Stochastic quantization of conformally coupled scalar in AdS

Dileep P. Jatkar[a1] and Jae-Hyuk Oh[b2]

Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad-211019, India[a]

Department of Physics, Hanyang University, Seoul 133-791, Korea[b]

Abstract

We explore the relation between stochastic quantization and holographic Wilsonian renormalization group flow further by studying conformally coupled scalar in AdS_{d+1}. We establish one to one mapping between the radial flow of its double trace deformation and stochastic 2-point correlation function. This map is shown to be identical, up to a suitable field re-definition of the bulk scalar, to the original proposal in arXiv:1209.2242.

[1]e-mail:dileep@hri.res.in

[2]e-mail:jack.jaehyuk.oh@gmail.com

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1 Introduction and Summary

The AdS/CFT correspondence relates d dimensional field theory to $d+1$ dimensional theory of gravity. This relation has been explored in great detail over the years in various context. Stochastic quantization[1, 2, 3] is a formalism which studies non-equilibrium dynamics of d dimensional field theory which evolves along stochastic time variable. Resulting theory is interpreted as a $d+1$ dimensional field theory. There were proposals relating AdS/CFT corresponding to the stochastic quantization in the past[6, 7, 8, 9].

Recently, we proposed a specific relation between AdS/CFT and Stochastic quantization. In [10], we proposed that the Hamiltonian governing the holographic Wilsonian renormalization equations[4, 5] in the AdS/CFT correspondence is equal to the Fokker-Planck Hamiltonian of the stochastic system. It in turn implies the stochastic time is identified with the radial variable in the AdS space. We also showed that our proposal works for theories which are invariant under Weyl rescaling[3]. Using this relationship it was shown that the Stochastic quantization correctly reproduces the radial evolution of the double trace coupling for the boundary theory.

This proposal is based on the direct analogy between the holographic RG equation,

$$\partial_\epsilon \psi_H(\phi, r) = -\int_{r=\epsilon} d^d x \mathcal{H}_{RG}(-\frac{\delta}{\delta\phi}, \phi)\psi_H(\phi, r), \tag{1.1}$$

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[3]In [10], we have dealt with theories which are invariant under the scaling of the background metric as $g_{\mu\nu} \to \lambda(r)g_{\mu\nu}$, where μ and ν are spacetime indices and $\lambda(r)$ is an arbitrary radial coordinate r-dependent function.

where, $\mathcal{H}_{RG}$ is Legendre transform of the bulk action in AdS space, $\psi_H = e^{-S_B}$ and S_B is the boundary effective action and on the stochastic side, the Fokker-Planck equation

$$\partial_t \psi_S(\phi, t) = - \int d^d x \mathcal{H}_{FP}(\frac{\delta}{\delta \phi}, \phi) \psi_S(\phi, t), \tag{1.2}$$

where, H_{FP} is the Fokker-Planck Hamiltonian, which can be derived from the Fokker-Planck action by Legendre transform. The stochastic wave-functional is written in terms of the probability distribution $P(\phi, t)$ and the classical action S_c as

$$\psi_S(\phi, t) = P(\phi, t) e^{\frac{S_c(\phi(t))}{2}}. \tag{1.3}$$

In fact, the relation between the boundary effective action obtained by solving Hamilton-Jacobi equations derived from the bulk action and stochastic 2-point correlator obtained from the solution of Langevin equation addressed in [10] is given by

$$\langle \phi_p(t) \phi_{-p}(t) \rangle_H^{-1} = \langle \phi_p(t) \phi_{-p}(t) \rangle_S^{-1} - \frac{1}{2} \frac{\delta^2 S_c}{\delta \phi_p \delta \phi_{-p}}, \tag{1.4}$$

where $\langle \phi_p(t) \phi_{-p}(t) \rangle_S$ is stochastic 2-point correlation function, $\langle \phi_p(t) \phi_{-p}(t) \rangle_H^{-1} = \frac{\delta^2 S_B}{\delta \phi_p \delta \phi_{-p}}$ and the stochastic time 't' is identified to the radial coordinate 'r' in AdS space . From the Fokker-Planck approach, it is also shown that

$$S_B = \int_{t_0}^t dt' d^d p \; \mathcal{L}_{FP}(\phi(t'), \partial \phi(t'); t'), \tag{1.5}$$

where $\mathcal{L}_{FP}$ is called Fokker-Planck Lagrangian density. This relation with the boundary effective action is consistent with (1.4).

