|f'(w)\right|^p (1-|w|^2)^{p-2} dA(w) \int_{\Delta} \frac{\left(1-|w|^2\right)^2}{|1-z\bar{w}|^4} K(1-|z|^2) dA(z). \end{aligned} \tag{5}$$

The inner integral above is nothing but the Berezin transform of the function $K(1-|z|^2)$ at the point w. The desired estimate now follows from Lemma 2.1
We can now prove the main result of this section

Theorem 2.3 *Suppose K satisfies condition (1) and satisfies all conditions of Theorems 2.1 and 2.2, then an analytic function f in Δ belongs to $Q_K(p, p-2)$ if and only if*

$$\int_\Delta \int_\Delta \frac{\left|f(z)-f(w)\right|^p}{|1-z\bar{w}|^4}\left(1-|z|^2\right)^{p-2} K(1-|z|^2))dA(z)dA(w) < \infty. \qquad (6)$$

Proof. $f \in Q_K(p, p-2)$ if and only if

$$\sup_{a\in\Delta} \int_\Delta \left|f'(z)\right|^p \left(1-|z|^2\right)^{p-2} K(1-|\varphi_a(z)|)dA(z) < \infty.$$

By a change of variables, we have $f \in Q_K(p, p-2)$ if and only if

$$\sup_{a\in\Delta} \int_\Delta \left|(f\circ\varphi_a)'(z)\right|^p \left(1-|\varphi_a(z)|^2\right)^{p-2} K\left(1-|z|^2\right)dA(z)$$

Replacing f by $f\circ\varphi_a$ in Theorems 2.1 and 2.2, we conclude that $f \in Q_K(p, p-2)$ iff

$$\sup_{a\in\Delta} \int_\Delta \int_\Delta \frac{\left|f\circ\varphi_a(z)-f\circ\varphi_a(w)\right|^p}{|1-z\bar{w}|^4}\left(1-|z|^2\right)^{p-2} K\left(1-|z|^2\right)dA(z)dA(w) < \infty.$$

Changing variables and simplifying the result, we find that the double integral above is the same as

$$\sup_{a\in\Delta} \int_\Delta \int_\Delta \frac{\left|f(z)-f(w)\right|^p}{|1-z\bar{w}|^4}\left(1-|z|^2\right)^{p-2} K(1-|\varphi_a(z)|^2)dA(z)dA(w) < \infty.$$

Therefore, $f \in Q_K(p, p-2)$ iff the condition (6) holds.

3 Bergman metric and $Q_K(p, q)$ spaces

In this section we give two closely related characterizations of $Q_K(p, q)$ spaces, one in terms of the Berezin transform and the other in terms of certain class of analytic functions in Bergman metric.
Given a function $f \in L^p(\Delta, dA)$ it is customary to write

$$S(f)(z) = (B(|f|^p) - |Bf(z)|^p)^{\frac{1}{p}}.$$

It easy to check that

$$\begin{aligned} (S(f)(z))^p &= \int_\Delta |f\circ\varphi_z(w) - Bf(z)|^p dA(w) \\ &= \int_\Delta |f(w) - Bf(z)|^p \frac{(1-|z|^2)^2}{|1-z\bar{w}|^4} dA(w). \end{aligned}$$

If the function f is analytic, then it is easy to see that $Bf = f$, so that

$$\begin{aligned} (S(f)(z))^p &= \int_\Delta |f\circ\varphi_z(w) - f(z)|^p dA(w) \\ &= \int_\Delta |f(w) - f(z)|^p \frac{(1-|z|^2)^2}{|1-z\bar{w}|^4} dA(w). \end{aligned}$$

We can now reformulate Theorem 3.1 as follows

Theorem 3.1 *If K satisfies condition (1), then an analytic function f in Δ belongs to $Q_K(p, p-2)$ iff*

$$\sup_{a\in\Delta} \int_\Delta (S(f)(z))^p \left(1-|z|^2\right)^{p-2} K(1-|\varphi_a(z)|^2)d\tau(z) < \infty, \qquad (7)$$

where

$$d\tau(z) = \frac{dA(z)}{(1-|z|^2)^2}$$

is the Möbius invariant measure on the unit disk.

Proof. From Theorem 3.1

$$I_a(f)=\int_\Delta\int_\Delta \frac{\big|f(z)-f(w)\big|^p}{|1-z\bar{w}|^4}\big(1-|z|^2\big)^q K(1-|\varphi_a(z)|^2)dA(z)dA(w)$$

we rewrite it as an iterated integral

$$I_a(f)=\int_\Delta \big(1-|z|^2\big)^p K(1-|\varphi_a(z)|^2)\,d\tau(z)\int_\Delta \frac{\big|f(z)-f(w)\big|^p}{|1-z\bar{w}|^4}\,dA(w)$$

or

$$I_a(f)=\int_\Delta \big(1-|z|^2\big)^{p-2} K(1-|\varphi_a(z)|^2)\,d\tau(z)\int_\Delta \big|f(z)-f(w)\big|^p\frac{\big(1-|z|^2\big)^2}{|1-z\bar{w}|^4}\,dA(w)$$

According to the calculations preceding this theorem, we have

$$I_a(f)=\int_\Delta (S(f)(z))^p\big(1-|z|^2\big)^{p-2}K(1-|\varphi_a(z)|^2)d\tau_z$$

This proves the desired result.

Now, fix a positive radius R and denote by

$$A_R(f)(z)=\frac{1}{|D(z,R)|}\int_{\Delta(z,R)} f(w)dA(w)$$

the a verge of f over the Bergman metric ball $D(z,R)$. For $p\geq 1$, we define

$$S_R(f)(z)=\left[\frac{1}{|D(z,R)|^p}\int_\Delta |f(w)-A_R(f)(z)|^p dA(w)\right]^{\frac{1}{p}}.$$

It is easy to verify that

$$(S_R(f)(z))^p=A_R(|f|^p)(z)-|A_R(f)(z)|^p.$$

Now, we prove the following theorem:

Theorem 3.2 *If K satisfies condition (1), then an analytic function f in Δ belongs to $Q_K(p,p-2)$ if and only if*

$$\sup_{a\in\Delta}\int_\Delta (S_R(f)(z))^p\big(1-|z|^2\big)^{p-2}K(1-|\varphi_a(z)|^2)d\tau(z)<\infty, \tag{8}$$

where R is any fixed positive radius.

Proof. There exists a positive constant C which is depending on R only such that

$$S_R(f)(z)\leq C\,S(f)(z),\quad z\in\Delta,$$

where f is any function in $L^p(\Delta,dA)$. Therefore, condition (6) implies condition (7).
On the other hand, since $D(0,R)$ is an Euclidean disk centered at the origin, we can find a positive constant C which is depending on R only such that

$$|f'(0)|^p\leq C\int_{D(0,R)}|f(w)-C|^p dA(w)$$

for all analytic f in Δ and all complex constants C.
Replace f by $f\circ\varphi_z$ and replace C by $A_R(f)(z)$ then

$$(1-|z|^2)^p|f'(z)|^p\leq C\int_{D(0,R)}|f\circ\varphi_z(w)-A_R(f)(z)|^p dA(w)$$

Make an obvious change of variables on the right hand side, we obtain

$$(1-|z|^2)^p\,|f'(z)|^p \leq C\int_{D(z,R)} |f(w)-A_R(f)(z)|^p \frac{\left(1-|z|^2\right)^2}{|1-z\bar{w}|^4}dA(w).$$

Since

$$\frac{\left(1-|z|^2\right)^2}{|1-z\bar{w}|^4} \sim \frac{1}{\left(1-|z|^2\right)^2} \sim \frac{1}{|D(z,R)|}$$

for $w \in \Delta(z,R)$, we can find another positive constant C such that

$$(1-|z|^2)^p\,|f'(z)|^p \leq C\,(S_R(f)(z))^p, \quad z\in\Delta$$

It follows that for each $a \in \Delta$ that

$$\sup_{a\in\Delta}\int_\Delta |f'(z)|^p\left(1-|z|^2\right)^q K(1-|\varphi_a(z)|^2)dA(z) \leq C\sup_{a\in\Delta}\int_\Delta (S_R(f)(z))^p\left(1-|z|^2\right)^{p-2}K(1-|\varphi_a(z)|^2)d\tau(z).$$

This shows that the condition (7) implies $f \in Q_K(p,p-2)$.

Recall from [5] that a positive Borel measure μ on Δ is called a K-Carleson measure if

$$\sup_I \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)d\mu(z) < \infty,$$

where the supremum is taken over all sub-arcs $I \subset \partial\Delta$. Here, for a sub-arcs I of $\partial\Delta$, $|I|$ is the length of I and $S(I) = \{r\xi : \xi \in I, 1-|I| < r < 1\}$ is the corresponding Carlesson square. Also, A positive Borel measure μ on Δ is called a vanishing $K-$ Carleson measure if

$$\lim_{|z|\to 1^-}\int_\Delta K\left(\frac{1-|z|}{|I|}\right)d\mu(z) = 0.$$

Theorem 3.3 *Suppose K satisfies the following two conditions:*

(a) There exists a constant $C > 0$ such that $K(2t) \leq CK(t)$ for all $t > 0$.

(b) The auxiliary function φ_k has the property that

$$\int_0^1 \varphi_k(s)\frac{ds}{s} < \infty.$$

Let μ be a positive Borel measure on Δ. Then μ is a K-Carleson measure if and only if

$$\sup_{a\in\Delta}\int_\Delta K(1-|\varphi_a(z)|^2)d\mu(z) < \infty.$$

Proof. Since $Q_K(p,q)$ is defined by the condition

$$\sup_{a\in\Delta}\int_\Delta |f'(z)|^p\left(1-|z|^2\right)^q K(1-|\varphi_a(z)|^2)dA(z) < \infty,$$

we see that $f \in Q_K(p,q)$ if and only if the measure $\left(1-|z|^2\right)^q|f'(z)|^p dA(z)$ is a K-Carleson measure. The following Corollary gives two analogous characterizations.

Corollary 3.1 *Suppose K satisfies condition (1) and conditions (a) and (b) in Theorem 3.3. Let $R > 0$ be a fixed radius. Then the following conditions are equivalent for an analytic function f in Δ.*

(a) The function f belong to $Q_K(p,p-2)$.

(b) The measure $d\mu(z) = (S(f)(z))^p(1-|z|^2)^{p-2}d\tau(z)$ is a K-Carleson measure.

(c) The measure $d\nu(z) = (S_R(f)(z))^p(1-|z|^2)^{p-2}d\tau(z)$ is a K-Carleson measure.

Proof. This is a direct consequence of Theorems 3.1,3.2, and 3.3.
The little-oh version of the above result can be stated as follows:

Theorem 3.4 *Suppose K satisfies condition (1) and $R > 0$ is a fixed, then the following conditions are equivalent for all analytic functions f in Δ.*

(1) $f \in Q_{K,0}(p, p-2)$

(2) $$\lim_{|z|\to 1^-} \int_\Delta \int_\Delta \frac{\left|f(z)-f(w)\right|^p}{|1-z\bar{w}|^4} \left(1-|z|^2\right)^{p-2} K(1-|\varphi_a(z)|^2) dA(z) dA(w) = 0$$

(3) $$\lim_{|z|\to 1^-} \int_\Delta \int_\Delta (S(f)(z))^p \left(1-|z|^2\right)^{p-2} K(1-|\varphi_a(z)|^2) d\tau(z) = 0$$

(4) $$\lim_{|z|\to 1^-} \int_\Delta \int_\Delta (S_R(f)(z))^p \left(1-|z|^2\right)^{p-2} K(1-|\varphi_a(z)|^2) d\tau(z) = 0.$$

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Coefficient bounds for certain subclasses of close-to-convex functions of Janowski type

Wasim Ul-Haq[1,a], Attiya Nazneen[a], Muhammad Arif[a], Nasir Rehman[b]

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In this work, we aim to determine the coefficient estimates for functions in certain subclasses of close-to-convex functions of Janowski type and related functions of complex order, which are here defined by means of Cauchy-Euler type non-homogeneous differential equation. Several interesting consequences of our results are also observed.

Key words: Analytic functions; Close-to-convex; Coefficient Estimates.
Subject classification: 30C45, 30C50.

† 1

E-mail addresses: wasim474@hotmail.com (W. Ul-Haq), anawkum@yahoo.com (A. Nazneen), marifmaths@yahoo.com nasirzainy1@hotmail.com (N. Rehman)

1 Introduction

We denote by $\mathcal{A}$ the class of functions $f(z)$ which are analytic in the open unit disc $E = \{z : |z| < 1\}$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

[1]Corresponding author.
a. Department of Mathematics, Abdul Wali Khan University Mardan, Khyber Pakhtunkhwa, Pakistan
b. Department of Mathematics and Statistics, Allama Open Iqbal University, Islamabad,Pakistan

Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent. Also let $\mathcal{S}_\gamma^*$, $\mathcal{C}_\gamma$, $\mathcal{K}_\gamma$ and $\mathcal{Q}_\gamma$ be the subclasses of $\mathcal{A}$ consisting of all functions which are starlike, convex, close-to-convex and quasi convex of complex order γ ($\gamma \neq 0$) respectively, for details see [3, 5, 7]. We note that for $0 < \gamma \leq 1$, these classes coincide with the well known classes of starlike, convex and close-to-convex of order $1-\gamma$. Recently Altintaş et al.[1] considered the following class of functions denoted by $\mathcal{SC}(\gamma, \lambda, A, B)$ and defined as:

$$\mathcal{SC}(\gamma,\lambda,A,B) = \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{\gamma}\left(\frac{z[(1-\lambda)f(z) + \lambda z f'(z)]'}{(1-\lambda)f(z) + \lambda z f'(z)} - 1, \right) \prec \frac{1+Az}{1+Bz}, z \in E \right\}, \quad (2)$$

where $-1 \leq B < A \leq 1$, $0 \leq \lambda \leq 1$, $\gamma \in \mathtt{C} - \{0\}$. Note that the classes $\mathcal{SC}(1,0,A,B) = \mathcal{S}^*[A,B]$ and $\mathcal{SC}(1,1,A,B) = \mathcal{C}[A,B]$ were introduced by Janowski [4] and are called classes of Janowski starlike and Janowski convex functions respectively. Also

$$\mathcal{SC}(\gamma,0,1,-1) = \mathcal{S}_\gamma^*, \mathcal{SC}(\gamma,1,1,-1) = \mathcal{C}_\gamma.$$

Throughout the entire paper onward we assume the restrictions $-1 \leq B < A \leq 1, 0 \leq \lambda \leq 1$, $\gamma \in \mathtt{C} - \{0\}$ unless otherwise mentioned. Now we denote $\mathcal{KQ}(\gamma, \lambda, A, B)$ be the class of functions $f(z) \in \mathcal{A}$ if there exist a function $g(z) \in \mathcal{SC}(1, \lambda, A, B)$ such that

$$1 + \frac{1}{\gamma}\left(\frac{z[(1-\lambda)f(z) + \lambda z f'(z)]'}{(1-\lambda)g(z) + \lambda z g'(z)} - 1 \right) \prec \frac{1+Az}{1+Bz}, z \in E.$$

As special choices we have the following relationships

$$\begin{aligned} \mathcal{KQ}(1,0,A,B) &= \mathcal{K}[A,B], \quad \mathcal{KQ}(1,1,A,B) = \mathcal{Q}[A,B], \textit{see [Noor, [6]]} \\ \mathcal{KQ}(\gamma,0,1,-1) &= \mathcal{K}_\gamma, \quad \mathcal{KQ}(\gamma,1,1,-1) = \mathcal{Q}_\gamma. \end{aligned}$$

Motivated from the recent work of Srivastava et al. [9] and Altintaş et al. [2] the main purpose of our investigation is to derive coefficient estimates of a subfamily $\mathcal{DK}(\gamma, \lambda, A, B, m; \mu)$ of $\mathcal{A}$, which consists of functions $f(z)$ in $\mathcal{A}$ satisfying the following Cauchy Euler type non homogenous differential equation

$$z^m \frac{d^m w}{dz^m} + {}^mC_1(\mu+m-1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}} + \cdots {}^mC_m w \prod_{j=0}^{m-1}(\mu+j) = h(z)\prod_{j=0}^{m-1}(\mu+j+1), \quad (3)$$

where $w = f(z)$, $h(z) \in \mathcal{KQ}(\gamma, \lambda, A, B)$, $\mu \in \mathtt{R} - (-\infty, -1]$, $m \in N^* = \{2, 3, \cdots\}$ for details we refer to [2, 8, 9, 10, 11]. The following result which is due to Altintaş et al. [2] is essential in deriving our main results.

Lemma 1. [2]. *Let $f(z) \in \mathcal{SC}(\gamma, \lambda, A, B)$ and be of the form* (1). *Then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}\left[j + \frac{2|\gamma|(A-B)}{1-B}\right]}{(n-1)![1+\lambda(n-1)]}, \; n \in N^*.$$

2 Coefficient Estimates for functions in the class $\mathcal{KQ}(\gamma, \lambda, A, B)$

We first establish the below result for the functions in the class $\mathcal{KQ}(\gamma, \lambda, A, B)$.
Theorem 1. *Let* $f(z) \in \mathcal{KQ}(\gamma, \lambda, A, B)$ *and be defined by* (1). *Then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}\left[j+\frac{2(A-B)}{1-B}\right]}{n![1+\lambda(n-1)]} + \frac{2|\gamma|}{n[1+\lambda(n-1)]}\frac{A-B}{1-B}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}\left[j+\frac{2(A-B)}{1-B}\right]}{(n-k-1)!},\ n \in N^*. \quad (4)$$

Proof. Since $f(z) \in \mathcal{KQ}(\gamma, \lambda, A, B)$, then there exists $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belonging to the class $\mathcal{SC}(1, \lambda, A, B)$ such that

$$1 + \frac{1}{\gamma}\left(\frac{zF'(z)}{G(z)} - 1\right) \prec \frac{1+Az}{1+Bz},\ for z \in E,$$

where $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ and $G(z) = z + \sum_{n=2}^{\infty} B_n z^n$, with

$$A_n = [1 + \lambda(n-1)]a_n,\ B_n = [1 + \lambda(n-1)]b_n. \quad (5)$$

Let

$$1 + \frac{1}{\gamma}\left(\frac{zF'(z)}{G(z)} - 1\right) = q(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,\ for z \in E, \quad (6)$$

Since $q(z) \prec \frac{1+Az}{1+Bz}$, $z \in E$, we find that by definition of subordination

$$q(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, w(0) = 0;\ |w(z)| < 1.$$

Therefore, we have

$$|w(z)| = \left|\frac{q(z) - 1}{A - Bq(z)}\right| < 1, \quad q(z) = u + iv,$$

which further implies that

$$2u(1 - AB) > 1 - A^2 + (1 - B^2)(u^2 + v^2).$$

Also, since $|q(z)|^2 \geq (Req(z))^2$, we have

$$(1 - B^2)u^2 - 2u(1 - AB) + 1 - A^2 < 0 \Longrightarrow Req(z) > \frac{1-A}{1-B}. \quad (7)$$

From (6) and (7), we find that

$$|c_n| \leq 2\left(\frac{A-B}{1-B}\right),\ n \in \mathbb{N}. \quad (8)$$

Then from (6), we obtain

$$nA_n = B_n + \gamma\left[c_{n-1} + \sum_{k=1}^{n-1} c_k B_{n-k}\right], n \geq 2$$

Now using Lemma 1 together with (5) and (8), we have

$$|A_n| \leq \frac{\prod_{j=0}^{n-2}\left[j+\frac{2(A-B)}{1-B}\right]}{n!} + \frac{2|\gamma|}{n}\frac{A-B}{1-B}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}\left[j+\frac{2(A-B)}{1-B}\right]}{(n-k-1)!},$$

and hence from the relation between $F(z)$ and $f(z)$ as in (5), we obtain the desired result. By assigning different specific values to the involved parameters $A, B,\ \gamma,\ \lambda$ in Theorem 1, we deduce the following interesting results. **Corollary 1.** *Let $f(z) \in \mathcal{KQ}(1,0,A,B) = \mathcal{K}[A,B]$ and be defined by* (1). *Then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}\left[j+\frac{2(A-B)}{1-B}\right]}{n!} + \frac{2}{n}\frac{A-B}{1-B}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}\left[j+\frac{2(A-B)}{1-B}\right]}{(n-k-1)!}\ n \in N^*.$$

Corollary 2. *Let $f(z) \in \mathcal{KQ}(\gamma,\lambda,1,-1)$ and be defined by* (1). *Then*

$$|a_n| \leq \frac{1}{[1+\lambda(n-1)]}\left[1+(n-1)|\gamma|\right],\ n \in N^*.$$

Corollary 3 [3]. *Let $f(z) \in \mathcal{KQ}(\gamma,0,1,-1) = \mathcal{K}(\gamma)$ and be defined by* (1). *Then*

$$|a_n| \leq 1+(n-1)|\gamma|,\ n \in N^*.$$

Corollary 4 [5]. *Let $f(z) \in \mathcal{KQ}(\gamma,1,1,-1) = \mathcal{Q}(\gamma)$ and be defined by* (1). *Then for $n \in N^* = \{2,3,4,\ldots\}$.*

$$|a_n| \leq \frac{1+(n-1)|\gamma|}{n},\ n \in N^*.$$

For $\gamma = 1$ in Corollary 2 and Corollary 3, we obtain the well-known coefficient estimates for close-to-convex and quasi convex functions. **Corollary 5.** *Let $f(z) \in \mathcal{KQ}(1-\alpha,\lambda,1-2\beta,-1)$ and be defined by* (1). *Then for $n \in N^*$*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}\left[j+2(1-\beta)\right]}{n![1+\lambda(n-1)]} + \frac{2(1-\alpha)(1-\beta)}{n[1+\lambda(n-1)]}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}\left[j+2(1-\beta)\right]}{(n-k-1)!}.$$

Corollary 6. *Let $f(z) \in \mathcal{KQ}(1-\alpha,0,1,-1) = \mathcal{K}(1-\alpha)$ and be defined by* (1). *Then*

$$|a_n| \leq n(1-\alpha)+\alpha,\quad n \in N^* = \{2,3,4,\ldots\}.$$

Corollary 7. *Let $f(z) \in \mathcal{KQ}(1-\alpha,1,1,-1) = \mathcal{Q}(1-\alpha)$ and be defined by* (1). *Then*

$$|a_n| \leq 1-\alpha+\frac{\alpha}{n},\ n \in N^* = \{2,3,4,\ldots\}.$$

3 Coefficient Estimates of the class $\mathcal{BK}(\gamma, \lambda, A, B; \mu)$

The theorem below is our main coefficient estimates for functions in the class $\mathcal{DK}(\gamma, \lambda, A, B, m; \mu)$.