In this paper we will extend our analysis to conformally coupled scalar[4]. As we did earlier, namely cases involving Weyl invariant theories, we will treat AdS metric as a fixed background except that in this case we will consider conformal coupling of the scalar field with spacetime scalar curvature. Since the background is maximally symmetric, conformal coupling terms shows up in the action as a mass term for the scalar field. Interestingly this mass falls within the window above the Breitenlohner-Freedman bound for any dimensional AdS space which allows alternative quantization of the scalar field in the AdS space[11, 12, 13, 14, 15, 16, 17]. We can therefore study double trace coupling obtained by carrying out alternate quantization. From the stochastic quantization point of view this example poses a new problem. The Langevin equation for this system turns out to have explicit stochastic time dependence. Nevertheless, as we will see, it is still possible to use the Langevin equation to determine equal time two-point correlation function. We will also be able to extract the Fokker-Planck action by eliminating the noise term using the Langevin equation.

It turns out that the above relations, proposed in [10], are still valid provided the classical action S_c is obtained in a more general way, which is the crucial ingredient in the above

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[4]Conformally coupled scalar theories have been discussed in the literature, especially in the AdS$_4$ context(see [18, 19]).

relation. In fact, one should be careful in choosing the classical action because in general there are divergences and one may need to add counter terms to regulate them. Similar issue arises for the classical action S_c in case of conformally coupled scalar in AdS space. In [10], it was proposed that $S_c = -2I_{os}(\phi_0)$, where **(1)** I_{os} **is bulk on-shell action computed on AdS boundary(at** $r = 0$, **where** r **is the radial coordinate of AdS space). Moreover, (2)there as no need to add counter term action in case of examples discussed in [10], because those examples involved Weyl invariant bulk actions only. It turns out that Weyl invariant bulk actions do not give rise to divergent terms at the AdS boundary.**[5]

Conformally coupled scalar action does give rise to divergences near AdS boundary since it is not exactly Weyl invariant theory even if it does enjoy certain scaling properties. Therefore the natural question that arises is how do we deal with these divergences. Our prescription is that **the bulk on-shell action,** $I_{os}(\phi(\epsilon))$ **is obtained at a certain radial cut-off,** $r = \epsilon$ **without adding any counter terms**, where $\phi(\epsilon)$ is the boundary value of the bulk scalar field at $r = \epsilon$ and then I_{os} should be written in terms of $\phi(\epsilon)$. **The classical action** S_c **is then defined using same relation,** $S_c(\phi(\epsilon)) = -2I(\phi(\epsilon))$ **but at the radial cut off.**

The new definition of the classical action makes sense since it correctly reproduces the classical actions for Weyl invariant cases, and so does the expected form of stochastic 2-point correlation functions. The on-shell action depends on radial cut-off $r = \epsilon$ explicitly in general, and that can be translated to the explicit stochastic time dependence of the classical action S_c defined on a certain time slice $t = \epsilon$ when the radial coordinate r is identified with t.

We will then show that same result can be derived in a more elegant way by doing field redefinition,

$$\phi(t, p) = \Omega(t) f_p(t), \tag{1.6}$$

where $\Omega(t)$ is a certain stochastic time t-dependent function[6]. Interesting feature of this field redefinition is that the Langevin dynamics in terms of $f_p(t)$ does not contain explicit dependence on the stochastic time. In fact in terms of $f_p(t)$ the system becomes quite similar to that studied in the Weyl invariant examples. This analysis gives result consistent with that obtained without doing the field redefinition. Thus while appropriate Langevin and Fokker-Planck descriptions can be derived even when there is explicit stochastic time dependence, we also can access conventional description by doing a field redefinition. In other words, we can retain essence of our proposed relation between AdS/CFT and stochastic quantization if we allow for field redefinition.

This paper is organized as follows, in section 2 we will discuss holographic Wilsonian Renormalization Group description of conformally coupled scalar in AdS_{d+1}. We solve for double trace deformation both for zero as well as non-zero momenta. To draw analogy with the field redefinition that is we will carry out while studying the Langevin dynamics, we will study effect of field redefinition on the AdS side. In section 3, we study stochastic quantization by first studying stochastic time dependent Langevin equation and the deriving the Fokker-Planck action. In section 4, we carry out the field redefinition and show that in the new

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[5]These issues are addressed in the conclusion section of [10].

[6]In fact, $\Omega(t)$ should be restricted by a certain differential equation so that using the field redefinition consistency between Langevin and Fokker-Planck approaches can be established.

variable, both the Langevin as well as the Fokker-Planck dynamics take canonical form and the original dictionary relating stochastic quantization to AdS/CFT can be applied without any modification.

2 Holographic Wilsonian renormalization group(HWRG) for conformally coupled scalar in AdS_{d+1}

In this section, we derive Hamilton-Jacobi equations for the holographic Wilsonian RG and their solutions for conformally coupled scalar in AdS_{d+1}.

2.1 Conformally coupled scalar and the radial flow of its double trace deformations

We start with the full bulk action for a scalar field propagating in AdS_{d+1} as

$$S = \int_{r>\epsilon} dr d^d x \sqrt{g} \mathcal{L}(\phi, \partial\phi) + S_B, \tag{2.1}$$

where S_B is the boundary effective action and the bulk Lagrangian density $\mathcal{L}$ is defined as

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^{\frac{2(d+1)}{d-1}}, \tag{2.2}$$

where $g_{\mu\nu}$ is Euclidean AdS_{d+1} metric, which is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{dr^2 + \sum_{i=1}^{d} dx^i dx^i}{r^2}. \tag{2.3}$$

$g^{\mu\nu}$ is the inverse metric, $\mu, \nu...$ run from 1 to $d+1$ whereas $i, j...$ run from 1 to d. ϵ is an arbitrary radial cut-off. The higher order interaction term in (2.2) is rather ill-defined since the power of it will be fractional in general. However, it is well defined in a certain bulk dimensions, for example, it becomes ϕ^4 interaction in AdS_4 and ϕ^3 in AdS_6 respectively[7]. In what follows, we will choose $m^2 = -\frac{d^2-1}{4}$ and will set $\lambda = 0$ to deal with free theory for a moment. We point out that there are two different merits when the mass of the scalar field is chosen to be $m^2 = -\frac{d^2-1}{4}$. Firstly, this mass value is in the window of mass square of the scalar field $-\frac{d^2}{4} \leq m^2 \leq -\frac{d^2}{4} + 1$. **In such a case, alternative quantization in the dual CFT defined on the boundary of AdS space is possible**, and then we have two different fixed points for the double trace deformation coupling in UV region. Secondly, it will show a scaling property that will be discussed in the next subsection. This allows us to deal with this theory from a different view point and provides a more rigorous way of defining relation between SQ and HWRG of this theory.

As usual, in order to derive the Hamilton-Jacobi type HWRG flow equation[4, 5], we take derivative of the bulk action(2.1) with respect to ϵ (the radial cutoff), and impose the condition

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[7]4-dimensional case is conformally coupled scalar in AdS_4. For detailed discussion, see [8]

that the full bulk action S does not depend on the radial cut-off. The Hamilton-Jacobi equation thus obtained is given by

$$\partial_\epsilon S_B = -\int_{r=\epsilon} d^d x \left[\frac{1}{\sqrt{g}g^{rr}} \left(\frac{\delta S_B}{\delta\phi(x)}\right)\left(\frac{\delta S_B}{\delta\phi(x)}\right) - \sqrt{g}\mathcal{L}(\phi, \partial\phi) \right]. \tag{2.4}$$

It is convenient to solve the above equation in momentum space by using the Fourier transform

$$\phi(x^\mu) = \frac{1}{(2\pi)^{d/2}} \int_{-\infty}^{\infty} d^d p\, e^{-ip_i x_i} \phi_p(r). \tag{2.5}$$