Theorem 2. *Let $f(z) \in \mathcal{DK}(\gamma, \lambda, A, B, m; \mu)$ and be defined by (1). Then for $n \in N^* = \{2, 3, 4, \ldots\}$*

$$|a_n| \leq \frac{\prod_{j=0}^{m-1}(\mu+j+1)}{\prod_{j=0}^{m-1}(\mu+j+n)} \left[\frac{\prod_{j=0}^{n-2}\left[j+\frac{2(A-B)}{1-B}\right]}{n![1+\lambda(n-1)]} + \frac{2|\gamma|}{n[1+\lambda(n-1)]} \frac{A-B}{1-B} \sum_{k=1}^{n-1} \frac{\prod_{j=0}^{n-k-2}\left[j+\frac{2(A-B)}{1-B}\right]}{(n-k-1)!} \right]. \tag{9}$$

Proof. Let $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{KQ}(\gamma, \lambda, \beta)$,so that

$$a_n = \frac{\prod_{j=0}^{m-1}(\mu+j+1)}{\prod_{j=0}^{m-1}(\mu+j+n)} b_n, \quad n \in N^*, \ \mu \in \mathrm{R} - (-\infty, -1].$$

Hence, by using Theorem 1, we immediately obtain the required inequality (9).

Corollary 8. *Let $f(z) \in \mathcal{DK}(\gamma, \lambda, 1-2\beta, -1, 2; \mu)$ and be defined by (1). Then for $n \in N^* = \{2, 3, 4, \ldots\}$*

$$|a_n| \leq \frac{(1+\mu)(2+\mu)}{(n+1+\mu)(n+\mu)} \left[\frac{\prod_{j=0}^{n-2}[j+2(1-\beta)]}{n![1+\lambda(n-1)]} + \frac{2|\gamma|(1-\beta)}{n[1+\lambda(n-1)]} \sum_{k=1}^{n-1} \frac{\prod_{j=0}^{n-k-2}[j+2(1-\beta)]}{(n-k-1)!} \right].$$

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Five-order Extrapolation Algorithms for Laplace Equation with Linear Boundary Condition*

Pan Cheng†, Zhi lin, Peng Xie
School of Science, Chongqing Jiaotong University,
Chongqing ,400074, PR.China

Abstract

Laplace equation with linear boundary condition will be converted into a boundary integral equation(BIE) with logarithmic singularity following potential theory. In this paper, a Sidi quadrature formula is introduced to approximate the logarithmic singularity integral operator with $O(h^3)$ approximate accuracy order. A similar approximate equation is also constructed for the logarithmic singular operator, which is based on coarse grid with mesh width $2h$. So an extrapolation algorithm is applied to approximate the logarithmic operator and the accuracy order is improved to $O(h^5)$. Moreover, the accuracy order is based on fine grid h. The convergence and stability are proved based on Anselone's collective compact and asymptotic compact theory. Furthermore, an asymptotic expansion with odd powers of the errors is presented with convergence rate $O(h^5)$. Using h^5-Richardson extrapolation algorithms(EAs), not only the approximation accuracy order can be improved to $O(h^7)$, but also an a posteriori error estimate can be obtained for constructing a self-adaptive algorithm. numerical examples are shown to verify its efficiency.

Keywords: boundary integral equation, Richardson extrapolation algorithm, Laplace equation, a posteriori error estimate

2000 MSC: 65N25, 65N38

1 Introduction

Laplace equation with linear boundary condition is defined as follows: to find non-zero deformation $\tilde{u}$ in the domain Ω and on the boundary Γ satisfying

$$\begin{cases} \triangle \tilde{u} = 0, \text{ in } \Omega, \\ \dfrac{\partial \tilde{u}}{\partial n} = -c\tilde{u}(x) + \tilde{f}(x), \text{ on } \Gamma, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded, simply connected domain with a smooth boundary Γ, $\partial/(\partial n)$ is an normal outward derivative on Γ, c is a positive constant, and $\tilde{f}(x)$ is a given function.

By means of potential theory, Eq.(1) will be transformed into a boundary integral equation(BIE) as follows[1,2,3]:

$$\alpha(y)\tilde{u}(y) - \int_\Gamma k^*(y,x)\tilde{u}(x)ds_x = \int_\Gamma h^*(y,x)\frac{\partial \tilde{u}(x)}{\partial n_x}ds_x, \quad y \in \Gamma, \tag{2}$$

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†Email-address: cheng_pass@sina.com(Pan Cheng)

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where $\alpha(y) = \theta(y)/(2\pi)$ is related to the interior angle $\theta(y)$ of Ω at point $y \in \Gamma$, in particular, when y is on a smooth part of the boundary Γ, $\alpha(y) = 1/2$, and $h^*(y, x)$ is the fundamental solution:

$$\begin{cases} h^*(y,x) = -\dfrac{1}{2\pi} \ln |x-y|, \\ k^*(y,x) = -\dfrac{\partial h^*(y,x)}{\partial n}, \end{cases} \tag{3}$$

where $|x - y|$ is the distance between points x and y.

The left terms in Eq.(2) are smooth integrals and the right hand side term is characterized as a logarithmic singularity. Various numerical methods have been proposed for dealing with the singularity, such as Galerkin methods in Stephan and Wendland[4], Chandler[5], Sloan and Spence[6], and Amini and Nixon[7], collocation methods in Elschner and Graham[8] and Yan[9], quadrature methods in Sidi and Israeli[10], Saranen[11], Huang and Lü[12,13] and combined Trefftz methods in Li[14].

Extrapolation algorithms (EAs) based on asymptotic expansion about errors are effective parallel algorithms, which possesses high accuracy degree, good stability and almost optimal computational complexity. Cheng et al.[15,16] harnessed extrapolation algorithms to obtain high accuracy order for Steklov eigenvalue in Laplace equations with smoothed and polygonal boundary condition. Huang and Lü established extrapolation algorithms for solving the Steklov eigenvalue problems[3],the Helmholtz equations[17] and the Laplace equations[18] with accuracy order $O(h^3)$. After the Extrapolation algorithms, the accuracy order of the approximate solution will be improved to $O(h^5)$.

A quadrature method[19,20] is presented for solving the boundary integral equation, in which the generation of the discrete matrixes does not require any calculations of singular integrals. The logarithmic integral kernel is approximated by extrapolation algorithms derived from Sidi's quadrature rule. An asymptotic expansion about the error is obtained with convergence rate $O(h^5)$.

Note that the five order approximate solution is obtained directly and is based on fine grid h. Although there are some papers[17−20] also obtain the same accuracy order, there are three main priority for our paper: firstly, those accuracy orders are based on fine grid; secondly, because the accuracy order is not derived from the extrapolation algorithms but from the directly calculation, so there are not any errors generated from the extrapolation algorithms; finally, when an linear equation with n order is solved, there are n approximate solutions u_h can be obtained on boundary Γ with accuracy order $O(h^5)$, while not $n/2$ values from extrapolation method.

This paper is organized as follows: In Section 2 a Sidi's quadrature method is recombined to approximate integral equations for solving the approximate solution; In Section 3 an asymptotically compact theory is provided for stability and convergence, and an asymptotic expansion for approximate solution is shown with convergence rate $O(h^5)$; In Section 4 the Richardson extrapolation algorithms are applied to improve the accuracy order to $O(h^7)$; In Section 5 numerical examples illustrate the calculate progress.

2 Five order approximate methods

Assume that Γ is a smooth closed curve described by a regular parameter mapping $x(s) = (x_1(s),\ x_2(s)) : [0, 2\pi] \to \Gamma$, satisfying $|x'(s)|^2 = |x_1'(s)|^2 + |x_2'(s)|^2 > 0$. Let $C^{2m}[0, 2\pi]$ denote the set of $2m$ times differentiable periodic functions with the periodic 2π and $x_i(s) \in C^{2m}[0, 2\pi], i = 1, 2$. Define the following integral operators on $C^{2m}[0, 2\pi]$:

$$\begin{cases} (Ku)(s) = 2\displaystyle\int_0^{2\pi} k(t,s)\frac{\partial u(t)}{\partial n}dt \\ (Hu)(s) = 2\displaystyle\int_0^{2\pi} h(t,s)u(t)dt. \end{cases}$$

where $u(t) = \tilde{u}(x_1(t), x_2(t))$, $k(t,s) = k^*(x(t), x(s))\ |x'(t)|$ and $h(t,s) = h^*(x(t), x(s))|x'(t)|$. Because

$$h(t,s) = -\frac{1}{2\pi} \ln |x(t) - x(s)||x'(t)|,$$

so $h(t,s)$ is a logarithmic weak singular kernel and $k(t,s)$ is a smooth kernel. Then Eq.(2) is equivalent to

$$(I - K)u - cHu = Hf \tag{4}$$

where I is an identity operator, and $f = \tilde{f}(x(t))$.

Let $h = 2\pi/n$ ($n \in N$ is supposed to be an even number and so $n/2 \in N$) be the mesh width and $t_j = s_j = jh$, $(j = 0, 1, \ldots, n-1)$ be the nodes. In order to approximate the integral operators K and H, a Lemma is obtained:

Lemma 1:[19] *Consider the integral $\int_0^{2\pi} G(x)dx$ with integral kernel $G(x)$. Assume that the functions $g(x)$, $\tilde{g}(x)$ are $2m$ times differentiable on $[0, 2\pi]$. Also assume that the integral kernel $G(x)$ are periodic function with period 2π. Then the following conclusion can be drawn:*

(a). If $G(x) = g(x)/(x-t) + \tilde{g}(x)$, and $Q_n[G] = h\sum_{j=1, x_j \neq t}^{n} G(x_j)$, then

$$E_n[G] = h[\tilde{g}(t) + g'(t)] + O(h^{2m}) \quad \text{as} \quad h \longrightarrow 0,$$

where $E_n[G] = \int_0^{2\pi} G(x)dx - Q_n[G]$ in all cases;

(b). If $G(x) = g(x)(x-t)^s + \tilde{g}(x), s > -1$, and $Q_n[G] = h\sum_{j=1, x_j \neq t}^{n} G(x_j) + h\tilde{g}(t) - 2\zeta(-s)g(t)h^{s+1}$, then

$$E_n[G] = -2\sum_{\mu=1}^{m-1} \frac{\zeta(-s-2\mu)}{(2\mu)!} g^{(2\mu)}(t)h^{2\mu+s+1} + O(h^{2m}), \quad \text{as} \quad h \to 0;$$

where $\varsigma(t)$ is the Riemann zeta function.

(c). If $G(x) = g(x)(x-t)^s \log|x-t| + \tilde{g}(x), s > -1$, and $Q_n[G] = h\sum_{j=1, x_j \neq t}^{n} G(x_j) + h\tilde{g}(t) + 2[\zeta'(-s) - \zeta(-s)\log h]g(t)h^{s+1}$, then

$$E_n[G] = -2\sum_{\mu=1}^{m-1} [\zeta'(-s-2\mu) - \zeta(-s-2\mu)\log h] \frac{g^{(2\mu)}(t)}{(2\mu)!} h^{2\mu+s+1} + O(h^{2m}), \quad \text{as} \quad h \to 0;$$

Especially, when $s = 0$, then $\zeta'(0) = -(1/2)\log(2\pi)$, and we have

$$Q_n[G] = h\sum_{j=1, x_j = t}^{n} G(x_j) + h\tilde{g}(t) + \log\Big(\frac{h}{2\pi}\Big)g(t)h,$$

then

$$E_n[G] = 2\sum_{\mu=1}^{m-1} \zeta'(-2\mu)\frac{g^{(2\mu)}(t)}{(2\mu)!} h^{2\mu+1} + O(h^{2m}), \quad \text{as} \quad h \to 0.$$

Since K is a smooth integral operator with period 2π, we obtain a high accuracy approximation when set $g(x) \equiv 0$ in case (a) of Lemma 1:

$$(K_h u)(s) = h\sum_{j=0}^{n-1} k(t_j, s)u(t_j), \tag{5}$$

with the error estimate

$$(Ku)(s) - (K_h u)(s) = O(h^{2m}). \tag{6}$$

For the logarithmic weak singular operator H, the continuous approximation of its kernel $h_n(t,\tau)$ is defined as:

$$h_n(t,s)=\begin{cases} h(t,s), & |t-s|\geq h, \\ \ln\left(\frac{h}{2\pi}|x'(s)|\right), & |t-s|<h, \end{cases} \tag{7}$$

so its approximation operator can be obtained when set $\tilde{g}(x)\equiv 0$ and $s=0$ in case (c) of Lemma 1:

$$(H_h u)(s)=h\sum_{j=0}^{n-1} h_n(t_j,s)u(t_j), \tag{8}$$

which has the following error estimate:

$$(Hu)(s)-(H_h u)(s)=2h^3\frac{\varsigma'(-2)}{2!}u^{(2)}+2\sum_{\mu=2}^{m-1}\frac{\varsigma'(-2\mu)}{(2\mu)!}u^{(2\mu)}(s)h^{2\mu+1}+O(h^{2m}). \tag{9}$$

We can find that there is an asymptotic expansion with accuracy order $O(h^3)$ for the logarithmic singular operator. In order to improve the accuracy order from $O(h^3)$ to $O(h^5)$, a coarse grid $2h=2\pi/(n/2)=4\pi/n$ is obtained. The approximate operator based on coarse grid $2h$ is shown as:

$$(H_{2h}u)(s)=2h\sum_{j=0}^{n-1} h_n(t_j,s)u(t_j)\vartheta_j,$$

where

$$\vartheta_j=\begin{cases} 0, & \text{j is an odd number,} \\ 1, & \text{j is an even number.} \end{cases}$$

The error estimate is:

$$\begin{aligned}(Hu)(s)-(H_{2h}u)(s)&=2(2h)^3\frac{\varsigma'(-2)}{2!}u^{(2)} \\ &+2\sum_{\mu=2}^{m-1}\frac{\varsigma'(-2\mu)}{(2\mu)!}u^{(2\mu)}(s)(2h)^{2\mu+1}+O((2h)^{2m}).\end{aligned} \tag{10}$$

An extrapolation algorithm is used to counteract the item $O(h^3)$ in Eqs (9) and (10):

$$(J_h u)(s)=\frac{8}{7}(H_h u)(s)-\frac{1}{7}(H_{2h}u)(s).$$

The error for the approximate operator will be improved from $O(h^3)$ to $O(h^5)$:

$$(Hu)(s)-(J_h u)(s)=\sum_{\mu=2}^{m-1}\eta_\mu h^{2\mu+1}+O(h^{2m}), \tag{11}$$

where η_μ is some coefficients combination of the item $h^{2\mu+1}$. So the accuracy order is not only improved to $O(h^5)$, but also built on the fine grid h.

Thus we obtain the numerical approximate equations of Eq.(4):

$$(I-K_h)u_h-cJ_h u_h=J_h f_h, \tag{12}$$

where K_h and J_h are discrete matrices of order n corresponding to the operators K and H, respectively.

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3 Asymptotical compact convergence

According to the logarithmic capacity theory[3], the eigenvalues of K and K_h do not include 1. Then the Eqs. (4) and (12) can be rewritten as follows: find $u \in C[0, 2\pi]$ satisfying

$$(I - L)u = \varphi, \tag{13}$$

and find u_h satisfying

$$(I - L_h)u_h = \varphi_h, \tag{14}$$

where $L = c(I-K)^{-1}H$, $L_h = c(I-K_h)^{-1}J_h$, $\varphi = (I-K)^{-1}\ Hf$ and $\varphi_h = (I-K_h)^{-1}J_hf_h$.

Theorem 1. *The approximate operator sequence $\{L_h\}$ is an asymptotical compact*[21,22] *sequence and convergent to L in $C[0, 2\pi]$, i,e.*

$$L_h \overset{a.c}{\to} L, \tag{15}$$

where $\overset{a.c}{\to}$ means the asymptotically compact convergence.

This proof can be obtained similarly as the proofs in the papers[15,16].

Corollary[13,15] **1.** *Under the assumption of Theorem 1, we obtain*

$$\begin{cases} \|(L_h - L)L\| \to 0 \\ \|(L_h - L)L_h\| \to 0, \ \textit{as}\, h \to 0. \end{cases}$$

4 Asymptotic expansions of the approximate solutions

Theorem 2. *Suppose $u(s) \in C^{(2m)}[0, 2\pi]$, then we have the following asymptotic expansion*

$$(L_h - L)u(s) = \sum_{j=2}^{m-1} \psi_j(s)h^{2j+1} + O(h^{2m}), \tag{16}$$

where $\psi_j(s) \in C^{(2m-2j)}, j = 2, \ldots, m-1$, are functions independent of h.

Proof. According to properties of the approximate operators, there is

$$(Ku)(s) - (K_hu)(s) = O(h^{2m}). \tag{17}$$

and

$$(Hu)(s) - (J_hu)(s) = \sum_{j=2}^{m-1} \eta_j(s)h^{2j+1} + O(h^{2m}). \tag{18}$$

We consider the relationship between L_h and L:

$$\begin{aligned}(L_h - L)u &= c(I - K_h)^{-1}J_hu - c(I - K)^{-1}Ju \\ &= c(I - K_h)^{-1}J_hu - (I - K)^{-1}J_hu + c(I - K)^{-1}J_hu - (I - K)^{-1}Ju \\ &= c[(I - K_h)^{-1} - (I - K)^{-1}]J_hu + c(I - K)^{-1}(J_h - H)u \\ &= c(I - K)^{-1}(K_h - K)(I - K_h)^{-1}J_hu + c(I - K)^{-1}(J_h - H)u\end{aligned}$$

Substituting the errors of Eqs. (19) and (20) into the above equation, and setting $\psi_j(s) = c(I - K)^{-1}\eta_j(s)$, we complete the proof of Theorem 2. □

Theorem 3. *Suppose $x(t), g(t) \in C^{2m}[0, 2\pi]$, Then there exists functions $\bar{\omega}_l \in C^{2m-2l}[0, 2\pi], l = 1, \ldots, m$ independent of h, such that*

$$(u - u_h)|_{t=t_j} = \sum_{l=2}^{m-1} h^{2l+1}\bar{\omega}_l|_{t=t_j} + O(h^{2l}) \tag{19}$$

Proof. Because $(I-K)^{-1}$ is exist, and $J_h \overset{a.c}{\to} H$, so there is an asymptotic expansion for function φ:

$$(\varphi - \varphi_h)|_{t=t_j} = h^5\omega_2|_{t=t_j} + h^7\omega_3|_{t=t_j} + \ldots + O(h^{2m}), \tag{20}$$

where $\omega_l \in C^{2m-2l}[0, 2\pi], l = 2, \ldots, m-1$.

Because u and u_h satisfy Eqs. (13) and (14) respectively, we obtain

$$\begin{aligned}(I - L_h)&(u_h - u)|_{t=t_j}\\ &= \Big[(I-L_h)u_h - (I-L)u + (I-L)u - (I-L_h)u\Big]\Big|_{t=t_j}\\ &= (\varphi_h - \varphi)|_{t=t_j} + (L - L^h)u|_{t=t_j} = h^5\phi_2|_{t=t_j} + \ldots + O(h^{2m}),\end{aligned} \tag{21}$$

where $\phi_l = \omega_l + \psi_l, l = 2, \ldots, m-1$.

Define an auxiliary equation

$$(I-L)\bar{\omega}_l = \phi_l, \quad l = 2, \ldots, m-1, \tag{22}$$

and its approximate equation

$$(I-L_h)\bar{\omega}_{lh} = \phi_{lh}, \quad l = 2, \ldots, m-1. \tag{23}$$

Substituting Eq. (25) into Eq. (23), we obtain

$$(I-L_h)(u_h - u - \sum_{l=2}^{m-1} h^{2l+1}\bar{\omega}_{lh})|_{t=t_j} = O(h^{2m}). \tag{24}$$

Noticing $\bar{\omega}_{lh} \in C^{2m-2l}[0, 2\pi]$, we have

$$(\bar{\omega}_l - \bar{\omega}_{lh})(t_i) = O(h^{2m-2l}). \tag{25}$$

When substitute $\bar{\omega}_{lh}$ by $\bar{\omega}_l$ and consider the asymptotic compact properties[21], we obtain

$$\big(u_h - u - \sum_{l=2}^{m-1} h^{2l+1}\bar{\omega}_l\big)|_{t=t_j} = O(h^{2m}), \tag{26}$$

so the proof is completed. □

The asymptotic expansion in Eq. (21) implies that the Richardson extrapolation[23] can be applied to improve the accuracy order. A higher accuracy order $O(h^7)$ can be obtained by computing some approximation on Γ in parallel. It can be described as follows:

Taking h and $h/2$ to solve Eq. (12) in parallel, we obtain that $u_h(t_i), u_{h/2}(t_i)$ are the solutions on Γ. According to the asymptotic expansion, we obtain

$$u_h^*(t_i) = \frac{1}{31}(32u_{h/2}(t_i) - u_h(t_i)), \tag{27}$$

and the error is $|u_h^*(t_i) - u(t_i)| = O(h^7)$.

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Moreover, using $|u_h^*(t_i) - u(t_i)| = O(h^7)$, we obtain a posteriori error estimate

$$\begin{aligned}
&|u(t_i) - u_{h/2}(t_i)| \\
&\quad \le |u(t_i) - \frac{1}{32}(32u_{h/2}(t_i) - u_h(t_i))| \\
&\quad\quad + \frac{1}{31}|u_{h/2}(t_i) - u_h(t_i)| \\
&\quad \le \frac{1}{31}|u_{h/2}(t_i) - u_h(t_i)| + O(h^7).
\end{aligned}$$

Note that the upper limitation $\frac{1}{31}|u_{h/2}(t_i) - u_h(t_i)|$ can be used to construct self-adaptive algorithms.

5 Numerical examples

In this section, we consider some computational aspects of the approximate equation and present two examples to illustrate the accelerated convergence of the extrapolation algorithms.

Example 1[24]**:** Consider the boundary value problem satisfying

$$\begin{cases} \triangle \tilde{u} = 0, \text{ in } \Omega, \\ \dfrac{\partial \tilde{u}}{\partial n} = -\tilde{u}(x) + \tilde{f}(x), \text{ on } \Gamma, \end{cases} \tag{28}$$

where $\tilde{f}(x) = 1$ and Ω is the region

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 < 1, \tag{29}$$

with $(a, b) = (1, 2)$. The boundary Γ can be described as: $x_1 = \cos t, x_2 = 2\sin t, 0 \le t \le 2\pi$. So the analyzed solution will be obtained as $u(x) \equiv 1$.

This problem is calculated in paper [24] by Nyström method. The results is listed in Table 1 and it shows that the convergent rate is three order. The denotes in Table 1 represent the following means: $t_i = 2i\pi/10$, with $i = 1, \dots, 10$; e_i is the errors at t_i; and $\texttt{rate} = \log_2 \frac{e_i(h)}{e_i(h/2)}$.

Table 1: Errors of the Nyström solutions in paper [24].

t_i	e_i with $h = \frac{2\pi}{10}$	e_i with $h = \frac{2\pi}{20}$	rate	e_i with $h = \frac{2\pi}{40}$	rate
0.628319	0.121881E-02	0.131313E-03	3.21	0.162984E-04	3.01
1.256637	-0.241312E-02	-0.350908E-03	2.78	-0.439971E-04	3.00
1.884956	-0.241325E-02	-0.350658E-03	2.78	-0.442431E-04	2.99
2.513274	0.121870E-02	0.131397E-03	3.21	0.162478E-04	3.02
3.141593	0.163276E-02	0.189617E-03	3.11	0.236295E-04	3.00
3.769911	0.121862E-02	0.132229E-03	3.20	0.171674E-04	2.95
4.398230	-0.241311E-02	-0.351662E-03	2.78	-0.435820E-04	3.01
5.026548	-0.241343E-02	-0.351146E-03	2.78	-0.437735E-04	3.00
5.654867	0.121874E-02	0.131003E-03	3.22	0.163326E-04	3.00
6.283185	0.163256E-02	0.189412E-03	3.11	0.236443E-04	3.00

We calculate the boundary numerical solutions u_h on Γ following Eq. (12). The boundary is divided into $5 * 2^n$ with $n = 0, 1, 2, \dots$ pieces For convenience, we introduce some denotes:

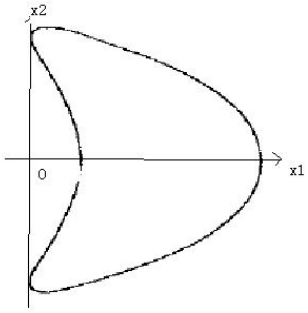

Figure 1: Boomerang-shaped domain for numerical example 2.

$e^h(P) = |u_h(P) - u(P)|$ is the error of the displacement; $r^h(P) = \log_2 e^h(P)/e^{h/2}(P)$ is the error ratio; $\bar{e}^h(P) = |u_h^*(P) - u(P)|$ is the error after Richardson extrapolation, and $p^h(P) = \frac{1}{31}|u_{h/2}(P) - u_h(P)|$ is a posteriori error estimate.

Table 2 lists the approximate values of $u_h(P)$ at points $P_1 = (a\cos 0, b\sin 0)$, $P_2 = (a\cos(\pi/5), b\sin(\pi/5))$ and $P_3 = (a\cos(2\pi/5), b\sin(2\pi/5))$.

Table 2: The errors, errors ratio of e^h, r^h and a posteriori error estimate p^h, at points $P = P_1, P_2, P_3$.

n	5	10	20	40	80
$e^h(P_1)$	2.043E-04	6.117E-06	1.848E-07	5.634E-09	1.728E-10
$r^h(P_1)$		5.062	5.049	5.036	5.027
$p^h(P_1)$		6.133E-06	1.852E-07	5.634E-09	1.727E-10
$e^h(P_2)$	7.203E-04	2.126E-05	6.419E-07	1.959E-08	6.061Ee-10
$r^h(P_2)$		5.102	5.050	5.034	5.014
$p^h(P_2)$		2.226E-05	6.428E-07	1.966E-08	6.061Ee-10
$e^h(P_3)$	4.726E-04	1.378E-05	4.096E-07	1.256E-08	3.886E-10
$r^h(P_3)$		5.100	5.073	5.028	5.014
$p^h(P_3)$		1.389E-05	4.106E-07	1.257E-08	3.886E-10

From Table 2, we can numerically see $r^h \approx 5$, that means the convergent rate is almost five order, which agrees with Theorem 3 very well.

Table 3. the errors $e^h(\theta), \tilde{e}^h(\theta)$ and errors ratio $r^h(\theta)$ when $\theta_1 = 0, \theta_2 = \pi/5$ on Γ.

n	5	10	20	40	80
$e^h(\theta_1)$	6.138E-4	1.779E-5	5.264E-7	1.585E-8	4.863E-10
$r^h(\theta_1)$		5.109	5.079	5.053	5.027
$p^h(\theta_1)$		1.791E-5	5.288E-7	1.585E-8	4.854E-10
$e^h(\theta_2)$	5.413E-4	1.574E-5	4.690E-7	1.413E-8	4.351E-10
$r^h(\theta_2)$		5.104	5.068	5.052	5.022
$p^h(\theta_2)$		1.632E-5	4.728E-7	1.403E-8	4.350E-10

Example 2[15]: Consider another boundary value problem with a non-convex boomerang-shaped cross section boundary. Similar problem is discussed for Helohmotz equation with nonlinear boundary condition in the same domain in paper [15]. The boundary Γ is illustrated in Fig.1 and described by the parametric representation:

$$x(t) = (x_1(t), x_2(t)) = (\cos t + 0.65\cos 2t + 0.65, 1.5\sin t), \quad 0 \le t \le 2\pi.$$

We set $c = 2$ and $f = (1.5\cos t + \sin t + 1.3\sin 2t)/\sqrt{w} + 2(\cos(t) + 0.65\cos(2t) + 1.5\sin(t))$ with $w = (1.5\cos t)^2 + (\sin t + 1.3\sin 2t)^2$. Then the analytic solution is $u(t) = x_1(t) + x_2(t) = \cos t + 0.65\cos 2t + 0.65 + 1.5\sin t, t \in [0, 2\pi]$.

In Table 3 we list some errors of the $u_h(y)$ on Γ computed by formulae (14) and then the u_h at arbitrary point in Ω can be obtained following Eq.(15). We also use the denotes as used in Table 1. Evidently, from Table 3, a similar conclusion can be obtained as example 1 done.

Conclusion

Generally, there are three main advantages for the Sidi's quadrature method:

(1) Evaluating each element of discretization matrices is very simple and straightforward, which does not require any singular integrals;

(2)We can obtain a high accuracy order $O(h^5)$ and an asymptotic expansion of the errors with odd powers, which are based on fine grid h. Harnessing the Richardson extrapolation algorithms, a higher accuracy order $O(h^7)$ can be obtained.

(3)The accuracy order $O(h^5)$ of the approximate solution is obtained directly, which avoid the errors derived from the extrapolation algorithms as some articles have done.

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Sufficient conditions for univalence obtained by using first order nonlinear strong differential subordinations

Georgia Irina Oros
Department of Mathematics, University of Oradea
Str. Universităţii, No.1, 410087 Oradea, Romania
E-mail: georgia_oros_ro@yahoo.co.uk

Abstract

The concept of differential subordination was introduced in [3] by S.S. Miller and P.T. Mocanu and the concept of strong differential subordination was introduced in [1] by J.A. Antonino and S. Romaguera. This last concept was applied in the special case of Briot-Bouquet strong differential subordination. In [5] the authors have developed the general theory of strong differential subordinations following the general theory introduced in [3]. In [6], the special case of first order linear strong differential subordinations was studied. Now, we study another special case, the first order nonlinear strong differential subordinations.

Keywords: analytic function, differential subordination, strong subordination, first order linear, first order nonlinear.
2000 Mathematical Subject Classification: 30C45, 34A30.

1 Introduction

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in U. For n a positive integer and $a \in \mathbb{C}$, let $\mathcal{H}[a,n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\}$. Let A be the class of functions f of the form $f(z) = z + a_2z^2 + a_3z^3 + \dots,\ z \in U$.

In adition, we need the classes of convex, alpha-convex, close-to-convex and starlike (univalent) functions given respectively by $K = \{f \in A : \mathrm{Re}\, zf''(z)/f'(z) + 1 > 0\}$, $M_\alpha = \{f \in A : \frac{f(z)f'(z)}{z} \neq 0,\ \mathrm{Re}\,(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in U\}$, $C = \{f \in A : \mathrm{Re}\, f'(z) > 0,\ z \in U\}$, and $S^* = \{f \in A : \mathrm{Re}\, zf'(z)/f(z) > 0\}$.

Definition 1.1 *[1], [2], [3] Let $H(z,\xi)$ be analytic in $U \times \overline{U}$ and $f(z)$ analytic and univalent in U. The function $H(z,\xi)$ is strongly subordinate to $f(z)$, written $H(z,\xi) \prec\prec f(z)$ if for each $\xi \in \overline{U}$, $H(z,\xi)$ is subordinate to $f(z)$.*

Remark 1.1 *(i) Since $f(z)$ is analytic and univalent Definition 1.1 is equivalent to $H(0,\xi) = f(0)$ and $H(U \times \overline{U}) \subset f(U)$.*

(ii) If $H(z,\xi) \equiv H(z)$ then the strong subordination becomes the usual notion of subordination.

Definition 1.2 *[4], [5, Definition 2.2.b, p. 21] We denote by Q the set of functions q that are analytic and injective in $\overline{U} \setminus E(q)$, where $E(q) = \left\{\zeta \in \partial U;\ \lim\limits_{z\to\zeta} q(z) = \infty\right\}$ and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.*

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Definition 1.3 *[5, Definition 4] Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n a positive integer. The class of admissible functions $\psi_n[\Omega,q]$ consists of those functions $\psi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\psi(r,s;z,\xi) \notin \Omega \qquad \text{(A)}$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $z \in U$, $\xi \in \overline{U}$, $\zeta \in \partial U \setminus E(f)$ and $m \geq n \geq 1$.

Remark 1.2 *The function* $q(z) = M\frac{Mz+a}{M+\overline{a}z}$, *with* $M > 0$ *and* $|a| < M$, *satisfies* $\Delta = q(U) = U_M = U(0, M)$, $q(0) = a$, $E(q) = \emptyset$ *and* $q \in Q$. *If* $a = 0$, *then (A) simplifies to*

$$\psi(Me^{i\theta}, Ke^{i\theta}; z, \xi) \notin \Omega \tag{A'}$$

whenever $K \geq nM$, $z \in U$, $\xi \in \overline{U}$ *and* $\theta \in \mathbb{R}$.

Remark 1.3 *The function* $q(z) = \frac{a+\overline{a}z}{1-z}$ *with* Re $a > 0$, *satisfies* $q(U) = \Delta$, $q(0) = a$, $E(q) = \{1\}$ *and* $q \in Q$. *If* $a = 1$, *then (A) simplifies to*

$$\psi(\rho i, \sigma, z, \xi) \notin \Omega, \tag{A''}$$

when $\rho, \sigma \in \mathbb{R}$, $\sigma \leq -\frac{n}{2}(1 + \rho^2)$, $z \in U$, $\xi \in \overline{U}$ *and* $n \geq 1$.

Lemma 1.1 *[3], [4, Lemma 2.2.d, p. 24] Let* $q \in Q(a)$, *with* $q(0) = a$ *and* $p(z) = a + a_n z^n + \ldots$ *analytic in* U, *with* $p(z) \not\equiv a$, $n \geq 1$. *If* p *is not subordinate to* q, *then there exist points* $z_0 = r_0 e^{i\theta_0} \in U$ *and* $\zeta_0 \in \partial U \setminus E(q)$, *and an* $m \geq n \geq 1$ *for which* $p(U_{r_0}) \subset q(U)$

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$.

Definition 1.4 *[6] A strong differential subordination of the form* $A(z,\xi)zp'(z) + B(z,\xi)p(z) \prec\prec h(z)$, $z \in U$, $\xi \in \overline{U}$, *where* $A(z,\xi)zp'(z) + B(z,\xi)p(z)$ *is analytic in* U *for all* $\xi \in \overline{U}$ *and* h *is an analytic and univalent function in* U, *is called first order linear strong differential subordination.*

2 Main results

Definition 2.1 *A strong differential subordination of the form*

$$A(z,\xi)zp'(z) + B(z,\xi)p(z) + C(z,\xi)p^2(z) + D(z,\xi) \prec\prec h(z), \tag{1}$$

where $A(z,\xi)zp'(z) + B(z,\xi)p(z) + C(z,\xi)p^2(z) + D(z,\xi)$ *is analytic in* U *for all* $\xi \in \overline{U}$ *and* h *is an analytic and univalent function in* U, *is called first order nonlinear strong differential subordination.*

Remark 2.1 *If* $C(z,\xi) = D(z,\xi) = 0$ *then (1) becomes a linear strong differential subordination studied in [6].*

Remark 2.2 *If* $A(z,\xi) = A(z)$, $B(z,\xi) = B(z)$, $C(z,\xi) = C(z)$, $D(z,\xi) = D(z)$ *then (1) becomes a nonlinear differential subordination studied in [7].*

Next, we find conditions for the functions p, A, B, C, D and h such that (1) holds.

Theorem 2.1 *Let* $p \in \mathcal{H}[0,n]$, $A, B, C : U \times \overline{U} \to C$ *with*

$$\text{Re } A(z,\xi) \geq 0, \quad \text{Re } [A(z,\xi) + B(z,\xi)] \geq 1 + M|C(z,\xi)| \tag{2}$$

and $A(z,\xi)zp'(z) + B(z,\xi)p(z) + C(z,\xi)p^2(z)$ *an analytic function in* U *for all* $\xi \in \overline{U}$. *Then*

$$A(z,\xi)zp'(z) + B(z,\xi)p(z) + C(z,\xi)p^2(z) \prec\prec Mz \tag{3}$$

implies $p(z) \prec Mz$, $M > 0$, $z \in U$.

Proof. Let $\psi(r,s;z,\xi) : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ given by Definition 1.3. For $r = p(z)$, $s = zp'(z)$, $z \in U$ we have

$$\psi(r,s;z,\xi) = A(z,\xi)s + B(z,\xi)s + C(z,\xi)r^2. \tag{4}$$

Then (3) becomes

$$\psi(r,s;z,\xi) \prec\prec Mz, \quad z \in U, \ \xi \in \overline{U}. \tag{5}$$

If we consider $h(z) = Mz$, $M > 0$ then $h(U) = U(0,M)$ and (5) is equivalent to

$$\psi(r,s;z,\xi) \in U(0,M), \quad z \in U, \ \xi \in \overline{U}. \tag{6}$$

Suppose that p is not subordinated to $h(z) = Mz$. Then, from Lemma 1.1, we have that there exist $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$, $\theta_0 \in \mathbb{R}$ and $\zeta_0 \in \partial U$ with $|\zeta_0| = 1$, such that $p(z_0) = h(\zeta_0) = Me^{i\theta_0}$, $z_0 p'(z_0) = m\zeta_0 h'(\zeta_0) = Ke^{i\theta_0}$, $K \geq nM$.

By replacing r with $p(z_0) = h(\zeta_0) = Me^{i\theta_0}$ and s with $z_0p'(z_0) = m\zeta_0 h'(\zeta_0) = Ke^{i\theta_0}$ in (4) and taking into account the conditions from (2), we have
$|\psi(p(z_0), z_0p'(z_0); z_0, \xi)| = |\psi(Me^{i\theta_0}, Ke^{i\theta_0}; z_0, \xi)| = |A(z_0, \xi)Ke^{i\theta_0} + B(z_0, \xi)Me^{i\theta_0} + C(z_0, \xi)M^2e^{2i\theta_0}| = |A(z_0, \xi)K+B(z_0, \xi)M+C(z_0, \xi)M^2e^{2i\theta_0}| \geq |A(z_0, \xi)K+B(z_0, \xi)M|-M^2|C(z_0, \xi)| \geq \mathrm{Re}\ [A(z_0, \xi)K+B(z_0, \xi)M]-M^2|C(z_0, \xi)| \geq K\mathrm{Re}\ A(z_0, \xi) + M\mathrm{Re}\ B(z_0, \xi) - M^2|C(z_0, \xi)| \geq nM\mathrm{Re}\ A(z_0, \xi) + M\mathrm{Re}\ B(z_0, \xi) - M^2|C(z_0, \xi)| \geq M\mathrm{Re}\ [A(z_0, \xi) + B(z_0, \xi)] - M^2|C(z_0, \xi)| \geq M,$
which contradicts (6). This means the assumption made is false, hence $p(z) \prec Mz$, $M > 0$, $z \in U$. ■

Example 2.1 *Let* $A(z, \xi) = z + \xi + 4$, $B(z, \xi) = 3z - 2\xi + 12 - 8i$, $C(z, \xi) = 2z - 3\xi + 1 - \sqrt{3}i$, $M = \frac{1}{2}$. *Since* $z \in U$, $\xi \in \overline{U}$, *we have* $\mathrm{Re}\ A(z, \xi) \geq 2$, $\mathrm{Re}\ B(z, \xi)| \geq 7$, $|C(z, \xi)| \leq 16$, $\mathrm{Re}\ [A(z, \xi) + B(z, \xi)] \geq 9$.

From Theorem 2.1, we obtain: If $p \in [0, n]$, $n \in \mathbb{N}$, *and* $(z + \xi + 4)zp'(z) + (3z - 2\xi + 12 - 8i)p(z) + (2z - 2\xi + 1 - \sqrt{3}i)p^2(z)$ *is a function of* z, *analytic in* U *for all* $\xi \in \overline{U}$, *then* $(z+\xi+4)zp'(z)+(3z-2\xi+12-8i)p(z) +(2z - 3\xi + i - \sqrt{3}i)p^2(z) \prec\prec \frac{z}{2}$, $z \in U$, $\xi \in \overline{U}$, *implies* $p(z) \prec \frac{z}{2}$, $z \in U$.

Theorem 2.2 *Let* $p \in [0, n]$, $A, B, C, D : U \times \overline{U} \to C$ *with*

$$\mathrm{Re}\ A(z, \xi) \geq 0, \quad \mathrm{Re}\ C(z, \xi) \geq 0, \quad \frac{n}{2}\mathrm{Re}\ A(z, \xi) \geq \mathrm{Re}\ D(z, \xi) \tag{7}$$

and $\mathrm{Im}\ B(z, \xi) \leq 2\sqrt{\left[\frac{n}{2}\mathrm{Re}\ A(z, \xi) + \mathrm{Re}\ C(z, \xi)\right] \left[\frac{n}{2}\mathrm{Re}\ A(z, \xi) - \mathrm{Re}\ D(z, \xi)\right]}$.

If $A(z, \xi)zp'(z)+B(z, \xi)p(z)+C(z, \xi)p^2(z)+D(z, \xi)$ *is analytic in* U *for all* $\xi \in \overline{U}$ *and satisfies the inequality*

$$\mathrm{Re}\ [A(z, \xi)zp'(z) + B(z, \xi)p(z) + C(z, \xi)p^2(z) + D(z, \xi)] > 0 \tag{8}$$

then $\mathrm{Re}\ p(z) > 0$, $z \in U$.

Proof. Let $\psi(r, s; z, \xi) : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ given by Definition 1.3. For $r = p(z)$, $s = zp'(z)$, $z \in U$ we have

$$\psi(r, s; z, \xi) = A(z, \xi)s + B(z, \xi)r + C(z, \xi)r^2 + D(z, \xi), \quad z \in U, \ \xi \in \overline{U}. \tag{9}$$

Then (8) becomes

$$\mathrm{Re}\ \psi(r, s; z, \xi) > 0, \quad z \in U, \ \xi \in \overline{U}. \tag{10}$$

If we consider $h(z) = \frac{1+z}{1-z}$ then $h(U) = \{w \in \mathbb{C};\ \mathrm{Re}\ w > 0\}$ and (10) is equivalent to

$$\psi(r, s; z, \xi) \prec\prec \frac{1+z}{1-z}, \quad z \in U, \ \xi \in \overline{U}. \tag{11}$$

Suppose that p is not subordinated to $h(z) = \frac{1+z}{1-z}$. Then, from Lemma 1.1, we have that there exist $z_0 = r_0e^{i\theta_0}$, $\theta_0 \in \mathbb{R}$ and $\zeta_0 \in \partial U$ such that $p(z_0) = h(\zeta_0) = \rho i$, $\rho \in \mathbb{R}$, $z_0p'(z_0) = m\zeta_0h'(\zeta_0) = \sigma$, $\sigma \in \mathbb{R}$, $\sigma \leq -\frac{n}{2}(1 + \rho^2)$.

By replacing r with ρi and s with σ in (9) and using the conditions given by (7) we obtain
$\mathrm{Re}\ \psi(p(z_0), z_0p'(z_0); z_0, \xi) = \mathrm{Re}\ \psi(\rho i, \sigma; z_0, \xi) = \mathrm{Re}\ [A(z_0, \xi)\sigma+B(z_0, \xi)\rho i-\rho^2C(z_0, \xi)+D(z_0, \xi)] = \sigma\mathrm{Re}\ A(z_0, \xi)-\rho\mathrm{Im}\ B(z_0, \xi)-\rho^2\mathrm{Re}\ C(z_0, \xi)+\mathrm{Re}\ D(z_0, \xi) \geq -\frac{n}{2}(1+\rho^2)\mathrm{Re}\ A(z_0, \xi)-\rho\mathrm{Im}\ B(z_0, \xi)-\rho^2\mathrm{Re}\ C(z_0, \xi)+\mathrm{Re}\ D(z_0, \xi) \geq -\rho^2\left[\frac{n}{2}\mathrm{Re}\ A(z_0, \xi) + \mathrm{Re}\ C(z_0, \xi)\right] - \rho\mathrm{Im}\ B(z_0, \xi) - \frac{n}{2}\mathrm{Re}\ A(z_0, \xi) + \mathrm{Re}\ D(z_0, \xi) \leq 0,$
which contradicts (10). This means the assumption made is false, hence $p(z) \prec \frac{1+z}{1-z}$, $z \in U$, which is equivalent to $\mathrm{Re}\ p(z) > 0$, $z \in U$. ■

Theorem 2.3 *Let* $p \in \mathcal{H}[1, n]$, $A, B, C, D : U \times \overline{U} \to C$ *with*

$$\mathrm{Re}\ A(z, \xi) \geq 0, \quad \mathrm{Re}\ C(z, \xi) \geq 0, \quad \frac{n}{2}\mathrm{Re}\ A(z, \xi) \geq \mathrm{Re}\ D(z, \xi) + 1 \tag{12}$$

and $\mathrm{Im}\, B(z, \xi) \leq 2\sqrt{\left[\frac{n}{2}\mathrm{Re}\, A(z, \xi)+\mathrm{Re}\, C(z, \xi)\right]\left[\frac{n}{2}\mathrm{Re}\, A(z, \xi)-\mathrm{Re}\, D(z, \xi)-1\right]}$.