The HWRG equation in the momentum space then becomes

$$\partial_\epsilon S_B = -\int_{r=\epsilon} d^d p \left[\frac{1}{2\sqrt{g}g^{rr}} \left(\frac{\delta S_B}{\delta\phi_p}\right)\left(\frac{\delta S_B}{\delta\phi_{-p}}\right) - \frac{1}{2}\sqrt{g}g^{ij}p_i p_j \phi_p \phi_{-p} + \frac{d^2-1}{8}\sqrt{g}\phi_p\phi_{-p} \right], \tag{2.6}$$

where the AdS$_{d+1}$ metric $g^{ij} = r^2 \delta_{ij}$ and δ_{ij} is the Kronecker delta function. To solve this equation, we propose the following form of the boundary effective action:

$$S_B = \Lambda(\epsilon) + \int \frac{d^d p}{(2\pi)^d} \sqrt{\gamma}\mathcal{J}(\epsilon, p)\phi_{-p} - \int \frac{d^d p}{2(2\pi)^d} \sqrt{\gamma}\mathcal{D}(\epsilon, p)\phi_p\phi_{-p}, \tag{2.7}$$

where $\mathcal{D}$ is the "double-trace" coupling, $\mathcal{J}$ is the boundary source term and Λ is the boundary cosmological constant. Substituting this ansatz into Eq.(2.6) and comparing the coefficients of expansion in the boundary fields ϕ_p, we get the following three equations

$$\partial_\epsilon \Lambda(\epsilon) = -\frac{1}{2}\int \frac{d^d p}{(2\pi)^{2d}} \frac{1}{\sqrt{g}g^{rr}}J(\epsilon, -p)J(\epsilon, p), \tag{2.8}$$

$$\partial_\epsilon J(\epsilon, p) = \frac{1}{\sqrt{g}g^{rr}(2\pi)^d}J(\epsilon, -p)D(\epsilon, p), \tag{2.9}$$

$$\text{and } \partial_\epsilon D(\epsilon, p) = \frac{1}{\sqrt{g}g^{rr}(2\pi)^d}D(\epsilon, p)D(\epsilon, -p) - (2\pi)^d\sqrt{g}\left(r^2\delta_{ij}p_i p_j - \frac{d^2-1}{4}\right), \tag{2.10}$$

where $J(\epsilon, p) \equiv \sqrt{\gamma}\mathcal{J}(\epsilon, p)$, $D(\epsilon, p) \equiv \sqrt{\gamma}\mathcal{D}(\epsilon, p)$ and $\gamma = \frac{g(\epsilon)}{g_{rr}(\epsilon)}$ is the induced metric on the $r = \epsilon$ hyper-surface.

As demonstrated as in [5], the solution of double trace coupling, D is given by

$$D(\epsilon, p) = -(2\pi)^d \frac{\Pi_\phi}{\phi}, \tag{2.11}$$

where

$$\Pi_\phi = \sqrt{g}g^{rr}\partial_r\phi = \frac{\delta S_B}{\delta\phi} \tag{2.12}$$

is canonical momentum of ϕ and it satisfies

$$\partial_r \Pi_\phi = \sqrt{g}\left(r^2|p|^2 - \frac{d^2-1}{4}\right)\phi_p, \tag{2.13}$$

in the classical gravity limit of the bulk theory.

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Double trace deformation: zero momentum solution To examine the double trace deformation term $\mathcal{D}$, we need to solve bulk equations of motion for the conformally coupled scalar. The bulk equation of motion is given by

$$0 = g^{\mu\nu}\nabla_\mu\partial_\nu\phi(x) + \frac{d^2-1}{4}\phi(x) - \frac{\lambda(d+1)}{2(d-1)}\phi^{\frac{d+3}{d-1}}, \qquad (2.14)$$

where ∇_μ is covariant derivative. In fact, this is also given by combining Eq.(2.12) and Eq.(2.13) in momentum space in the limit $\lambda = 0$,

$$0 = \partial_r^2\phi_p - \frac{d-1}{r}\partial_r\phi_p + \left(\frac{d^2-1}{4r^2} - p^2\right)\phi_p, \qquad (2.15)$$

where $p^2 = \sum_{i,j=1}^d p_i p_j \delta_{ij}$. In the zero momentum limit, $p_i = 0$, the most general solution is given by

$$\phi = a_1 r^{\frac{d-1}{2}} + a_2 r^{\frac{d+1}{2}}, \qquad (2.16)$$

where a_1 and a_2 are arbitrary constants. Using the solution of Hamilton-Jacobi equation (2.11), the double trace coupling becomes

$$\mathcal{D}(r) = \frac{D(r)}{\sqrt{\gamma}} = -(2\pi)^d \frac{d-1}{2}\left(\frac{\frac{d+1}{d-1}r + \chi}{r + \chi}\right), \qquad (2.17)$$