If $A(z, \xi)zp'(z)+B(z, \xi)p(z)+C(z, \xi)p^2(z)+D(z, \xi)$ *is analytic in* U *for all* $\xi \in \overline{U}$ *and satisfies the nonlinear strong differential subordination*

$$A(z, \xi)zp'(z) + B(z, \xi)p(z) + C(z, \xi)p^2(z) + D(z, \xi) \prec\prec z \tag{13}$$

then $p(z) \prec \frac{1+z}{1-z}$, $z \in U$.

Proof. Let ψ given by (9). For $r = p(z)$, $s = zp'(z)$, (13) becomes

$$\psi(r, s; z, \xi) \prec\prec z, \quad z \in U,\ \xi \in \overline{U}. \tag{14}$$

If we consider $h(z) = z$, $z \in U$, then $q(U) = U$ and from (14) we have

$$\psi(r, s; z, \xi) \in U, \quad z \in U,\ \xi \in \overline{U}, \tag{15}$$

which is equivalent to

$$-1 < \text{Re } \psi(r, s; z, \xi) < 1, \quad z \in U,\ \xi \in \overline{U}. \tag{16}$$

Suppose that p is not subordinated to $q(z) = \frac{1+z}{1-z}$. Then, from Lemma 1.1 we have that there exist $z_0 = r_0 e^{i\theta_0}$, $\theta_0 \in \mathbb{R}$ and $\zeta_0 \in \partial U$, such that $p(z_0) = q(\zeta_0) = \rho i$, $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) = \sigma$, $\sigma \in \mathbb{R}$, $\sigma \leq -\frac{n}{2}(1+\rho^2)$.

By replacing r with ρi and s with σ in (9) and using the conditions given by (12), we have:

Re $\psi(r, s; z_0, \xi) =$ Re $\psi(\rho i, \sigma; z_0, \xi) =$ Re $[A(z_0, \xi)\sigma + B(z_0, \xi)\rho i - C(z_0, \xi)\rho^2 + D(z_0, \xi)] = \sigma$Re $A(z_0, \xi) -$ ρIm $B(z_0, \xi) - \rho^2$Re $C(z_0, \xi) +$ Re $D(z_0, \xi) \leq -\frac{n}{2}(1+\rho^2)$Re $A(z_0, \xi) - \rho$Im $B(z_0, \xi) - \rho^2$Re $C(z_0, \xi) +$ Re $D(z_0, \xi)$ $\leq -\rho^2 \left[\frac{n}{2}\text{Re } A(z_0, \xi) + \text{Re } C(z_0, \xi)\right] - \rho\text{Im } B(z_0, \xi) - \frac{n}{2}\text{Re } A(z_0, \xi) + \text{Re } D(z_0, \xi) \leq -1$,
which contradicts (15). That means the assumption made was false, hence $p(z) \prec \frac{1+z}{1-z}$, $z \in U$. ■

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A note on the symmetric properties for the second kind twisted q-Euler polynomials

C. S. Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper, we introduce the second kind twisted q-Euler numbers and polynomials. By using these numbers and polynomials, we give some interesting relations between the power sums and the the second kind twisted Euler polynomials.

Key words : the second kind Euler numbers and polynomials, the second kind twisted Euler numbers and polynomials, the second kind twisted q-Euler numbers and polynomials, alternating sums

1. Introduction

Euler numbers, Euler polynomials, q-Euler numbers, q-Euler polynomials, the second kind Euler number and the second kind Euler polynomials were studied by many authors. Euler numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics(see for details [1-9]). In this paper, we introduce the second kind twisted q-Euler numbers and polynomials. In this paper, by using the symmetry of p-adic q-integral on $\mathbb{Z}_p$, we give recurrence identities the second twisted q-Euler polynomials and the power sums.

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $g \in UD(\mathbb{Z}_p)$, if the difference quotients

$$F_g(x,y) = \frac{g(x) - g(y)}{x - y}$$

have a limit $l = g'(a)$ as $(x,y) \to (a,a)$. For $g \in UD(\mathbb{Z}_p)$, Kim defined the fermionic p-adic integral on $\mathbb{Z}_p$ (see [1])

$$I_{-1}(g) = \lim_{q \to -1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{0 \le x < p^N} g(x)(-1)^x. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{1.2}$$

Let $T_p = \cup_{N \ge 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{\omega | \omega^{p^N} = 1\}$ is the cyclic group of order p^N. For $\omega \in T_p$, we denote by $\phi_\omega : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \omega^x$.

Let us define the second kind twisted q-Euler numbers $E_{n,q,\omega}$ and polynomials $E_{n,q,\omega}(x)$ as follows:

$$I_{-1}(\phi_\omega(y) q^y e^{(2y+1)t}) = \int_{\mathbb{Z}_p} \phi_\omega(y) q^y e^{(2y+1)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q,\omega} \frac{t^n}{n!}, \tag{1.3}$$

$$I_{-1}(\phi_\omega(y) q^y e^{(2y+1+x)t}) = \int_{\mathbb{Z}_p} \phi_\omega(y) q^y e^{(2y+1+x)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q,\omega} x) \frac{t^n}{n!}. \tag{1.4}$$

By (1.3) and (1.4), we obtain the following Witt's formula.

Theorem 1. For $\omega \in T_p$, we have

$$\int_{\mathbb{Z}_p} \phi_\omega(x) q^x (2x+1)^n d\mu_{-1}(x) = E_{n,q,\omega},$$

$$\int_{\mathbb{Z}_p} \phi_\omega(y) q^y (2y+1+x)^n d\mu_{-1}(y) = E_{n,q,\omega}(x).$$

Theorem 2. For any positive integer n, we have

$$E_{n,q,w}(x) = \sum_{k=0}^{n} \binom{n}{k} E_{k,q,w} x^{n-k}.$$

2. The alternating sums of powers of consecutive q-odd integers

In this section, we assume that $q \in \mathbb{C}$, with $|q| < 1$. Let ω be the p^N-th root of unity. By using (1.4), we give the alternating sums of powers of consecutive q-integers as follows:

$$\sum_{n=0}^{\infty} E_{n,q,\omega} \frac{t^n}{n!} = \frac{2e^t}{\omega q e^{2t}+1} = 2\sum_{n=0}^{\infty} (-1)^n \omega^n q^n e^{(2n+1)t}.$$

From the above, we obtain

$$-\sum_{n=0}^{\infty} (-1)^n \omega^n q^n e^{(2n+2k+1)t} + \sum_{n=0}^{\infty} (-1)^{n-k} \omega^{n-k} q^{(n-k)} e^{(2n+1)t} = \sum_{n=0}^{k-1} (-1)^{n-k} \omega^{n-k} q^{(n-k)} e^{(2n+1)t}.$$

By using (1.3)and (1.4), we obtain

$$-\frac{1}{2}\sum_{j=0}^{\infty} E_{j,q,\omega}(2k)\frac{t^j}{j!} + \frac{1}{2}(-1)^{-k}\omega^{-k}q^{-k}\sum_{j=0}^{\infty} E_{j,q,\omega}\frac{t^j}{j!} = \sum_{j=0}^{\infty}\left((-1)^{-k}\omega^{-k}q^{-k}\sum_{n=0}^{k-1}(-1)^n\omega^n q^n (2n+1)^j\right)\frac{t^j}{j!}.$$

By comparing coefficients $\frac{t^j}{j!}$ in the above equation, we obtain

$$\sum_{n=0}^{k-1} (-1)^n \omega^n q^n (2n+1)^j = \frac{(-1)^{k+1}\omega^k q^k E_{j,q,\omega}(2k) + E_{j,q,\omega}}{2}.$$

By using the above equation we arrive at the following theorem:

Theorem 3. Let k be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$ and ω be the p^N-th root of unity. Then we obtain

$$T_{j,q,\omega}(k-1) = \sum_{n=0}^{k-1} (-1)^n \omega^n q^n (2n+1)^j = \frac{(-1)^{k+1}\omega^k q^k E_{j,q,\omega}(2k) + E_{j,q,\omega}}{2}. \tag{2.1}$$

Remark 4. For the alternating sums of powers of consecutive odd integers, we have

$$\lim_{q\to 1} T_{j,q,\omega}(k-1) = \sum_{n=0}^{k-1} (-1)^n \omega^n (2n+1)^j = \frac{(-1)^{k+1}\omega^k E_{j,\omega}(2k) + E_{j,\omega}}{2},$$

where $E_{j,\omega}(x)$ and $E_{j,\omega}$ denote the second kind twisted Euler polynomials and the second kind twisted Euler numbers, respectively (see [5]).

3. The symmetry property of the q-deformed fermionic integral on $\mathbb{Z}_p$

In this section, we assume that $q \in \mathbb{C}_p$ and $\omega \in T_p$. In this section, we obtain recurrence identities the second twisted q-Euler polynomials and the alternating sums of powers of consecutive q-odd integers. By using (1.1), we have

$$I_{-1}(g_n) + (-1)^{n-1} I_{-1}(g) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k} g(k), \text{ (see [1], [2], [3], [5])},$$

where $n \in \mathbb{N}, g_n(x) = g(x+n)$. If n is odd from the above, we obtain

$$I_{-1}(g_n) + I_{-1}(g) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k} g(k). \tag{3.1}$$

It will be more convenient to write (3.1) as the equivalent integral form

$$\int_{\mathbb{Z}_p} g(x+n) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k} g(k). \tag{3.2}$$

Substituting $g(x) = \omega^x q^x e^{(2x+1)t}$ into the above, we obtain

$$\int_{\mathbb{Z}_p} \omega^{x+n} q^{x+n} e^{(2(x+n)+1)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x e^{(2x+1)t} d\mu_{-1}(x) = 2\sum_{j=0}^{n-1}(-1)^j \omega^j q^j e^{(2j+1)t}. \tag{3.3}$$

After some elementary calculations, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \omega^x q^x e^{(2x+1)t} d\mu_{-1}(x) &= \frac{2e^t}{\omega q e^{2t} + 1}, \\ \int_{\mathbb{Z}_p} \omega^{x+n} q^{x+n} e^{(2(x+n)+1)t} d\mu_{-1}(x) &= \omega^n q^n e^{2nt} \frac{2e^t}{\omega q e^{2t} + 1}. \end{aligned} \tag{3.4}$$

By using (3.3) and (3.4), we have

$$\int_{\mathbb{Z}_p} \omega^{x+n} q^{x+n} e^{(2(x+n)+1)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x e^{(2x+1)t} d\mu_{-1}(x) = \frac{2e^t(1+\omega^n q^n e^{2nt})}{\omega q e^{2t} + 1}.$$

From the above, we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} \omega^{x+n} q^{x+n} e^{(2(x+n)+1)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x e^{(2x+1)t} d\mu_{-1}(x) \\ &= \frac{2\int_{\mathbb{Z}_p} \omega^x q^x e^{(2x+1)t} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} \omega^{nx} q^{nx} e^{2ntx} d\mu_{-1}(x)}. \end{aligned} \tag{3.5}$$

By substituting Taylor series of $e^{(2x+1)t}$ into (3.3), we obtain

$$\begin{aligned} &\sum_{m=0}^{\infty}\left(\int_{\mathbb{Z}_p} \omega^{x+n} q^{x+n} (2x+1+2n)^m d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x (2x+1)^m d\mu_{-1}(x)\right)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(2\sum_{j=0}^{n-1}(-1)^j \omega^j q^j (2j+1)^m\right)\frac{t^m}{m!} \end{aligned}$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation, we obtain

$$\begin{aligned} &\omega^n q^n \sum_{k=0}^{m}\binom{m}{k}(2n)^{m-k}\int_{\mathbb{Z}_p} \omega^x q^x (2x+1)^k d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x (2x+1)^m d\mu_{-1}(x) \\ &= 2\sum_{j=0}^{n-1}(-1)^j \omega^j q^j (2j+1)^m \end{aligned}$$

By using (2.1), we have

$$\begin{aligned}&\omega^n q^n \sum_{k=0}^{m}\binom{m}{k}(2n)^{m-k}\int_{\mathbb{Z}_p}\omega^x q^x(2x+1)^k d\mu_{-1}(x)+\int_{\mathbb{Z}_p}\omega^x q^x(2x+1)^m d\mu_{-1}(x)\\&=2T_{m,q,\omega}(n-1).\end{aligned}\tag{3.6}$$

By using (3.5) and (3.6), we arrive at the following theorem:

Theorem 5. Let n be odd positive integer. Then we obtain

$$\frac{2\int_{\mathbb{Z}_p}\omega^x q^x e^{(2x+1)t}d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}\omega^{nx}q^{nx}e^{2ntx}d\mu_{-1}(x)}=\sum_{m=0}^{\infty}\left(2T_{m,q,\omega}(n-1)\right)\frac{t^m}{m!}.\tag{3.7}$$

Let w_1 and w_2 be odd positive integers. By using (3.7), we have

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\omega^{(w_1x_1+w_2x_2)}q^{(w_1x_1+w_2x_2)}e^{(w_1(2x_1+1)+w_2(2x_2+1)+w_1w_2x)t}d\mu_{-1}(x_1)d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}\omega^{w_1w_2x}q^{w_1w_2x}e^{2w_1w_2xt}d\mu_{-1}(x)}\\&=\frac{2e^{w_1t}e^{w_2t}e^{w_1w_2xt}(\omega^{w_1w_2}q^{w_1w_2}e^{2w_1w_2t}+1)}{(\omega^{w_1}q^{w_1}e^{2w_1t}+1)(\omega^{w_2}q^{w_2}e^{2w_2t}+1)}\end{aligned}\tag{3.8}$$

By using (3.7) and (3.8), after elementary calculations, we obtain

$$\begin{aligned}a&=\left(\frac{1}{2}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{(w_1(2x_1+1)+w_1w_2x)t}d\mu_{-1}(x_1)\right)\left(\frac{2\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{(2x_2+1)(w_2t)}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}\omega^{w_1w_2x}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}\sum_{m=0}^{\infty}E_{m,q^{w_1},\omega^{w_1}}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}T_{m,q^{w_2},\omega^{w_2}}(w_1-1)w_2^m\frac{t^m}{m!}\right).\end{aligned}\tag{3.9}$$

By using Cauchy product in the above, we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}E_{j,q^{w_1},\omega^{w_1}}(w_2x)w_1^jT_{m-j,q^{w_2},\omega^{w_2}}(w_1-1)w_2^{m-j}\right)\frac{t^m}{m!}\tag{3.10}$$

By using the symmetry in (3.9), we have

$$\begin{aligned}a&=\left(\frac{1}{2}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{(w_2(2x_2+1)+w_1w_2x)t}d\mu_{-1}(x_2)\right)\left(\frac{2\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{(2x_1+1)(w_1t)}d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}\omega^{w_1w_2x}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}\sum_{m=0}^{\infty}E_{m,q^{w_2},\omega^{w_2}}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}T_{m,q^{w_1},\omega^{w_1}}(w_2-1)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}E_{j,q^{w_2},w^{w_2}}(w_1x)w_2^jT_{m-j,q^{w_1},w^{w_1}}(w_2-1)w_1^{m-j}\right)\frac{t^m}{m!}\tag{3.11}$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (3.10) and (3.11), we arrive at the following theorem:

Theorem 6. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}w_1^{m-j}w_2^jE_{j,q^{w_2},\omega^{w_2}}(w_1x)T_{m-j,q^{w_1},\omega^{w_1}}(w_2-1)\\&=\sum_{j=0}^{m}\binom{m}{j}w_1^jw_2^{m-j}E_{j,q^{w_1},\omega^{w_1}}(w_2x)T_{m-j,q^{w_2},\omega^{w_2}}(w_1-1),\end{aligned}$$

where $E_{k,q,\omega}(x)$ and $T_{m,q,\omega}(k)$ denote the second kind twisted q-Euler polynomials and the alternating sums of powers of consecutive q-odd integers, respectively.

By using Theorem 2, we have the following corollary:

Corollary 7. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^{j}x^{j-k}E_{k,q^{w_2},\omega^{w_2}}T_{m-j,q^{w_1},\omega^{w_1}}(w_2-1)\\&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{j}w_2^{m-k}x^{j-k}E_{k,q^{w_1},\omega^{w_1}}T_{m-j,q^{w_2},\omega^{w_2}}(w_1-1).\end{aligned}$$

By using (3.8), we have

$$\begin{aligned}a&=\left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{(2x_1+1)w_1t}d\mu_{-1}(x_1)\right)\left(\frac{2\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{(2x_2+1)(w_2t)}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}\omega^{w_1w_2x}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{(2x_1+1)w_1t}d\mu_{-1}(x_1)\right)\left(2\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2j}e^{(2j+1)(w_2t)}\right)\\&=\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2j}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{\left(2x_1+1+w_2x+(2j+1)\frac{w_2}{w_1}\right)(w_1t)}d\mu_{-1}(x_1)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2j}E_{n,q^{w_1},\omega^{w_1}}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^n\right)\frac{t^n}{n!}.\end{aligned}\tag{3.12}$$

By using the symmetry property in (3.12), we also have

$$\begin{aligned}a&=\left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{(2x_2+1)w_2t}d\mu_{-1}(x_2)\right)\left(\frac{2\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{(2x_1+1)(w_1t)}d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}\omega^{w_1w_2x}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{(2x_2+1)w_2t}d\mu_{-1}(x_2)\right)\left(2\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1j}e^{(2j+1)(w_1t)}\right)\\&=\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1j}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{\left(2x_2+1+w_1x+(2j+1)\frac{w_1}{w_2}\right)(w_2t)}d\mu_{-1}(x_1)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1j}E_{n,q^{w_2},\omega^{w_2}}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^n\right)\frac{t^n}{n!}.\end{aligned}\tag{3.13}$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (3.12) and (3.13), we have the following theorem.

Theorem 8. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2j}E_{n,q^{w_1},\omega^{w_1}}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^n\\&=\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1j}E_{n,q^{w_2},\omega^{w_2}}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^n.\end{aligned}\tag{3.14}$$

Substituting $w_1=1$ into (3.14), we arrive at the following corollary.

Corollary 9. Let w_2 be odd positive integer. Then we obtain

$$E_{n,q,\omega}(x) = w_2^n \sum_{j=0}^{w_2-1} (-1)^j \omega^j q^j E_{n,q^{w_2},\omega^{w_2}} \left(\frac{x - w_2 + (2j+1)}{w_2} \right).$$

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Sufficient conditions for functions to be in a class of p-valent analytic functions

M. Arif[a],M. Ayaz[1,a],W. Haq[a], J. Iqbal[a]

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In this article, we obtain certain simple sufficiency criteria for a subclass of p-valent analytic functions. Many known results appear as special consequences of our work. Some applications of our work to the generalized Alexander integral operator is also given.

Key words: Spiral-like functions, convolution, integral operator.
Subject classification: 30C45, 30C50.

† 1

E-mail addresses: marifmaths(M. Arif),mayazmath@awkum.edu.pk(M.Ayaz), javedmath@yahoo.com(J.Iqbal), wasim474@hotmail.com(W.Haq)

1 Introduction

Let $A(p,n)$ denote the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{k=p+n}^{\infty}a_k z^k, (p\in \mathtt{N}=\{1,2,3,...\}), \tag{1}$$

which are analytic and multivalent in the open unit disk $\mathtt{U}=\{z:|z|<1\}$. For functions $f(z)$, $g(z)\in A(p,n)$ of the form (1), We define the convolution (Hadamard product) of $f(z)$ and $g(z)$ by

$$(f\star g)(z)=z^p+\sum_{k=p+n}^{\infty}a_k b_k z^k,\ (z\in \mathtt{U}). \tag{2}$$

[1]Corresponding author.
a. Department of Mathematics, Abdul Wali Khan University Mardan, Khyber Pakhtunkhwa, Pakistan

Also let $Q_\lambda(p,n,\alpha;g(z))$, λ is real with $|\lambda|<\frac{\pi}{2}$, $0\le\alpha<p$, $n\in\mathbb{N}$ and $p\in\mathbb{N}$, denote the subclass of $A(p,n)$ consisting of all functions $f(z)$ which is defined with the help of convolution by

$$\Re e^{i\lambda}\frac{z\left((f\star g)'(z)\right)}{(f\star g)(z)}>\alpha\cos\lambda,(z\in\mathbb{U}). \tag{3}$$

By suitably choosing $g(z)$ in (3), we obtain the subclasses $S_\lambda^*(p,n,\alpha)$ and $C_\lambda(p,n,\alpha)$ of $A(p,n)$ which are defined, respectively, by

$$\Re e^{i\lambda}\frac{zf'(z)}{f(z)}>\alpha\cos\lambda,(z\in\mathbb{U}), \tag{4}$$

$$\Re e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha\cos\lambda,(z\in\mathbb{U}). \tag{5}$$

We note that for $\lambda=0$, the classes $S_\lambda^*(p,n,\alpha)$ and $C_\lambda(p,n,\alpha)$ reduces to the classes $S_p^*(n,\alpha)$ and $C_p(n,\alpha)$ respectively studied by Goyal et al [1]. Further if we take $\alpha=0$, $p=1$ and $n=1$ in the classes (4) and (5), we obtain the class of spiral-like functions introduced by Spacek [2] and the class of Robertson functions studied by Robertson [3] respectively.

We will assume throughout our discussion, unless otherwise stated, that λ is real with $|\lambda|<\frac{\pi}{2}$, $0\le\alpha<p$, $n\in\mathbb{N}$ and $p\in\mathbb{N}$

2 Sufficient conditions for the class $Q_\lambda(p,n,\alpha;g(z))$

To obtain our main results, we need the following Lemma due to Mocanu [4].

Lemma 2.1. If $q(z)\in A(n)$ satisfies the condition

$$\left|q'(z)-1\right|<\frac{n+1}{\sqrt{(n+1)^2+1}}\quad(z\in\mathbb{U}), \tag{6}$$

then

$$q(z)\in S^*(n,0). \tag{7}$$

Theorem 2.1. If $f(z)\in A(p,n)$ satisfies

$$\left|\left(\frac{(f\star g)(z)}{z^p}\right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}}\left\{e^{i\lambda}\frac{z(f\star g)'(z)}{(f\star g)(z)}-\alpha\cos\lambda-ip\sin\lambda\right\}-(p-\alpha)\cos\lambda\right|$$

$$<\frac{n+1}{\sqrt{(n+1)^2+1}}(p-\alpha)\cos\lambda\ (z\in\mathbb{U}), \tag{8}$$

then $f(z)\in Q_\lambda(p,n,\alpha;g(z))$.

Proof. Let us set a function $p(z)$ by

$$p(z)=z\left(\frac{(f\star g)(z)}{z^p}\right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}}=z+\frac{e^{i\lambda}a_{p+n}b_{p+n}}{(p-\alpha)\cos\lambda}z^{n+1}+\dots \tag{9}$$

for $f(z),g(z)\in A(p,n)$. Then clearly (9) shows that $p(z)\in A(n)$.

Differentiating (9) logarithmically, we have

$$\frac{p'(z)}{p(z)}=\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}\left[\frac{(f\star g)'(z)}{(f\star g)(z)}-\frac{p}{z}\right]+\frac{1}{z} \tag{10}$$

which gives

$$\left|p'(z)-1\right| \tag{11}$$

$$=\left|\left(\frac{(f\star g)(z)}{z^p}\right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}}\frac{1}{(p-\alpha)\cos\lambda}\left\{e^{i\lambda}\frac{z(f\star g)'(z)}{(f\star g)(z)}-\alpha\cos\lambda-ip\sin\lambda\right\}-1\right|. \tag{12}$$

Thus using (8), we have

$$\left|p'(z)-1\right|\leq\frac{n+1}{\sqrt{(n+1)^2+1}},\ (z\in\mathtt{U}). \tag{13}$$

Hence, using Lemma 2.1, we have $p(z)\in S^*(n,0)$.

From (10), we can write

$$\frac{zp'(z)}{p(z)}=\frac{1}{(p-\alpha)\cos\lambda}\left[e^{i\lambda}\frac{z(f\star g)'(z)}{(f\star g)(z)}-(\alpha\cos\lambda+ip\sin\lambda)\right]. \tag{14}$$

Since $p(z)\in S^*(n,0)$, it implies that $\Re\frac{zp'(z)}{p(z)}>0$. Therefore, we get

$$\frac{1}{(p-\alpha)\cos\lambda}\left[\Re\left(e^{i\lambda}\frac{z(f\star g)'(z)}{(f\star g)(z)}\right)-\alpha\cos\lambda\right]=\Re\frac{zp'(z)}{p(z)}>0 \tag{15}$$

or

$$\Re\left(e^{i\lambda}\frac{z(f\star g)'(z)}{(f\star g)(z)}\right)>\alpha\cos\lambda. \tag{16}$$

and this implies that $f(z)\in Q_\lambda(p,n,\alpha;g(z))$. By taking $g(z)$ is an identity function and Koebe p-valent functions with $\lambda=0$ in Theorem 2.1, we obtain Corollary 2.2 and Corollary 2.3 respectively proved by Goyal et.al [1].

Corollary 2.2. If $f(z)\in A(p,n)$ satisfies

$$\left|\left(\frac{f(z)}{z}\right)^{\frac{1}{p-\alpha}}\left\{z^{\frac{1-\alpha}{p-\alpha}}\frac{f'(z)}{f(z)}-\alpha z^{\frac{1-p}{p-\alpha}}\right\}-p+\alpha\right|<\frac{n+1}{\sqrt{(n+1)^2+1}}(p-\alpha)\ (z\in\mathtt{U}), \tag{17}$$

for $0\leq\alpha<p$, then $f(z)\in S_p^*(n,\alpha)$.

Corollary 2.3. If $f(z)\in A(p,n)$ satisfies

$$\left|\left\{\frac{(f'(z))^{\alpha+1-p}}{pz^{p-1}}\right\}^{\frac{1}{p-\alpha}}\left\{zf''(z)+(1-\alpha)f'(z)\right\}-p+\alpha\right|<\frac{n+1}{\sqrt{(n+1)^2+1}}(p-\alpha)\ (z\in\mathtt{U}), \tag{18}$$

for $0\leq\alpha<p$, then $f(z)\in C_p(n,\alpha)$.

Further If we take $n=1$ and $p=1$ in Corollary 2.2 and Corollary 2.3, we get the following result proved by Uyanik et al [5].

Corollary 2.4. If $f(z)\in A$ satisfies

$$\left|\left(\frac{f(z)}{z}\right)^{\frac{1}{1-\alpha}}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}-1+\alpha\right|<\frac{2}{\sqrt{5}}(1-\alpha)\ (z\in\mathtt{U}), \tag{19}$$

for $0\leq\alpha<1$, then $f(z)\in S^*(\alpha)$.

Corollary 2.5. If $f(z) \in A$ satisfies

$$\left| \left(f'(z)\right)^{\frac{\alpha}{1-\alpha}} \left\{ f'(z) + \frac{1}{1-\alpha} z f''(z) \right\} - 1 \right| < \frac{2}{\sqrt{5}} \ (z \in \mathtt{U}), \tag{20}$$

for $0 \leq \alpha < 1$, then $f(z) \in C(\alpha)$.

Remark 2.1. If we put $\alpha = 0$ and $p = 1$ in Corollary 2.4 and Corollary 2.5, we get the result proved by Mocanu [6] and Nunokawa et al [7] respectively.

Theorem 2.6. If $p(z)$, given by (9), satisfies

$$\left|p''(z)\right| < \frac{n+1}{\sqrt{(n+1)^2+1}} \ (z \in E), \tag{21}$$

then $f(z) \in Q_\lambda(p, n, \alpha; g(z))$.

Proof. From (9), we have $p(z) \in A(n)$. Also

$$\begin{aligned} \left|p'(z) - 1\right| &= \left| \int_0^z p''(t)dt \right| \leq \int_0^{|z|} \left|h''(\rho e^{i\theta})\right| d\rho \qquad (22) \\ &\leq \frac{n+1}{\sqrt{(n+1)^2+1}} |z| \leq \frac{n+1}{\sqrt{(n+1)^2+1}}, \qquad (23) \end{aligned}$$

where we have used (21). This proves that $p(z)$ satisfies the condition of Lemma 2.1 and therefore $p(z) \in S^*(n, 0)$, which leads $f(z) \in Q_\lambda(p, n, \alpha; g(z))$. **Theorem 2.7.** If $f(z) \in A(p, n)$ satisfies

$$\begin{aligned} &\left| \left(\frac{(f \star g)(z)}{z^p} \right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}} \left[\left(\frac{(f \star g)'(z)}{(f \star g)(z)} \right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}} - \frac{p}{z} \right] \right| \\ &\qquad \leq \frac{(n+1)(p-\alpha)\cos\lambda}{2\sqrt{(n+1)^2+1}}, \end{aligned} \tag{24}$$

then $f(z) \in Q_\lambda(p, n, \alpha; g(z))$.

Proof. Let us define a function $p(z)$ by

$$p(z) = \int_0^z \left(\frac{(f \star g)(t)}{t^p} \right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}} dt. \tag{25}$$

Then

$$zp'(z) = z \left(\frac{(f \star g)(z)}{z^p} \right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}}. \tag{26}$$

Let $g(z) = zp'(z)$. Then $g(z) \in A(n)$. Consider

$$\left|g'(z) - 1\right| = \left|p'(z) + zp''(z) - 1\right| \leq \left|p'(z) - 1\right| + \left|zp''(z)\right| = \left| \int_0^z p''(t)dt \right| + \left|zp''(z)\right|$$

$$\begin{aligned} &\leq \int_0^{|z|} \left| \frac{e^{i\lambda}}{(p-\alpha)\cos\lambda} H(z) \right| dt + \left| \frac{e^{i\lambda}}{(p-\alpha)\cos\lambda} H(z) \right| |z| \\ &\leq \int_0^{|z|} \frac{(n+1)}{2\sqrt{(n+1)^2+1}} dt + \frac{(n+1)}{2\sqrt{(n+1)^2+1}} |z| < \frac{(n+1)}{\sqrt{(n+1)^2+1}}. \end{aligned} \tag{27}$$

with

$$H(z)=\left(\frac{(f\star g)(z)}{z^{p}}\right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}}\left[\left(\frac{(f\star g)'(z)}{(f\star g)(z)}\right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}}-\frac{p}{z}\right]. \tag{28}$$

Therefore, by using Lemma 2.1, we have

$$g(z)=zp'(z)\in S^{*}(n,0). \tag{29}$$

This means that $p(z)\in C(n,0)$, which implies that $f(z)\in Q_{\lambda}(p,n,\alpha;g(z))$.

3. Generalized Alexander Integral Operator

For $f(z),g(z)\in A(p,n)$, we consider

$$G(z)=\int_{0}^{z}\left(\frac{(f*g)(t)}{t^{p}}\right)^{\gamma}dt=z+\frac{\gamma a_{p+n}b_{p+n}}{n+1}z^{n+1}+\dots \tag{30}$$

Clearly $G(z)\in A(n)$ and when $p=1,\gamma=1$, $g(z)=\frac{z}{1-z}$, then (30) reduces to the well-known Alexander integral operator [8].

Theorem 3.1. If $\gamma\geq\frac{1}{p}$ and $f(z),g(z)\in A(p,n)$ satisfies

$$\left|\frac{\gamma\left((f*g)(z)\right)^{\frac{\gamma e^{i\lambda}}{\cos\lambda}}}{z^{\frac{p\gamma e^{i\lambda}}{\cos\lambda}+1}}\left(\frac{z\left((f*g)'(z)\right)}{(f*g)(z)}-p\right)\right|\leq\frac{(n+1)\cos\lambda}{2\sqrt{(n+1)^{2}+1}}, \tag{31}$$

then $f(z)\in Q_{\lambda}(p,n,0;g(z))$.

Proof. From (30), we get

$$G'(z)=\left(\frac{(f*g)(z)}{z^{p}}\right)^{\gamma}. \tag{32}$$

Differentiating (32), logarithmically, we get

$$\frac{G''(z)}{G'(z)}=\gamma\left(\frac{(f*g)'(z)}{(f*g)(z)}-\frac{p}{z}\right). \tag{33}$$

Then by simple computation, we have,

$$\begin{aligned}\left|G''(z)\left[G'(z)\right]^{\frac{e^{i\lambda}}{\cos\lambda}-1}\right| &= \left|\gamma\left(\frac{(f*g)(z)}{z^{p}}\right)^{\frac{\gamma e^{i\lambda}}{\cos\lambda}}\left(\frac{(f*g)'(z)}{(f*g)(z)}-\frac{p}{z}\right)\right|\\ &\leq \frac{(n+1)\cos\lambda}{2\sqrt{(n+1)^{2}+1}},\end{aligned}$$

where we have used (31). Therefore

$$\left|G''(z)\left[G'(z)\right]^{\frac{e^{i\lambda}}{\cos\lambda}-1}\right|\leq\frac{(n+1)\cos\lambda}{2\sqrt{(n+1)^{2}+1}} \tag{34}$$

By using Theorem 2.7 with $p=1$, $\alpha=0$ and $g(z)=\frac{z}{(1-z)^{2}}$, we have $G(z)\in C_{\lambda}(1,n,0)$.

From (33), we can write

$$\Re\left[e^{i\lambda}\left(1+\frac{zG''(z)}{G'(z)}\right)\right]=\gamma\Re e^{i\lambda}\left(\frac{z\left((f*g)'(z)\right)}{(f*g)(z)}\right)-p\gamma\cos\lambda+\cos\lambda, \tag{35}$$

or

$$Ree^{i\lambda}\left(\frac{z\left(\left(f*g\right)'\left(z\right)\right)}{\left(f*g\right)\left(z\right)}\right) > \left(p-\frac{1}{\gamma}\right)\cos\lambda \ \ (since G\left(z\right)\in C_{\lambda}\left(1,n,0\right)) \tag{36}$$

which shows that $f(z) \in Q_\lambda\left(p, n, 0; g\left(z\right)\right)$, where $\gamma \geq \frac{1}{p}$.

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ADDITIVE FUNCTIONAL INEQUALITIES IN PARANORMED SPACES

SEO-YUN YANG AND CHOONKIL PARK*

Abstract. In this paper, we investigate the following additive functional inequalities

$$\begin{aligned} \left\| \frac{1}{s} f(x) + \frac{1}{s} f(y) + f(z) + f(w) \right\| &\leq \left\| f\left(\frac{x+y}{s} + z + w \right) \right\|, \\ \left\| \frac{1}{s} f(x) + \frac{1}{s} f(y) + \frac{1}{s} f(z) + f(w) \right\| &\leq \left\| f\left(\frac{x+y+z}{s} + w \right) \right\| \end{aligned}$$

in paranormed spaces for a fixed integer s greater than 1. Furthermore, we prove the Hyers-Ulam stability of the above additive functional inequalities in paranormed spaces.

1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [3] and Steinhaus [26] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [5, 14, 16, 17, 25]). This notion was defined in normed spaces by Kolk [15].

We recall some basic facts concerning Fréchet spaces.

Definition 1.1. [28] Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that

(1) $P(0) = 0$;
(2) $P(-x) = P(x)$;
(3) $P(x + y) \leq P(x) + P(y)$ (triangle inequality)
(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a *paranormed space* if P is a *paranorm* on X.

The paranorm is called *total* if, in addition, we have

(5) $P(x) = 0$ implies $x = 0$.

A *Fréchet space* is a total and complete paranormed space.

The stability problem of functional equations originated from a question of Ulam [27] concerning the stability of group homomorphisms. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [21] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

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*Corresponding author.

In 1990, Th.M. Rassias [22] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [6] following the same approach as in Th.M. Rassias [21], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [6], as well as by Th.M. Rassias and Šemrl [23] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$ (cf. the books of P. Czerwik [2], D.H. Hyers, G. Isac and Th.M. Rassias [11]).

In 1982, J.M. Rassias [20] followed the innovative approach of the Th.M. Rassias' theorem [21] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. Găvruta [7] provided a further generalization of Th.M. Rassias' Theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings (see [12, 13, 18]).

In [8], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [24]. Fechner [4] and Gilányi [9] proved the Hyers-Ulam stability of the functional inequality (1.1).

Park, Cho and Han [19] proved the Hyers-Ulam stability of the following functional inequalities

$$\begin{aligned} \|f(x) + f(y) + f(z)\| &\leq \left\|2f\left(\frac{x+y+z}{2}\right)\right\|, \\ \|f(x) + f(y) + f(z)\| &\leq \|f(x+y+z)\|, \\ \|f(x) + f(y) + 2f(z)\| &\leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\|. \end{aligned}$$

We proved the Hyers-Ulam stability of the following functional inequalities

$$\left\|\frac{1}{s}f(x) + \frac{1}{s}f(y) + f(z) + f(w)\right\| \leq \left\|f\left(\frac{x+y}{s} + z + w\right)\right\|, \tag{1.2}$$

$$\left\|\frac{1}{s}f(x) + \frac{1}{s}f(y) + \frac{1}{s}f(z) + f(w)\right\| \leq \left\|f\left(\frac{x+y+z}{s} + w\right)\right\| \tag{1.3}$$

for a fixed integer s greater than 1.

In Section 2, we prove the Hyers-Ulam stability of the functional inequality (1.2) in paranormed spaces.

In Section 3, we prove the Hyers-Ulam stability of the functional inequality (1.3) in paranormed spaces.

Throughout this paper, assume that $(X, P(\cdot))$ is a total paranormed space and that $(Y, \|\cdot\|)$ is a Banach space.

2. Hyers-Ulam stability of the functional inequality (1.2)

In this section, we prove the Hyers-Ulam stability of the functional inequality (1.2) in paranormed spaces.

Proposition 2.1. *Let $f : X \to Y$ be a mapping such that*

$$\left\|\frac{1}{s}f(x) + \frac{1}{s}f(y) + f(z) + f(w)\right\| \leq \left\|f\left(\frac{x+y}{s} + z + w\right)\right\| \tag{2.1}$$

for all $x, y, z, w \in X$. Then f is additive.

Proof. Letting $x = y = z = w = 0$ in (2.1), we get

$$\left(\frac{2}{s} + 2\right) \|f(0)\| = \left\|\frac{2}{s} f(0) + 2f(0)\right\| \leq \|f(0)\|$$

and so

$$f(0) = 0.$$

Letting $x = y = 0$ and $w = -z$ in (2.1), we get

$$f(-z) = -f(z)$$

for all $z \in X$. Letting $x = -sz$ and $y = w = 0$ in (2.1), we get

$$f(sz) = f(sz) \qquad \& \qquad f\left(\frac{z}{s}\right) = \frac{1}{s} f(z)$$

for all $z \in X$. Letting $z = -\frac{x+y}{s}$ and $w = 0$ in (2.1), we get

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$. Thus f is additive. □

Note that $P(sx) \leq sP(x)$ for all $x \in X$.

Theorem 2.2. *Let r be a positive real number with $r < 1$, and $f : X \to Y$ be an odd mapping such that*

$$\begin{aligned} \left\|\frac{1}{s} f(x) + \frac{1}{s} f(y) + f(z) + f(w)\right\| &\leq \left\|f\left(\frac{x+y}{s} + z + w\right)\right\| \\ &+ P(x)^r + P(y)^r + P(z)^r + P(w)^r \end{aligned} \tag{2.2}$$

for all $x, y, z, w \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq s\left(\frac{s^r + 1}{s - s^r}\right) P(x)^r \tag{2.3}$$

for all $x \in X$.

Proof. Letting $y = w = 0$ and $z = -\frac{x}{s}$ in (2.2), we get

$$\left\|\frac{1}{s} f(x) - f\left(\frac{x}{s}\right)\right\| = \left\|\frac{1}{s} f(x) + f\left(-\frac{x}{s}\right)\right\| \leq P(x)^r + P\left(-\frac{x}{s}\right)^r$$

and so

$$\left\|\frac{1}{s} f(sx) - f(x)\right\| \leq P(sx)^r + P(-x)^r \leq (s^r + 1) P(x)^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{s^l} f(s^l x) - \frac{1}{s^m} f(s^m x)\right\| &\leq \sum_{j=l}^{m-1} \left\|\frac{1}{s^j} f(s^j x) - \frac{1}{s^{j+1}} f(s^{j+1} x)\right\| \\ &\leq (s^r + 1) \sum_{j=l}^{m-1} \frac{s^{rj}}{s^j} P(x)^r \end{aligned} \tag{2.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.4) that the sequence $\{\frac{1}{s^n} f(s^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{s^n} f(s^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{s^n} f(s^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.4), we get (2.3).

It follows from (2.2) that

$$\begin{aligned}
&\left\|\frac{1}{s}h(x) + \frac{1}{s}h(y) + h(z) + h(w)\right\| \\
&= \lim_{n\to\infty} \frac{1}{s^n} \left\|\frac{1}{s} f\left(s^n x\right) + \frac{1}{s} f\left(s^n y\right) + f\left(s^n z\right) + f\left(s^n y\right)\right\| \\
&\le \lim_{n\to\infty} \frac{1}{s^n} \left\|f\left(s^n\left(\frac{x+y}{s} + z + w\right)\right)\right\| + \lim_{n\to\infty} \frac{s^{nr}}{s^n}(P(x)^r + P(y)^r + P(z)^r + P(w)^r) \\
&= \left\|h\left(\frac{x+y}{s} + z + w\right)\right\|
\end{aligned}$$

for all $x, y, z, w \in X$. So

$$\left\|\frac{1}{s}h(x) + \frac{1}{s}h(y) + h(z) + h(w)\right\| = \left\|h\left(\frac{x+y}{s} + z + w\right)\right\|$$

for all $x, y, z, w \in X$. By Proposition 2.1, the mapping $h : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.3). Then we have

$$\begin{aligned}
\|h(x) - T(x)\| &= \frac{1}{s^n} \left\|h\left(s^n x\right) - T\left(s^n x\right)\right\| \\
&\le \frac{1}{s^n} \left(\left\|h\left(s^n x\right) - f\left(s^n x\right)\right\| + \left\|T\left(s^n x\right) - f\left(s^n x\right)\right\|\right) \\
&\le \frac{2s(s^r + 1)s^{nr}}{(s - s^r)s^n} P(x)^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (2.3). $\square$

3. Hyers-Ulam stability of the functional inequality (1.3)

In this section, we prove the Hyers-Ulam stability of the functional inequality (1.3) in paranormed spaces.

Proposition 3.1. *Let $f : X \to Y$ be a mapping such that*

$$\left\|\frac{1}{s}f(x) + \frac{1}{s}f(y) + \frac{1}{s}f(z) + f(w)\right\| \le \left\|f\left(\frac{x+y+z}{s} + w\right)\right\| \tag{3.1}$$

for all $x, y, z, w \in X$. Then f is additive.

Proof. Letting $x = y = z = w = 0$ in (3.1), we get

$$\left(\frac{3}{s} + 1\right) \|f(0)\| = \left\|\frac{3}{s}f(0) + f(0)\right\| \le \|f(0)\|$$

and so

$$f(0) = 0.$$

Letting $y = z = x$ and $w = -x$ in (3.1), we get

$$f(-x) = -f(x)$$

for all $x \in X$. Letting $w = -\frac{x}{s}$ and $y = z = 0$ in (3.1), we get

$$\frac{1}{s} f(x) = f\left(\frac{1}{s} x\right)$$

for all $x \in X$. Letting $z = -x - y$ and $w = 0$ in (3.1), we get

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. Thus f is additive. $\square$

Note that $P(sx) \le sP(x)$ for all $x \in X$.