where $\chi = \frac{a_1}{a_2}$. There are two different fixed points for the double trace coupling, $\mathcal{D}(r)$ at UV region, $r = 0$. When $\chi = 0$, the double trace coupling has $\mathcal{D}(r = 0) = -(2\pi)^d\left(\frac{d+1}{2}\right)$ at the UV region and it is a fixed point. Another fixed point is obtained when $\chi = \infty$. In this case, $\mathcal{D}(r = 0) = -(2\pi)^d\left(\frac{d-1}{2}\right)$. In the IR region, $r = \infty$, the fixed points exist. When $\chi = \infty$, $\mathcal{D}(r = \infty) = -(2\pi)^d\left(\frac{d-1}{2}\right)$ is fixed point. For the other generic value of χ including $\chi = 0$, $\mathcal{D}(r = \infty) = -(2\pi)^d\left(\frac{d+1}{2}\right)$ is fixed point.

Finally, the double trace deformation part of boundary effective action S_B is given by

$$S_B^{DT} = \frac{1}{2}\left(\frac{d-1}{2r^d}\right)\left(\frac{\frac{d+1}{d-1}r + \chi}{r + \chi}\right)\phi^2. \qquad (2.18)$$

Solution with non-zero momenta The most general solution of this equation of motion with non-zero momenta p_i is

$$\phi_p = r^{\frac{d-1}{2}}\left[\phi_0(p)\cosh(|p|r) + \phi_1(p)\sinh(|p|r)\right], \qquad (2.19)$$

where $|p|$ is norm of p_i, $\phi_0(p)$ and $\phi_1(p)$ are arbitrary momentum dependent functions. Conjugate momentum $\Pi_\phi(p)$ is obtained using its definition (2.12) as

$$\Pi_\phi(p) = \frac{\frac{d-1}{2}\phi_0(p) + |p|r\phi_1(p)}{r^{\frac{d+1}{2}}}\cosh(|p|r) + \frac{\frac{d-1}{2}\phi_1(p) + |p|r\phi_0(p)}{r^{\frac{d+1}{2}}}\sinh(|p|r). \qquad (2.20)$$

The double trace deformation coupling, $\mathcal{D}(r,p)$ is then given by

$$\mathcal{D}(r,p) = \frac{D(r,p)}{\sqrt{\gamma}} = -(2\pi)^d \left[\frac{d-1}{2} + |p|r \frac{\sinh(|p|r) + \tilde{\phi}(p)\cosh(|p|r)}{\cosh(|p|r) + \tilde{\phi}(p)\sinh(|p|r)} \right], \tag{2.21}$$

where $\tilde{\phi}(p) = \frac{\phi_1(p)}{\phi_0(p)}$. Finally, the double trace part of the boundary effective action S_B becomes

$$S_B = -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{\mathcal{D}(r,p)}{r^d} \phi_p \phi_{-p}, \tag{2.22}$$

where we have explicitly written down only the double trace deformation term in S_B and we will do the same for any S_B appearing hereafter unless stated otherwise.

2.2 Re-defined field and its relation with the original field ϕ

We start from the bulk action(2.1) and define a new field[8] $f(x^\mu)$ which is related to the original field ϕ by a field redefinition,

$$\phi(x^\mu) = \Omega(r)f(x^\mu), \tag{2.23}$$

where we will choose $\Omega(r) \equiv r^{\frac{d-1}{2}}$. Using this field re-definition, the bulk action(2.1) can be written as

$$S = \int_{r>\epsilon} dr d^d x \left(\frac{1}{2}\delta^{\mu\nu}\partial_\mu f(x)\partial_\nu f(x) + \frac{\lambda}{4} f^{\frac{2(d+1)}{d-1}}(x) \right) + \frac{d-1}{2} \int d^d x \left. \frac{f^2(x)}{2r} \right|_\epsilon^\infty + S_B, \tag{2.24}$$

where we have used a relation that $g_{\mu\nu} = r^{-2}\delta_{\mu\nu}$. Up to boundary terms (the second term in the action(2.24)), the bulk action becomes effectively that of a massless scalar field, $f(x)$ defined in $d+1$-dimensional flat Euclidean spacetime with $f^{\frac{2(d+1)}{d-1}}(x)$ interaction. Varying this bulk action with respect to $f(x)$ provides a bulk equation of motion as