Theorem 3.2. *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be an odd mapping such that*

$$\begin{aligned} \left\| \frac{1}{s} f(x) + \frac{1}{s} f(y) + \frac{1}{s} f(z) + f(w) \right\| &\le \left\| f\left(\frac{x+y+z}{s} + w\right)\right\| \\ &+ P(x)^r + P(y)^r + P(z)^r + P(w)^r \end{aligned} \tag{3.2}$$

for all $x, y, z, w \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le s\left(\frac{s^r + 1}{s - s^r}\right) P(x)^r \tag{3.3}$$

for all $x \in X$.

Proof. Letting $y = x = 0$ and $z = -\frac{x}{s}$ in (3.2), we get

$$\left\| \frac{1}{s} f(x) - f\left(\frac{x}{s}\right)\right\| = \left\| \frac{1}{s} f(x) + f\left(-\frac{x}{s}\right)\right\| \le P(x)^r + P\left(-\frac{x}{s}\right)^r$$

and so

$$\left\| \frac{1}{s} f(sx) - f(x)\right\| \le P(sx)^r + P(-x)^r \le (s^r + 1) P(x)^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{s^l} f(s^l x) - \frac{1}{s^m} f(s^m x)\right\| &\le \sum_{j=l}^{m-1} \left\| \frac{1}{s^j} f(s^j x) - \frac{1}{s^{j+1}} f(s^{j+1} x)\right\| \\ &\le (s^r + 1) \sum_{j=l}^{m-1} \frac{s^{rj}}{s^j} P(x)^r \end{aligned} \tag{3.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.4) that the sequence $\{\frac{1}{s^n} f(s^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{s^n} f(s^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{s^n} f(s^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.4), we get (3.3).

It follows from (3.2) that

$$\begin{aligned}
&\left\|\frac{1}{s}h(x) + \frac{1}{s}h(y) + h(z) + h(w)\right\| \\
&= \lim_{n\to\infty} \frac{1}{s^n} \left\|\frac{1}{s}f\left(s^n x\right) + \frac{1}{s}f\left(s^n y\right) + f\left(s^n z\right) + f\left(s^n y\right)\right\| \\
&\le \lim_{n\to\infty} \frac{1}{s^n} \left\|f\left(s^n\left(\frac{x+y}{s} + z + w\right)\right)\right\| + \lim_{n\to\infty} \frac{s^{nr}}{s^n}(P(x)^r + P(y)^r + P(z)^r + P(w)^r) \\
&= \left\|h\left(\frac{x+y}{s} + z + w\right)\right\|
\end{aligned}$$

for all $x, y, z, w \in X$. So

$$\left\|\frac{1}{s}h(x) + \frac{1}{s}h(y) + h(z) + h(w)\right\| = \left\|h\left(\frac{x+y}{s} + z + w\right)\right\|$$

for all $x, y, z, w \in X$. By Proposition 3.1, the mapping $h : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (3.3). Then we have

$$\begin{aligned}
\|h(x) - T(x)\| &= \frac{1}{s^n}\left\|h\left(s^n x\right) - T\left(s^n x\right)\right\| \\
&\le \frac{1}{s^n}\left(\left\|h\left(s^n x\right) - f\left(s^n x\right)\right\| + \left\|T\left(s^n x\right) - f\left(s^n x\right)\right\|\right) \\
&\le \frac{2s(s^r+1)s^{nr}}{(s-s^r)s^n}P(x)^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (3.3). □

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Seo-Yun Yang
Department of Mathematics, Hanyang University, Seoul 133-791, Korea
E-mail address: nana1301@nate.com

Choonkil Park
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea
E-mail address: baak@hanyang.ac.kr

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SOME IDENTITIES FOR BERNOULLI POLYNOMIALS INVOLVING CHEBYSHEV POLYNOMIALS

DAE SAN KIM, TAEKYUN KIM AND SANG-HUN LEE

Abstract. In this paper we derive some new and interesting identities for Bernoulli, Euler and Hermite polynomials associated with Chebyshev polynomials.

1. Introduction

The Bernoulli number are defined by the generating function to be

$$\frac{t}{e^t-1}=e^{Bt}=\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n, \quad \text{(see [3,13,14])}, \tag{1}$$

with the usual convention about replacing B^n by B_n.

As is well known, the Bernoulli polynomials are given by

$$B_n(x)=(B+x)^n=\sum_{l=0}^{n}\binom{n}{l}B_{n-l}x^l, \quad \text{(see [1-8])}. \tag{2}$$

From (1), we note that the recurrence relation for the Bernoulli numbers is given by

$$B_0=1, \quad (B+1)^n-B_n=\delta_{1,n}, \quad \text{(see [6-8])},$$

where $\delta_{m,n}$ is the Kronecker symbol.

By (2), we get

$$\frac{dB_n(x)}{dx}=n\sum_{l=0}^{n-1}\binom{n-1}{l}B_{n-1-l}x^l=nB_{n-1}(x). \tag{3}$$

Thus, by (3), we see that

$$\int B_n(x)dx=\frac{B_{n+1}(x)}{n+1}+C, \quad \text{(see [3])}, \tag{4}$$

where C is a some constant.

The Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt}=e^{E(x)t}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}, \tag{5}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$, (see [1,2,4,10,11]).

In the special case, $x=0$, $E_n(0)=E_n$ are called the n-th Euler numbers.

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It is well known [6, 15] that Hermite polynomials are given by the generating function to be

$$e^{2xt-t^2} = e^{H(x)t} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \tag{6}$$

with the usual convention about replacing $H^n(x)$ by $H_n(x)$.

From (6),we have

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x), \quad H_n(x) = (-1)^n H_n(-x). \tag{7}$$

By (1) and (2), we easily get

$$B_n(x) = \sum_{\substack{k=0\\k\neq 1}}^{n} \binom{n}{k} E_{n-k}(x), \quad \text{(see [1-15])}, \tag{8}$$

$$E_n(x) = -2\sum_{l=0}^{n} \binom{n}{l} \frac{E_{l+1}}{l+1} E_{n-l}(x), \tag{9}$$

and

$$x^n = \frac{1}{n+1}\big(B_{n+1}(x+1) - B_{n+1}(x)\big) = \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l} B_l(x). \tag{10}$$

The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by the relation

$$T_n(x) = \cos n\theta, \quad \text{when } x = \cos\theta, \quad \text{(see [9])}. \tag{11}$$

If the range of the variable x is the interval $[-1, 1]$, then the range of the corresponding variable θ can be taken as $[0, \pi]$. It is known that $\cos n\theta$ is a polynomial of degree n in $\cos\theta$, and indeed we are familiar with elementary formulas $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1, \cdots$.

Thus, by (11), we get

$$\begin{aligned} &T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \\ &T_4(x) = 8x^4 - 8x^2 + 1, \cdots . \end{aligned}$$

The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x defined by

$$U_n(x) = \sin{(n+1)\theta}/\sin\theta, \quad \text{when } x = \cos\theta, \quad \text{(see [9])}. \tag{12}$$

Thus, from (12), we have

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x, \cdots .$$

By (11), we see that $T_n(x)$ is a polynomial of degree n with integral coefficients and the leading coefficient 2^{n-1} $(n \geq 1)$ and 1 $(n = 0)$. It is not difficult to show that $U_n(x)$ is a polynomial of degree n with integral coefficients and the leading coefficient 2^n $(n \geq 0)$. $T_n(x)$ is a solution of $(1-x^2)y'' - xy' + n^2y = 0$ and $U_n(x)$ is a solution of $(1-x^2)y'' - 3xy' + n(n+2)y = 0$. It is well known [9] that the generating functions of $T_n(x)$ and $U_n(x)$ are given by

$$\frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \tag{13}$$

and

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n, \quad \text{for } |x| \le 1,\ |t| < 1. \tag{14}$$

From (11) and (12), we have

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{\pi}{2}, & \text{if } n = m > 0 \\ \pi, & \text{if } n = m = 0 \end{cases}, \tag{15}$$

and

$$\int_{-1}^{1} (1-x^2)^{1/2} U_n(x)U_m(x)dx = \frac{\pi}{2}\delta_{n,m}, \quad \text{(see [9])}. \tag{16}$$

The equations (15) and (16) are used to derive our main result in this paper.

The Rodrigues' formulae for $T_n(x)$ and $U_n(x)$ are known as follows:

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}(1-x^2)^{1/2}\left(\frac{d^n}{dx^n}(1-x^2)^{n-1/2}\right), \tag{17}$$

and

$$U_n(x) = \frac{(-1)^n 2^n (n+1)!}{(2n+1)!}(1-x^2)^{-1/2}\left(\frac{d^n}{dx^n}(1-x^2)^{n+1/2}\right). \tag{18}$$

The equations (17) and (18) are also used to derive our result related to orthogonality of Chebyshev polynomials.

From (11) and (12), we can easily derive the following equations (19) and (20):

$$T_n(x) = \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2}, \tag{19}$$

and

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}. \tag{20}$$

By the definitions of $T_n(x)$ and $U_n(x)$, we easily get

$$\frac{dT_n(x)}{dx} = nU_{n-1}(x), \quad \frac{dU_n(x)}{dx} = \frac{(n+1)T_{n+1}(x) - xU_n(x)}{x^2-1}. \tag{21}$$

From (21), we have

$$\int U_n(x)dx = \frac{T_{n+1}(x)}{n+1}, \quad \int T_n(x)dx = \frac{nT_{n+1}(x)}{n^2-1} - \frac{xT_n(x)}{n-1}. \tag{22}$$

In this paper we derive some new and interesting identities for Bernoulli, Euler and Hermite polynomials arising from the orthogonality of the Chebyshev polynomials for the inner product space with weighted inner product.

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2. Some identities for Bernoulli, Euler and Hermite polynomials involving Chebyshev polynomials

Let $\mathbf{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\}$. Then $\mathbf{P}_n$ is an inner product space with the weighted inner product

$$\langle p(x),\ q(x)\rangle = \int_{-1}^{1} \frac{p(x)q(x)}{\sqrt{1-x^2}}dx, \quad \text{where } p(x), q(x) \in \mathbf{P}_n.$$

From (15), we note that $\{T_0(x), T_1(x), \cdots, T_n(x)\}$ is an orthogonal basis for $\mathbf{P}_n$. Let us assume $p(x) \in \mathbf{P}_n$. Then $p(x)$ is generated by $\{T_0(x), T_1(x), \cdots, T_n(x)\}$ to be

$$p(x) = \sum_{k=0}^{n} C_k T_k(x). \tag{23}$$

By (15) and (23), we get

$$\begin{aligned} &C_k = \frac{\delta_k}{\pi}\int_{-1}^{1}\frac{T_k(x)p(x)}{\sqrt{1-x^2}}dx = \frac{\delta_k}{\pi}\frac{(-1)^k 2^k k!}{(2k)!}\int_{-1}^{1}\left(\frac{d^k}{dx^k}(1-x^2)^{k-1/2}\right)p(x)dx,\\ &\text{where } \delta_k = \begin{cases} 1, & \text{if } k=0 \\ 2, & \text{if } k>0. \end{cases}\end{aligned} \tag{24}$$

Let us take $p(x) = x^n \in \mathbf{P}_n$. From (24), we have

$$\begin{aligned} C_k &= \frac{(-1)^k 2^k k! \delta_k}{\pi(2k)!}\int_{-1}^{1}\left(\frac{d^k}{dx^k}(1-x^2)^{k-1/2}\right)x^n dx \\ &= \frac{(-1)^k 2^k k!}{\pi(2k)!}\delta_k(-1)^k\frac{n!}{(n-k)!}\int_{-1}^{1}(1-x^2)^{k-1/2}x^{n-k}dx. \end{aligned} \tag{25}$$

It is easy to show that

(26)

$$\begin{aligned} &\int_{-1}^{1}(1-x^2)^{k-1/2}x^{n-k}dx = \frac{(1+(-1)^{n-k})}{2}\int_{0}^{1}(1-y)^{k-1/2}y^{\frac{n-k+1}{2}-1}dy \\ &= \frac{(1+(-1)^{n-k})}{2}\frac{\Gamma(k+1/2)\Gamma(\frac{n-k+1}{2})}{\Gamma(\frac{k+n+2}{2})} = \frac{(1+(-1)^{n-k})}{2}\frac{(n-k)!(2k)!\pi}{2^{n+k}(\frac{n+k}{2})!(\frac{n-k}{2})!k!}. \end{aligned}$$

By (25) and (26), we get

$$C_k = \begin{cases} 0, & \text{if } n-k \equiv 1 \pmod 2 \\ \frac{n!\delta_k}{2^n(\frac{n+k}{2})!(\frac{n-k}{2})!}, & \text{if } n-k \equiv 0 \pmod 2. \end{cases} \tag{27}$$

From (27), we note that

$$x^n = \sum_{k=0}^{n} C_k T_k(x) = \frac{n!}{2^{n-1}}\sum_{\substack{1\le k\le n \\ k\equiv 1 \pmod 2}}\frac{T_k(x)}{(\frac{n+k}{2})!(\frac{n-k}{2})!}, \tag{28}$$

where $n \equiv 1 \pmod 2$.

For $n \equiv 0 \pmod 2$, we have

$$x^n = \frac{n!}{2^n}\left\{\frac{T_0(x)}{\left((\frac{n}{2})!\right)^2} + 2\sum_{\substack{2\le k\le n \\ k\equiv 0 \pmod 2}}\frac{T_k(x)}{(\frac{n+k}{2})!(\frac{n-k}{2})!}\right\}. \tag{29}$$

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Let us take $p(x) = B_n(x) \in \mathbf{P}_n$. Then

$$\begin{aligned} C_k &= \frac{(-1)^k 2^k k! \delta_k}{\pi(2k)!} \int_{-1}^{1} \left(\left(\frac{d}{dx}\right)^k (1-x^2)^{k-1/2} \right) B_n(x) dx \\ &= \frac{(-1)^k 2^k k! \delta_k}{\pi(2k)!} (-1)^k \frac{n!}{(n-k)!} \int_{-1}^{1} (1-x^2)^{k-1/2} B_{n-k}(x) dx \qquad (30) \\ &= \frac{2^k k! \delta_k}{\pi(2k)!} \frac{n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} \int_{-1}^{1} (1-x^2)^{k-1/2} x^l dx. \end{aligned}$$

Now, we compute $\int_{-1}^{1}(1-x^2)^{k-1/2}x^l dx$.

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^{k-1/2} x^l dx &= (1+(-1)^l) \int_{0}^{1} (1-x^2)^{k-1/2} x^l dx \qquad (31) \\ &= \begin{cases} 0, & \text{if } l \equiv 1 \pmod 2 \\ \frac{l!(2k)!\pi}{2^{2k+l}(\frac{2k+l}{2})!(\frac{l}{2})!k!}, & \text{if } l \equiv 0 \pmod 2. \end{cases} \end{aligned}$$

By (30) and (31), we get

$$\begin{aligned} C_k &= \frac{2^k k! \delta_k}{\pi(2k)!} \times \frac{n!}{(n-k)!} \times \frac{(2k)!\pi}{2^{2k}k!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathrm{mod}\ 2)}} \binom{n-k}{l} B_{n-k-l} \frac{l!}{2^l(\frac{2k+l}{2})!(\frac{l}{2})!} \qquad (32) \\ &= \frac{n!\delta_k}{2^k(n-k)!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathrm{mod}\ 2)}} \frac{\binom{n-k}{l} B_{n-k-l} l!}{2^l(\frac{2k+l}{2})!(\frac{l}{2})!}. \end{aligned}$$

Therefore, by (32), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, we have*

$$B_n(x) = n! \sum_{0 \le k \le n} \left(\frac{\delta_k}{2^k(n-k)!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathit{mod}\ 2)}} \frac{\binom{n-k}{l} B_{n-k-l} l!}{2^l(\frac{2k+l}{2})!(\frac{l}{2})!} \right) T_k(x).$$

By the same method, we can derive the following identity:

$$E_n(x) = n! \sum_{0 \le k \le n} \left(\frac{\delta_k}{2^k(n-k)!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathrm{mod}\ 2)}} \frac{\binom{n-k}{l} E_{n-k-l} l!}{2^l(\frac{2k+l}{2})!(\frac{l}{2})!} \right) T_k(x).$$

Let us take $p(x) = H_n(x) \in \mathbf{P}_n$. From (24), we have

$$\begin{aligned} C_k &= \frac{(-1)^k 2^k k! \delta_k}{\pi(2k)!} \int_{-1}^{1} \left(\frac{d^k}{dx^k} (1-x^2)^{k-1/2} \right) H_n(x) dx \\ &= \frac{(-1)^k 2^k k! \delta_k}{(2k)!\pi} \times (-1)^k 2^k \frac{n!}{(n-k)!} \int_{-1}^{1} (1-x^2)^{k-1/2} H_{n-k}(x) dx \qquad (33) \\ &= \frac{2^{2k} k! \delta_k n!}{(2k)!(n-k)!\pi} \sum_{l=0}^{n-k} \binom{n-k}{l} H_{n-k-l} 2^l \int_{-1}^{1} (1-x^2)^{k-1/2} x^l dx, \end{aligned}$$

where H_{n-k-l} is the $(n-k-l)$th Hermite number.

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By (31) and (33), we get

$$C_k = n!\delta_k \sum_{\substack{0\le l\le n-k \\ l\equiv 0 \pmod 2}} \frac{H_{n-k-l}}{(n-k-l)!(\frac{2k+l}{2})!(\frac{l}{2})!}. \tag{34}$$

Therefore, by (34), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, we have*

$$H_n(x) = n! \sum_{0\le k\le n} \left(\delta_k \sum_{\substack{0\le l\le n-k \\ l\equiv 0\ (mod\ 2)}} \frac{H_{n-k-l}}{(n-k-l)!(\frac{2k+l}{2})!(\frac{l}{2})!} \right) T_k(x).$$

Let $\mathbf{P}_n^* = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\}$. Then $\mathbf{P}_n^*$ is an inner product space with the weighted inner product $\langle p(x),\ q(x)\rangle = \int_{-1}^{1} \sqrt{1-x^2}p(x)q(x)dx$, where $p(x), q(x) \in \mathbf{P}_n$. Then $\{U_0(x), U_1(x), \cdots, U_n(x)\}$ is an orthogonal basis for the inner product space $\mathbf{P}_n^*$.
For $p(x) \in \mathbf{P}_n^*$, let

$$p(x) = \sum_{k=0}^{n} C_k U_k(x), \tag{35}$$

where

$$\begin{aligned} C_k &= \frac{2}{\pi}\langle p(x),\ U_k(x)\rangle = \frac{2}{\pi}\int_{-1}^{1} (1-x^2)^{1/2} U_k(x)p(x)dx \\ &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+1/2} \right) p(x)dx. \end{aligned} \tag{36}$$

Let us assume that $p(x) = x^n \in \mathbf{P}_n^*$. Then, by (36), we get

$$\begin{aligned} C_k &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+1/2} \right) x^n dx \\ &= \frac{(-1)^k 2^{2k+1}(k+1)!}{(2k+1)!\pi} \times \frac{(-1)^k n!}{(n-k)!} \int_{-1}^{1} (1-x^2)^{k+1/2} x^{n-k} dx. \end{aligned} \tag{37}$$

It is easy to show that

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^{k+1/2} x^{n-k} dx &= (1+(-1)^{n-k}) \int_{0}^{1} (1-x^2)^{k+1/2} x^{n-k} dx \\ &= \begin{cases} 0, & \text{if } n-k \equiv 1 \pmod 2 \\ \frac{(n-k)!(2k+2)!\pi}{2^{n+k+2}(\frac{n+k+2}{2})!(\frac{n-k}{2})!(k+1)!}, & \text{if } n-k \equiv 0 \pmod 2). \end{cases} \end{aligned} \tag{38}$$

Therefore, by (37) and (38), we obtain the following proposition.

Proposition 2.3. *For $n \in \mathbb{Z}_+$, we have*

$$x^n = \frac{n!}{2^n} \sum_{\substack{0\le k\le n \\ k\equiv n\ (mod\ 2)}} \frac{k+1}{(\frac{n+k+2}{2})!(\frac{n-k}{2})!} U_k(x).$$

Let us consider $p(x) = B_n(x) \in \mathbf{P}_n^*$. From (36), we have

$$\begin{aligned} C_k &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+1/2} \right) B_n(x)dx \\ &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \times \frac{(-1)^k n!}{(n-k)!} \int_{-1}^{1} (1-x^2)^{k+1/2} B_{n-k}(x)dx \\ &= \frac{2^{k+1}(k+1)!}{(2k+1)!\pi} \times \frac{n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} \int_{-1}^{1} (1-x^2)^{k+1/2} x^l dx. \end{aligned} \tag{39}$$

It is not difficult to show that

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^{k+1/2} x^l dx &= (1+(-1)^l) \int_{0}^{1} (1-x^2)^{k+1/2} x^l dx \\ &= \begin{cases} 0, & \text{if } l \equiv 1 \pmod 2 \\ \frac{(2k+2)!l!\pi}{2^{2k+2+l}(\frac{2k+2+l}{2})!(k+1)!(\frac{l}{2})!}, & \text{if } l \equiv 0 \pmod 2). \end{cases} \end{aligned} \tag{40}$$

By (39) and (40), we get

$$C_k = \frac{(k+1)n!}{2^k} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \frac{B_{n-k-l}}{(n-k-l)!2^l(\frac{2k+l+2}{2})!(\frac{l}{2})!}. \tag{41}$$

Therefore, by (41), we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$, we have*

$$B_n(x) = n! \sum_{0 \le k \le n} \left(\frac{k+1}{2^k} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (mod\ 2)}} \frac{B_{n-k-l}}{2^l(n-k-l)!(\frac{2k+l+2}{2})!(\frac{l}{2})!} \right) U_k(x).$$

By the same method, we can derive the following identity:

$$E_n(x) = n! \sum_{0 \le k \le n} \left(\frac{k+1}{2^k} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \frac{E_{n-k-l}}{2^l(n-k-l)!(\frac{2k+l+2}{2})!(\frac{l}{2})!} \right) U_k(x).$$

Let us take $p(x) = H_n(x) \in \mathbf{P}_n^*$. Then $H_n(x) = \sum_{k=0}^{n} C_k U_k(x)$, with

$$\begin{aligned} C_k &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+1/2} \right) H_n(x)dx \\ &= \frac{2^{2k+1}(k+1)!n!}{(2k+1)!\pi(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l H_{n-k-l} \int_{-1}^{1} (1-x^2)^{k+1/2} x^l dx \\ &= n!(k+1) \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \frac{H_{n-k-l}}{(n-k-l)!} \times \frac{1}{(\frac{2k+l+2}{2})!(\frac{l}{2})!}. \end{aligned} \tag{42}$$

Thus, by (42) and (43), we get

$$H_n(x) = n! \sum_{0 \le k \le n} \left((k+1) \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \frac{H_{n-k-l}}{(n-k-l)!(\frac{2k+l+2}{2})!(\frac{l}{2})!} \right) U_k(x).$$

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Dae San Kim

Department of Mathematics,
Sogang University, Seoul 121-742, Republic of Korea

TaeKyun Kim

Department of Mathematics,
Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail: tkkim@kw.ac.kr, taekyun64@hotmail.com

Sang-Hun Lee

Division of General Education,
Kwangwoon University, Seoul 139-701, Republic of Korea

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An economical aggregation algorithm for algebraic multigrid (AMG)*

Liang-Jian Deng†, Ting-Zhu Huang‡, Xi-Le Zhao§
Liang Zhao¶, Si Wang‖
School of Mathematical Sciences,
University of Electronic Science and Technology of China,
Chengdu, Sichuan, 611731, P.R. China

Abstract

Aggregation-based AMG method is a widely studied technique of robustness for large-scale linear systems. Some previous aggregation algorithms, belonging to a part of aggregation-based AMG method, exhibit certain excellent properties. These aggregation methods, however, have to aggregate every grid points so that these methods lead expensive computation with grid points increasing. In the paper, a property that the aggregations hold particular structure associated with certain condition is discovered to damp the computation of aggregation algorithms. Meanwhile, this property is under the condition of the system matrix derived from the 9-point Finite Difference Method (FDM) and the particular setting of grid number. Furthermore, the conclusions about multilevel, such as the setting rule of grid number and corresponding theoretical analysis, are obtained from the extension of two level issues. Computational experiments demonstrate that the CPU time of new aggregation algorithm which generates the same aggregations with previous aggregation algorithms, keeps on a low level evidently, even for the linear systems of millions grade.

Key words: Aggregation-based AMG; Aggregation algorithms; Economical computation; Poisson-like equations; Helmholtz-like equations; Millions grade problems

1 Introduction

Several methods can be utilized to solve the large-scale sparse linear systems

$$Ax = b, \tag{1}$$

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†E-mail: liangjian1987112@126.com
‡E-mail: tingzhuhuang@126.com
§E-mail: xlzhao122003@163.com
¶E-mail: 391475@163.com
‖E-mail: wangsi1989@yahoo.cn

where $A \in \mathcal{R}^{N\times N}$ arises from the discretization of a scalar second-order elliptic partial differential equation (PDE). AMG method for large-scale linear systems is among the most efficient and convenient iterative methods [1, 2, 3, 4, 5, 6, 7, 8].

AMG method is composed with two parts: one is the *setup phase* and the other is the *solve phase*. *Setup phase* is associated with three parts on each level, i.e., defining coarse grids (aggregations), constructing transfer operators (i.e., prolongation and restriction operators), computing the linear systems on the coarse level, respectively. *Solve phase* involves a recursive process with solving the linear systems level by level and contains three parts mainly, i.e., the smoothing steps, the transferring of linear systems among levels and solving linear systems on the coarsest level, respectively.

AMG method is a recursive method of efficiency for large-scale linear systems with mainly recursive forms: V-cycle and W-cycle, for instance, in [9, 10, 11, 12]. It projects the large-scale problems, level by level, to the small-scale problems until the problems can be solved as accurate as possible. The most important issue is making the computed solutions approximate to the true solutions. We transfer the linear systems of different levels through the restriction and prolongation operators R and P. AMG method has the following relationship among levels

$$A_{i+1} = R_i A_i P_i, \qquad i = 1, 2, \cdots, l_{\max} - 1, \tag{2}$$

where $l_{\max}$ labels the number of levels. R_i is the restriction operator from the i-th level to the $(i+1)$-th level while P_i, $P_i = R_i^T$, is prolongation operator from the $(i+1)$-th level to the i-th level. The subscripts of A_i denote corresponding belonging levels, and the levels range from fine to coarse with i increasing.

AMG method mainly refers to classical AMG method, aggregation-based AMG method, adaptive AMG method and AMGe method [6, 7, 13, 14]. Classical AMG method is introduced by Brandt, McCormick, Ruge and Stüben [15, 16], it has been employed to solve linear systems whose coefficient matrices are M-matrices. For aggregation-based AMG method, the most critical step is the construction of the prolongation operators by the aggregation algorithms [17, 18, 19, 20, 21, 22] based on different definitions of strength of Connection. Adaptive AMG method utilizes a multigrid algorithm to enhance the efficiency of the prolongation, aiming to earn a more efficient AMG algorithm [23, 24]. The AMGe method, located in [25, 26], was first introduced as a measure to improve the robustness of AMG for the finite element problems. It is different from standard AMG method for requiring access to local element stiffness matrices (in addition to the assembled global stiffness matrices). The main differences among these AMG methods, e.g., classical AMG method, aggregation-based AMG method, adaptive AMG method and AMGe method, can be discriminated by the constructions of transfer operators and coarse grids, respectively. Particularly, transfer operators of aggregation-based AMG method can be generated by a classical aggregation algorithm, corresponding details in [17]. The motivation of our work is to acquire the aggregations, same as the aggregations of the algorithm in [17], with cheaper computation by utilizing the property discovered in this paper.

In this paper, we mainly focus on the *setup phase* and establish a novel construction, aiming to reduce the computation of constructing aggregations. During the process of generating aggregations, an excellent discovery is found that the aggregations , obtained under the condition of the square grid number satisfying $(3i+4) \times (3i+4)$, $i \in \mathcal{N}$, are symmetric. Besides, the system matrix on the finest level should be derived from the discretization of a scalar second-order PDE with 9-point FDM. Then we make use of these properties, the symmetry of aggregations and the relationship among subscripts of the grid points, to construct a new aggregation algorithm to decrease the computation. Computational experiments present that the new aggregation algorithm gains a lower consuming-time, besides, the same aggregations compared with previous aggregation algorithm in [17]. Particularly, we have to emphasize that this paper is mainly to improve the aggregation algorithm in the *setup phase*, aiming to gain more economical computation. The *solve phase*, meanwhile, keeps unchanged while our proposed method is applied in the *setup phase*.

The paper is organized as follows. In section 2, we introduce the basic scheme of AMG method and the classical aggregation algorithm. Section 3 is about the new aggregation algorithm based on 9-point FDM. Meanwhile, some theoretical analysis and conclusions on the parameter and grid number on finest level, respectively, are presented in this section. Section 4 shows the capability of our aggregation algorithm on some numerical experiments about 2D Poisson-like equation and 2D Helmholtz-like equation. A compact conclusion will be presented in section 5.

2 Aggregation-based AMG methods

Aggregation-based AMG method clusters the fine grid of unknowns to aggregations representing the unknowns on the coarse level. Different with other methods, aggregation-based AMG method constructs the transfer operators mentioned in section 1 by aggregating the unknowns on each level. The coarsening part in classic AMG method is realized mainly by the aggregation algorithm (i.e., the *setup phase* mentioned in section 1) generating prearranged conditions for *solve phase*, e.g., aggregations, transfer operators and linear systems on coarse level. Meanwhile, it is necessary to introduce the basic AMG scheme (See the following forma). Where $A_1 = A \in \mathcal{R}^{N\times N}$, $b_1 = b \in \mathcal{R}^N$,

$y_i = \mathrm{MGM}(x_0, b_i, i)$		
If $(i = m)$	Then	$y_m = \mathrm{Solve}(A_m, b_m, e_m)$
	Else	$x_i = \mathrm{Smooth}(A_i, b_i, x_0)$
		$r_{i+1} = R_i(b_i - A_i x_i)$
		$A_{i+1} = R_i A_i P_i$
		$d_{i+1} = \mathrm{MGM}(0, r_{i+1}, i+1)$
		$\hat{x}_i = x_i + P_i d_{i+1}$
		$x_i = \mathrm{Smooth}(A_i, b_i, \hat{x}_i)$

$x_0 = \mathbf{0} \in \mathcal{R}^N$ and transfer operators $R_i = P_i^T$. The above recursive process is called V-cycle while another recursive type of AMG is called W-cycle doing twice on the fifth row. Aggregation-based AMG is divide into two parts: one is *setup phase* and the other is *solve phase* mentioned in above section. The *setup phase* may be considered as the prearranged section of the *solve phase* for solving the linear system (1), i.e., aggregations, transfer operators (i.e., R_i and P_i) and coarse linear systems A_i on each

3

level, respectively, so the *setup phase* part acts actually an important role in the whole process of AMG.

2.1 The classical aggregation algorithm

Before giving the new algorithm, it is necessary to introduce the classical aggregation algorithm coming from [17, 27]. The following content is about the graph $G_{A_l}(V_l, E_l)$ of the system matrices A_l on the l-th level. We have to emphasize that the goal of illustrating this classical algorithm is to present that our new algorithm generates the same aggregations with the classical algorithm.

The system matrix A, generating the graph $G_A(V, E)$, is generally gained by handling the PDE with different methods of discretization, e.g., 5-point FDM and 9-point FDM, etal. In this section, some definitions about graph theory are summarized again.

Definition 1 ([27]). *Corresponding to a sparse matrix A with symmetric sparsity pattern (i.e., $a_{i,j} \neq 0 \Leftrightarrow a_{j,i} \neq 0$), let $G_A(V, E)$ be the graph that consists of a set $V = \{v_1, v_2, v_3, \cdots, v_n\}$ of n ordered vertices (nodes, unknowns), and a set of edges E such that the edge $e_{i,j} \in E$ exists (connecting v_i and v_j) if and only if $a_{i,j} \neq 0, i \neq j$.*

For a vertex v_i, the set of neighbor vertices N_i is defined in the following form,

$$N_i = \{v_j \in V | e_{i,j} \in E\}, \tag{3}$$

$|N_i|$ denotes the number of the elements in the set N_i. The degree of a vertex v_i is $\deg(v_i) = |N_i|$.

For example, if the matrix is

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix},$$

and the $G_A(V, E)$ of the matrix A is shown in Figure 1.

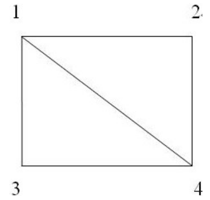


Figure 1: The matrix graph

The following contents introduce the classical aggregation algorithm and the construction of aggregations $\{A_i^l\}_{i=1}^{N_{l+1}}$ (i.e., the i-th aggregation on the l-th level), only depending on the l-th level system matrix A_l.

For a given parameter $\theta \in (0, 1]$, the strongly coupled neighborhood of the node v_i on the l-th level is defined as

$$N_i^l(\theta) = \left\{v_j \in V_l \middle| |a_{i,j}| \geq \theta\sqrt{a_{i,i}a_{j,j}}\right\}. \tag{4}$$

The classical aggregation algorithm, proposed by P. Vaněk, J. Mandel and M. Brezina [17] and utilized by Wagner [27], is presented in the following part.

Algorithm 1 Let a $N_l \times N_l$ matrix A_l with the corresponding graph $G_{A_l}(V_l, E_l)$ and $\theta \in (0,1]$ be given. The following Aggregation $(G_{A_l}(V_l, E_l))$ generates a disjoint covering $\left\{A_i^l\right\}_{i=1}^{N_{l+1}}$ of the set $V = \{v_1, v_2, v_3, \cdots, v_{N_l}\}$.

Aggregation($G_{A_l}(V_l, E_l)$)
{
  initialization:
  $U = \{v_i \in V_l | N_i^l(0) \neq \{v_i\}\}$;
  j=0;
  step 1:
  $for(v_i \in U)$
  {
  $if(N_i^l(\theta) \subset U)$
    $\{j++; A_j^l = N_i^l(\theta); U = U \backslash A_j^l;\}$
  }
  end

  step 2:
  $for(z \leq j) \quad \widetilde{A}_z^l = A_z^l; \quad end$
  $for(v_i \in U)$
  {
    $for(z \leq j)$
    {
      $if(N_i^l(\theta) \cap \widetilde{A}_z^l \neq \{\}) \quad \{A_z^l = A_z^l \cup \{v_i\}; U = U \backslash \{v_i\}; break;\}$
    }
    end
  }
  end

  step 3:
  $for(v_i \in U)$
  {
    $j++; A_j^l = N_i^l(\theta) \cap U; U = U \backslash A_j^l;$
  }
  end
}

In the part of initialization, the set U does not contain all nodes, meanwhile, isolated nodes are not aggregated. In step 1, disjoint strongly coupled neighborhoods are selected as the initial approximation of the covering. Step 2 adds remaining nodes $v_i \in U$ to one of the sets A_z^l to which the node v_i is strongly connected if any such set exists. Finally, in step 3, the still remaining nodes $v_i \in U$ are clustered into aggregations that consist of subsets of strongly coupled neighborhoods.

The above algorithm acts crucial role for AMG method due to generating the prearranged information that mentioned at the beginning of this section. The above algorithm, however, runs slowly because it has to aggregate every point in the domain and judge whether the points belong to certain aggregation. *Can we accelerate the process of above algorithm by some particular constructions?* Fortunately, next section will introduce the new discovery about the 9-point FDM based aggregation algorithm. We draw this inspiration of the discovery to develop a completely different algorithm with Algorithm 1.

3 The new aggregation algorithm

In this section, the discovery about the aggregations is illustrated clearly, meanwhile, the aggregation algorithm, according to the discovery, obtains the aggregations without through Algorithm 1 entirely but a new way of more economical computation. In classical algorithm (i.e., Algorithm 1), the final aggregations have to be gained by aggregating every point while the new way only needs to satisfy the particular condition about number of grids.

The new aggregation algorithm is based on the following definition of strongly coupled neighborhood, i.e., the eq. (4). If the problems are from the discretization of 9-point FDM, the $N_i(\theta)$, strongly coupled neighborhood of the node v_i, contains eight nodes around the v_i, e.g., $N_{10}(\theta) = \{v_2, v_3, v_4, v_9, v_{11}, v_{16}, v_{17}, v_{18}\}$ (Figure 2). Besides, the parameter θ must ensure existent according to the results in subsection 3.2.

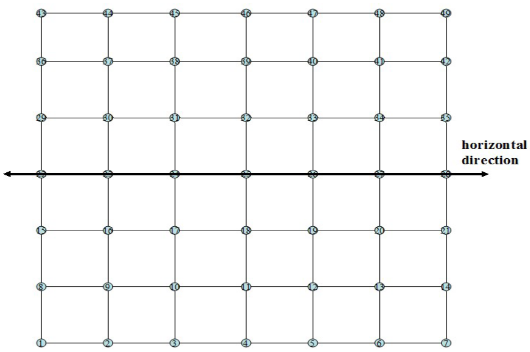
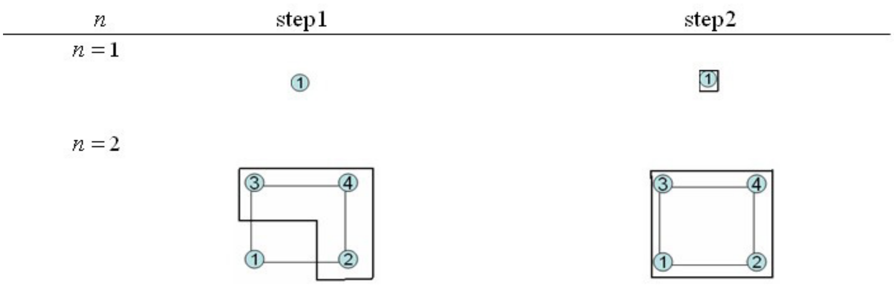


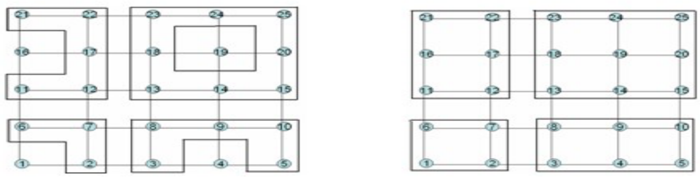
Figure 2: The instruction figure



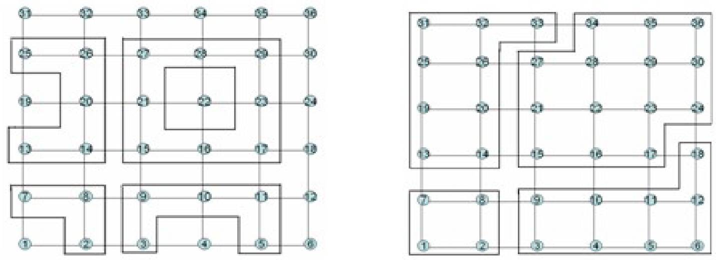
$n = 3$

- $n = 4$ $(i.e, n = 4 + 3 \times 0)$

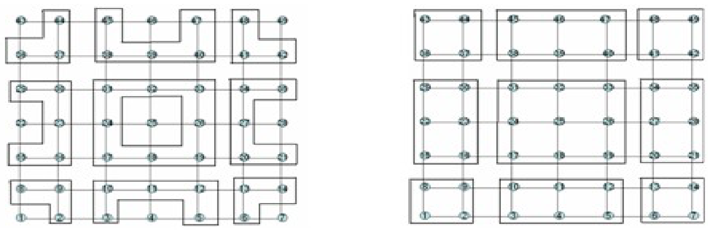
$n = 5$



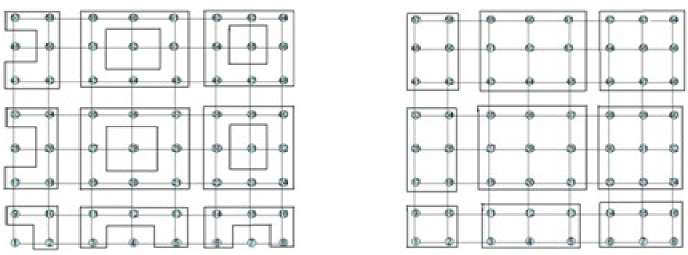
$n = 6$



- $n = 7$ $(i.e, n = 4 + 3 \times 1)$



$n = 8$



7

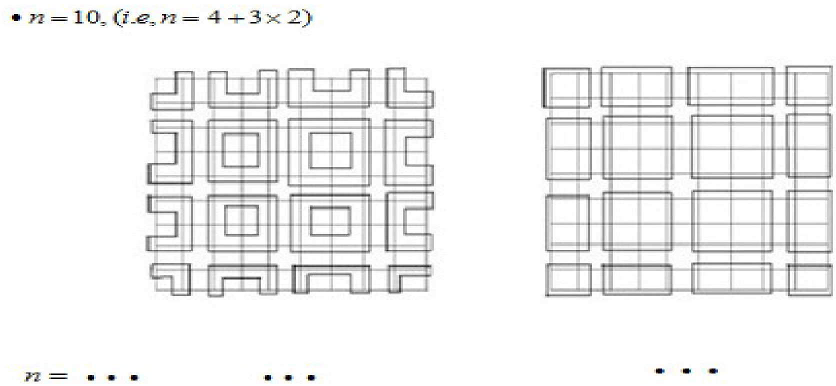


Figure 3: The process of constructing aggregations according to Algorithm 1 with 9-point FDM

3.1 The discovery for constructing aggregations based on 9-point FD-M

To demonstrate our new aggregation algorithm clearly, we give the Figure 3 with small grid number n. From Figure 3, we learn that

1. When $n = 2 + 3x, (x = 0, 1, 2, \cdots)$, the aggregations are symmetrical about back diagonal direction, and the number of aggregations is $(1 + x)^2$.

2. When $n = 4 + 3x, (x = 0, 1, 2, \cdots)$, the aggregations are symmetrical about horizontal direction and vertical direction (See Figure 2), and the number of final aggregations is $(2 + x)^2$.

Particularly, when $n = 4 + 3x, (x = 0, 1, 2, \cdots)$ (See $\bullet$ of Figure 3), there is the property of symmetry so that the aggregations can be gained by fixed scheme easily when the grid number is set as $n = 4+3x, (x = 0, 1, 2, \cdots)$ (See details in the following algorithm). We note the subscripts from left to right and then from down to up (See Figure 2) to illustrate our algorithm clearly.

Based on the discovery, the finally aggregation algorithm is given as follows.

Algorithm 2. Consider matrix $A \in R^{N_l \times N_l}(N_l = n_l^2, n_l = 4 + 3x, x = 0, 1, 2, \cdots)$ and corresponding graph $G_{A_l}(V_l, E_l)$ and $\theta \in (0, 1]$ being given. Then Aggregation $(G_{A_l}(V_l, E_l))$ generates a disjoint covering $\left\{A_i^l\right\}_{i=1}^{N_{l+1}}$ of the set $V = \{v_1, v_2, v_3, \cdots, v_{N_l}\}$.

Aggregation$(G_{A_l}(V_l, E_l))$
{
/* firstly, we have the relation: $n_l = 4 + 3x, (x = 0, 1, 2, \cdots), A_{k,j}^l = A_{(k-1)(x+2)+j}^l$ (See the following paragraph)*/

/***step 1:** get four angle's aggregations (See Figure 4 (a)) */

Get $A_{1,1}^l \quad A_{1,(x+2)}^l \quad A_{(x+2),1}^l \quad A_{(x+2),(x+2)}^l$

8

/* **step 2:** get the aggregations of upper boundary and lower boundary (See Figure 4 (b) (c)) */

for j=2:(x+1)
/* get the aggregations of lower boundary */

$$A^l_{1,j} = \{V_{3(j-1)}, V_{3(j-1)+1}, V_{3(j-1)+2}, V_{3(j-1)+n}, V_{3(j-1)+n+1}, V_{3(j-1)+n+2}\};$$

/* get the aggregations of upper boundary */

$$\begin{aligned} A^l_{(x+2),j} = \{&V_{3(j-1)+(3x+2)\cdot n}, V_{3(j-1)+(3x+2)\cdot n+1}, V_{3(j-1)+(3x+2)\cdot n+2},\\ &V_{3(j-1)+(3x+2)\cdot n+n}, V_{3(j-1)+(3x+2)\cdot n+n+1}, V_{3(j-1)+(3x+2)\cdot n+n+2}\}; \end{aligned}$$

end
/* **step 3:** get the aggregations of left boundary and right boundary and central zone (See Figure 4 (d) (e) (f)) */

for k=2:(x+1)
/* get the aggregations of left boundary (See Figure 4 (d))*/

$$\begin{aligned} A^l_{k,1} = \{&V_{(3k-4)\cdot n+1}, V_{(3k-4)\cdot n+2},\\ &V_{(3k-4)\cdot n+n+1}, V_{(3k-4)\cdot n+n+2},\\ &V_{(3k-4)\cdot n+2n+1}, V_{(3k-4)\cdot n+2n+2}\}; \end{aligned}$$

/* get the aggregations of right boundary (See Figure 4 (e))*/

$$\begin{aligned} A^l_{k,(x+2)} = \{&V_{(3k-4)\cdot n+3x+3}, V_{(3k-4)\cdot n+3x+4},\\ &V_{(3k-4)\cdot n+3x+3+n}, V_{(3k-4)\cdot n+3x+4+n},\\ &V_{(3k-4)\cdot n+3x+3+2n}, V_{(3k-4)\cdot n+3x+4+2n}\}; \end{aligned}$$

/* get the aggregations of central zone (See Figure 4 (f))*/

for j=2:(x+1)

$$\begin{aligned} A^l_{k,j} = \{&V_{(3k-4)\cdot n+3(j-1)}, V_{(3k-4)\cdot n+3(j-1)+1}, V_{(3k-4)\cdot n+3(j-1)+2},\\ &V_{(3k-4)\cdot n+3(j-1)+n}, V_{(3k-4)\cdot n+3(j-1)+n+1}, V_{(3k-4)\cdot n+3(j-1)+n+2},\\ &V_{(3k-4)\cdot n+3(j-1)+2n}, V_{(3k-4)\cdot n+3(j-1)+2n+1}, V_{(3k-4)\cdot n+3(j-1)+2n+2}\}; \end{aligned}$$

end
end
}
where the aggregation algorithm is under the condition that PDEs are discretilized by 9-point FDM when $n = 4 + 3x, (x = 0, 1, 2, \cdots)$. We utilize a useful formula $A_{k,j} = A_{(k-1)(x+2)+j}$, matching the Algorithm 2 for two-dimensional $A_{k,j}$. The formula is easy to be proved. Seeing Figure 5, we learn that $A_1 = A_{1,1}$, $A_2 = A_{1,2}$, $A_3 = A_{1,3}$, etc. By $n = 4+3x, (x = 0, 1, 2, \cdots)$, the aggregations' number of every row is $x+2$ and

the total number of $k-1$ rows is $(k-1)(x+2)$. So $A_{k,j} = A_{(k-1)(x+2)+j}$ is proved easily. We can obtain the aggregations of boundary and central zone easily through step 2 and step 3 of the above Algorithm 2, respectively. Finally, this algorithm generates the same aggregations with classical algorithm (i.e., Algorithm 1).

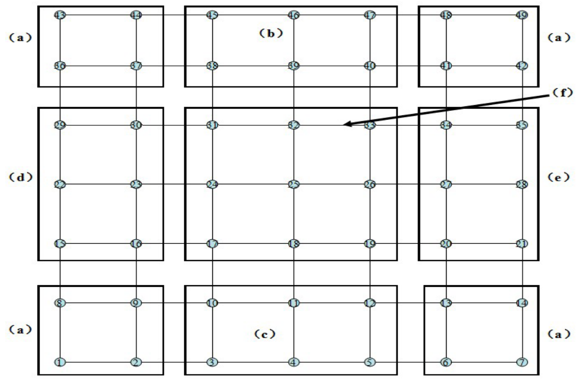


Figure 4: The instruction figure

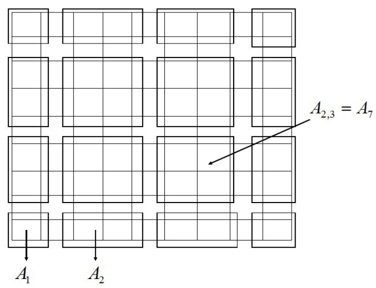


Figure 5: The instruction figure of formula $A_{k,j} = A_{(k-1)(x+2)+j}$

3.2 About the parameter θ

The parameter $\theta \in (0, 1]$ in equation (4) plays a significant role, because θ can decide the node v_j whether belongs to certain strongly coupled neighborhood $N_i(\theta)$ of node v_i. For example, if the parameter θ is smaller enough, then the corresponding strongly coupled neighborhood of node v_i will contain more nodes. Moreover, maybe the finally aggregations by the Algorithm 2 are changed obviously with slightly change of $\theta \in (0, 1]$, so it is necessary to discuss the parameter in equation (4).

Due to the discretization method (i.e., 9-point FDM) of corresponding problems, we hope the $N_i(\theta)$, the strongly coupled neighborhood of node v_i, contains the corresponding eight nodes of around v_i. We will demonstrate the existence of parameter θ firstly for this condition.

Theorem 1. *Let the strongly coupled neighborhood of node v_i be defined as equation (4). Consider the coefficient matrix $A \in R^{N\times N}$ arising from 9-point FDM, if $a_{ij} \neq$*

$0, (i \neq j)$, *then, there must exist* $\theta \in (0, 1]$, *such that* $N_i(\theta)$ *contains the corresponding eight nodes of around* v_i, $i = 1, 2, 3, \cdots, N$.

Proof. We learn that the coupled neighborhood of v_i is defined as (4)

$$N_i^l(\theta) = \left\{v_j \in V_l \middle| |a_{i,j}| \geq \theta\sqrt{a_{i,i}a_{j,j}}\right\}.$$

According to known conditions by 9-point FDM, all diagonal elements of matrix A are the nonzero (i.e, $a_{ii} \neq 0$), so above definition can be written as follows

$$0 < \theta \leq \frac{|a_{i,j}|}{\sqrt{a_{i,i} \cdot a_{j,j}}},$$

by $\theta \in (0, 1]$, we have

$$\theta \in (0, 1] \cap (0, \frac{|a_{i,j}|}{\sqrt{a_{i,i} \cdot a_{j,j}}}].$$

So we are sure that there must exist a θ such that $N_i(\theta)$ contains the corresponding eight nodes of around v_i, $i = 1, 2, 3, \cdots, N$. □

For 9-point FDM, the Theorem 1 can ensure the $N_i(\theta)$ containing the corresponding eight nodes of around v_i so that the Algorithm 2 is available. Furthermore, the simplified corollary will be presented in following part.

Corollary 1. *There exists the following definition of strongly couple neighborhood of node* v_i *where coefficient matrix* $A \in R^{N\times N}$ *arising from 9-point FDM*

$$N_i^l(\theta) = \left\{v_j \in V_l \middle| a_{i,j} > 0\right\}, \tag{5}$$

such that $N_i(\theta)$ *contains the corresponding eight nodes of around* v_i, $i = 1, 2, 3, \cdots, N$.

Proof. Due to the 9-point FDM, the coefficient matrix $A \in R^{N\times N}$ is a nine diagonal matrix that every row of A has only nine nonzero elements including $a_{i,i} \neq 0$. According to the new definition (5) and 9-point FDM, it is easy to know that $N_i(\theta)$ contains the corresponding eight nodes of around v_i, $i = 1, 2, 3, \cdots, N$. □

3.3 Extending to multilevel

This section mainly makes a discussion about extending the proposed Algorithm 2 to multilevel. According to section 3.1, if the grid number on the fine level is $(3x + 4) \times (3x+4)$ $(x = 0, 1, 2, \cdots)$, then the grid number on the next coarse level is $(x+2)\times(x+2)$ under the condition that one aggregation on the fine level generates only one grid node on the coarse level. In order to extend the relationship from two level to multilevel, two significant conclusions are given in the following analysis.

Theorem 2. *Let the number of levels of multigrid be L and assuming the gird number is* $n_L = 3i + 4$ *(i.e., the square grid is* $(3i + 4) \times (3i+4)$*) on the coarsest level. If the girds number* n *on the finest level satisfies the following equation*

$$n = (\sum_{j=1}^{L-1} 3^j \cdot 2) + 3^L \cdot i + 4, \tag{6}$$

then the Algorithm 2 can be extended to multilevel.

Proof. Since the number of levels is L and the gird number on the coarsest level L is $n_L = 3i + 4$. According to the conclusions of section 3.1, the grid number on the $(L-1)$-th level should be

$$n_{L-1} = 3(n_L - 2) + 4 = 3(3 \cdot i + 4 - 2) + 4 = 3^2 \cdot i + 3 \cdot 2 + 4,$$

and the grid number on the $(L-2)$-th level should be

$$\begin{aligned} n_{L-2} &= 3(n_{L-1} - 2) + 4 = 3(3^2 \cdot i + 3 \cdot 2 + 4 - 2) + 4 \\ &= 3^3 \cdot i + 3^2 \cdot 2 + 3 \cdot 2 + 4 = (\sum_{j=1}^{2} 3^j \cdot 2) + 3^3 \cdot i + 4, \end{aligned}$$

and similar to above, the grids number on the $(L-3)$-th level should be

$$\begin{aligned} n_{L-3} &= 3(n_{L-2} - 2) + 4 = 3(3^3 \cdot i + 3^2 \cdot 2 + 3 \cdot 2 + 4 - 2) + 4 \\ &= 3^3 \cdot i + 3^2 \cdot 2 + 3 \cdot 2 + 4 = (\sum_{j=1}^{3} 3^j \cdot 2) + 3^4 \cdot i + 4, \end{aligned}$$

it is easy to extend to the finest level by mathematical induction, the grids number on the finest level should satisfy

$$n_1 = 3(n_2 - 2) + 4 = 3((\sum_{j=1}^{L-2} 3^j \cdot 2) + 3^{L-1} \cdot i + 4 - 2) + 4 = (\sum_{j=1}^{L-1} 3^j \cdot 2) + 3^L \cdot i + 4,$$

and $n = n_1$ is the grids number satisfying Algorithm 2 on the finest level and it also extends the Algorithm 2 to multilevel. □

Theorem 2 requires the grid number on coarsest level being $n_L = 3i + 4, i = 0, 1, 2, \cdots$, moreover, we can also extend to the arbitrary grids number $n_L = i, i = 0, 1, 2, \cdots$, on coarsest level.

Corollary 2. *For arbitrary grid number $n_L = i, i = 0, 1, 2, \cdots$, on the coarsest level, if the number of levels is L, then the gird number n on the finest level satisfy*

$$n = (\sum_{j=1}^{L-2} 3^j \cdot 2) + 3^{L-1} \cdot (i - 2) + 4, \tag{7}$$

then the Algorithm 2 can be extended to multilevel.

Proof. It is easy to gain the Corollary 2 by replacing $n_L = 3i+4$ in Theorem 2 with $n_L = i$. □

4 Computational experiments

All experimental problems are discretized by 9-point FDM. Before our experiments, we will introduce the 9-point FDM briefly.

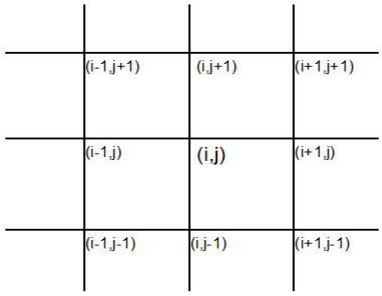


Figure 6: The instruction figure

We can get the 9-point FDM from the 5-point FDM in which one point (i, j) is only relevant to its adjacent four points (See Figure 6), i.e., $(i-1, j), (i+1, j), (i, j-1), (i, j+1)$. For example, if the elliptic PDE in a square domain is Poisson equation

$$-\Delta u = -(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = f(x, y), (x, y) \in \mathcal{R}^{[a,b]\times[a,b]}, \tag{8}$$

if $h_1 = h_2 = (b-a)/(n+1)$, then the 5-point FDM scheme can be obtained as follows

$$-\Delta_h u_{i,j} = \frac{1}{h^2}(-u_{i,j+1} - u_{i,j-1} - u_{i+1,j} - u_{i-1,j} + 4u_{i,j}) = f_{i,j}, \tag{9}$$

We define the vector $u_h = [u_{11}, u_{21}, \cdots, u_{n,1}; \cdots; u_{1,n}, u_{2,n}, \cdots, u_{n,n}]^T$ and assume zero boundary, then the finally linear system is obtained by (9)

$$\frac{1}{h^2} H u_h = g, \tag{10}$$

where

$$H = \begin{pmatrix} B & -I & & & \\ -I & B & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & B & -I \\ & & & -I & B \end{pmatrix}, B = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix},$$

where the right hand vector is known beforehand and I is the identity matrix.

Then we rotate the coordinate system with $\pi/4$ so that the point (i, j) is relevant to its adjacent four points (See Figure 6), i.e., $(i-1, j+1), (i-1, j-1), (i+1, j-1), (i+1, j+1)$. By this rotation, another 5-point FDM scheme is as follows

$$-\bar{\Delta}_h u_{i,j} = \frac{1}{2h^2}(-u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} - u_{i-1,j-1} + 4u_{i,j}) = f_{i,j}, \tag{11}$$

and it also gains the similar linear system with (10) but the wider bandwidth of coefficient matrix.

Combining with the above two 5-point FDM scheme (9) and (11), the finally 9-point FDM scheme of Poisson equation [10] is determined,

$$-(\frac{2}{3}\Delta_h + \frac{1}{3}\bar{\Delta}_h)u_{ij} = f_{ij} + \frac{h^2}{12}\Delta_h f_{ij}, \tag{12}$$

where this scheme has smaller truncation error of $O(h^4)$ than 5-point FDM scheme.

Besides, some notations are necessary to be introduced. The t_i where $i = 1, 2, \cdots, L-1$, is just the CPU time of constructing aggregations by Algorithm 2 on the i-th level. $T_i, i = 1, 2, \cdots, L-1$, represents the total CPU time of generating prolongation operators by the following equation (13) and the coefficient matrices by equation (2) on the i-th level, respectively.

$$P^l_{ij} = \begin{cases} 1, & i \in A^l_j, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Moreover, the dimension N of coefficient matrix on the finest level, i.e., n^2, is computed by equation (6). Next, we will present two examples to demonstrate the efficiency of our algorithm.

4.1 Example 1: Poisson-like equation

First example is a 2D Poisson-like equation containing two scalars $\alpha, \beta \in \mathcal{R}$, it can be written in the form of

$$-(\alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^2 u}{\partial y^2}) = f(x,y), (x,y) \in \Omega = [a,b] \times [a,b], \tag{14}$$

where the $f(x,y) \in \mathcal{R}$ is an arbitrary function and the boundary condition is

$$u(a,y) = u(b,y) = u(x,a) = u(x,b) = C \in \mathcal{R}. \tag{15}$$

It is easy to obtain the finally coefficient matrix $A \in \mathcal{R}^{N\times N}$ arising from the 9-point FDM of equation (14). A is nine diagonal matrix

$$A = \begin{pmatrix} B & R & & & \\ R & B & R & & \\ & \ddots & \ddots & \ddots & \\ & & R & B & R \\ & & & R & B \end{pmatrix}, B = \begin{pmatrix} e & k & & & \\ k & e & k & & \\ & \ddots & \ddots & \ddots & \\ & & k & e & k \\ & & & k & e \end{pmatrix}, R = \begin{pmatrix} p & q & & & \\ q & p & q & & \\ & \ddots & \ddots & \ddots & \\ & & q & p & q \\ & & & q & p \end{pmatrix},$$

where $e = 12(\alpha+\beta) - 4\alpha\beta, k = 2\alpha\beta - 6\alpha, p = 2\alpha\beta - 6\beta, q = -\alpha\beta$.

In Table 1, we choose the L being 3 levels and set the grid number n_L on the coarsest level being 64, 94 and 124, respectively. According to Theorem 2, the dimensions of linear systems on the finest level are 322624, 702244, 1227664, respectively. The CPU time t, constructing aggregation on each level, is very short and not exceeding 0.4s for the large-scale matrix with dimension 1227664 while the classical algorithm exceeds 1000s. The total time for the dimension with 322624, 702244, 1227664 is about 10.824s, 49.501s, 148.483s, respectively.

Table 1: CPU time for Poisson-like equation by our method with 3 levels

	t		T		Total
N	t_1	t_2	T_1	T_2	t+T
322624	0.102	0.009	10.525	0.188	10.824
702244	0.192	0.018	48.601	0.690	49.501
1227664	0.320	0.033	146.160	1.970	148.483

Table 2: CPU time for Poisson-like equation by our method with 4 levels

		t			T		Total
N	t_1	t_2	t_3	T_1	T_2	T_3	t+T
237169	0.081	0.006	0.0008	5.65	0.108	0.007	5.853
795664	0.215	0.021	0.003	62.150	0.849	0.027	63.265
1682209	0.436	0.045	0.005	274.050	3.644	0.077	278.257

In Table 2, L is 4 and the grid number n_L is 19, 34, 49, respectively. The dimensions of linear systems on the finest level are 237169, 795664, 1682209, respectively, according to Theorem 2. The consuming time t for constructing aggregations on each level is not exceeding 0.5s for the large-scale matrix with dimension 1682209 while the classical algorithm can not compute the consuming time. Total time for the dimension with 237169, 795664, 1682209 is about 5.853s, 63.265s, 278.257s, respectively. The two tables with different maximal levels illustrate that the Algorithm 2 is indeed a novel and fast method for the *setup phase* of aggregation-base AMG method.

From Table 1 and Table 2, it is easy to learn that the total time does not only contain the time t, constructing aggregations by Algorithm 2, but also the time T, constructing prolongation operators and generating the system matrices on each level. Furthermore, the mainly cost of total time is clearly T_1, because the matrices, keeping largest dimension on the finest level, are referred to vast matrix-matrix multiplication according to equation (2). The time on other levels are shorter seriously than the finest level and decreased evidently.

4.2 Example 2: Helmholtz equation

The second example containing a scalars $\omega \in \mathcal{R}$ is a 2D Helmholtz equation, the form of this equation is as follows

$$-(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - \omega^2 u = f(x,y), (x,y) \in \Omega = [a,b] \times [a,b], \tag{16}$$

where the $f(x,y) \in \mathcal{R}$ can be also an arbitrary function and the boundary condition is

$$u(a,y) = u(b,y) = u(x,a) = u(x,b) = C \in \mathcal{R}, \tag{17}$$

Table 3: Time consuming for Helmholtz equation by our method with 4 levels

	t			T			Total
N	t_1	t_2	t_3	T_1	T_2	T_3	t+T
237169	0.085	0.008	0.001	5.84	0.119	0.009	6.062
795664	0.275	0.026	0.004	62.450	0.836	0.030	63.621
1682209	0.442	0.055	0.006	280.150	3.744	0.087	284.484

where $\omega \in R$ is a determined scalar, $h = (b-a)/(n+1)$. Similar to section 4.1, A is also a nine diagonal matrix

$$A = \begin{pmatrix} B & R & & & \\ R & B & R & & \\ & \ddots & \ddots & \ddots & \\ & & R & B & R \\ & & & R & B \end{pmatrix},$$

and

$$B = \begin{pmatrix} 20-2h^2\omega^2 & -4 & & & \\ -4 & 20-2h^2\omega^2 & -4 & & \\ & \ddots & \ddots & \ddots & \\ & & -4 & 20-2h^2\omega^2 & -4 \\ & & & -4 & 20-2h^2\omega^2 \end{pmatrix}, R = \begin{pmatrix} -4 & -1 & & & \\ -1 & -4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & -4 & -1 \\ & & & -1 & -4 \end{pmatrix},$$

In this example, ω, set to be 0.2, is utilized for all experiments. Actually the linear system generated by 9-point FDM has the same form with example 1, so the CPU time for the same dimension problem is almost not different. In Table 3, similar to Table 2, L is set to be 4 and n_L is also 19, 34, 49, respectively. The dimensions of linear systems on the finest level are 237169, 795664, 1682209, respectively. From Table 3, it is easy to learn that our algorithm speeds much less time while the classical one can not finish the *setup phase* within 1000s. Besides, the number of nonzero elements (NNZ) of system matrices on each level is presented in Table 4 where the NNZ1, NNZ2, NNZ3 and NNZ4 represent the number of nonzero elements on level 1, 2, 3 and 4, respectively.

Furthermore, we will try some larger scale system matrices to illustrate our Algorithm 2 all alone, i.e., the prolongation operators and system matrices on each level is out of range in the following test. L and is chosen to be 3 and n_L is set to be 229, 379, 604, 754, 904, respectively, i.e., the dimensions of the system matrices on the finest level are 4214809, 11580409, 29463184, 45941284 and 66064384, respectively. The finally shown results are in Table 5 clearly and indeed quite attractive.

5 Conclusion

This paper describes a new algorithm for constructing the aggregations in the *setup phase* of aggregation-based AMG method. The new algorithm, utilizing some particular

Table 4: NNZ on each level for Helmholtz equation by our method with 4 levels

N	NNZ1	NNZ2	NNZ3	NNZ4
237169	2128681	237169	26569	3025
795664	7150276	795664	88809	10000
1682209	15124321	1682209	187489	21025

Table 5: Time consuming of constructing aggregations by Algorithm 2 with 3 levels

N	4214809	11580409	29463184	45941284	66064384
t_1	1.028	2.815	7.002	10.844	15.553
t_2	0.091	0.260	0.713	0.112	1.620
Total	1.119	3.075	7.715	10.956	17.173

settings, e.g., the particular grid number on the finest level according to Theorem 2 and the discretization with 9-point FDM, is different with the any previous aggregation algorithms. During the process of constructing aggregations, the symmetry of the aggregations was discovered if the number of square grid satisfies the conditions of equation (5), (6) and (7). Moreover, some theoretical and practical conclusions such as Theorem 1, etal., were also illustrated in this paper. Computational experiments for Poisson-like equation and Helmholtz-like equation presented that the new aggregation algorithm captured the perfect results in the CPU time even for millions grade problems.

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