$$0 = \delta^{\mu\nu}\partial_\mu\partial_\nu f(x) - \frac{\lambda(d+1)}{2(d-1)} f^{\frac{d+3}{d-1}}(x), \tag{2.25}$$

which, of course, reproduces Eq(2.14) once we substitute the field redefinition Eq.(2.23) into it. An interesting observation is that once we define a new boundary effective action as

$$S_B' = S_B - \frac{d-1}{2} \int_{r=\epsilon} d^d x \frac{f^2(x)}{2r}, \tag{2.26}$$

then *"massive scalar with mass $m^2 = -\frac{d^2-1}{4}$, $\phi^{\frac{2(d+1)}{d-1}}$ interaction and the boundary action S_B in Euclidean AdS_{d+1} becomes precisely the same with massless scalar field with $f^{\frac{2(d+1)}{d-1}}$ interaction defined in flat Euclidean upper half of the spacetime with the boundary term S_B' in the classical gravity limit"*. In the following discussion, we will set $\lambda = 0$ so that we will be dealing with free field $f(x)$.

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[8]The properties of this redefined field are discussed in [8].

Holographic Wilsonian renormalization group in terms of the field $f(x)$: Recalling in terms of the new field $f(x)$, our action is that of a free massless field. As a result, our starting point is

$$S = \frac{1}{2} \int_{r>\epsilon} dr d^d x \delta^{\mu\nu} \partial_\mu f(x) \partial_\nu f(x) + S'_B. \tag{2.27}$$

The Hamilton-Jacobi equation in momentum space derived from this action becomes

$$\partial_\epsilon S'_B = -\frac{1}{2} \int_{r=\epsilon} d^d p \left[\left(\frac{\delta S'_B}{\delta f_p} \right) \left(\frac{\delta S'_B}{\delta f_{-p}} \right) - |p|^2 f_p f_{-p} \right], \tag{2.28}$$

and the ansatz of S'_B as

$$S'_B = \Lambda'(\epsilon) + \int \frac{d^d p}{(2\pi)^d} \sqrt{\gamma} \mathcal{J}'(\epsilon, p) \phi_{-p} - \int \frac{d^d p}{2(2\pi)^d} \sqrt{\gamma} \mathcal{D}'(\epsilon, p) \phi_p \phi_{-p}. \tag{2.29}$$

one can get an equation of the double trace coupling, $D'(r, p) \equiv \sqrt{\gamma} \mathcal{D}'(r, p)$ and its solution as

$$\partial_\epsilon D'(\epsilon, p) = \frac{1}{(2\pi)^d} D'(\epsilon, p) D'(\epsilon, -p) - (2\pi)^d |p|^2, \tag{2.30}$$

$$D'(\epsilon, p) = -(2\pi)^d \frac{\Pi_f}{f_p}, \tag{2.31}$$

where

$$\Pi_f = \partial_r f_p = \frac{\delta S'_B}{\delta f_{-p}}, \tag{2.32}$$

is the canonical momentum of the re-defined field, f_{-p}. Since equation of motion of $f_p(r)$ is given by

$$(\partial_r^2 - |p|^2) f_p = 0, \tag{2.33}$$

its solutions are

$$f_p = b_1 + b_2 r, \quad \text{for zero momentum case, } p_i = 0, \tag{2.34}$$
$$= f_0(p) \cosh(|p|r) + f_1(p) \sinh(|p|r), \quad \text{for nonzero momentum case,} \tag{2.35}$$

where b_1, b_2, $f_0(p)$ and $f_1(p)$ are arbitrary constants but the last two are momentum dependent. Properties of fixed points of the double trace coupling have similar behavior as the massless scalar field defined in 2-dimensional Euclidean space(See examples in [10] for detailed discussion.), and we will not discuss it here. We just list the precise forms of the double trace part of the boundary effective action for further discussion:

$$S'_B = \frac{1}{2} \left(\frac{\tilde{b}}{1 + \tilde{b}r} \right) f^2, \quad \text{for zero momentum, } p_i = 0, \tag{2.36}$$

$$= \frac{1}{2} \int |p| d^d p \left(\frac{\sinh(|p|r) + \tilde{f}_p \cosh(|p|r)}{\cosh(|p|r) + \tilde{f}_p \sinh(|p|r)} \right) f_p f_{-p}, \quad \text{for nonzero momentum,} \tag{2.37}$$

where $\tilde{b} = \frac{b_2}{b_1}$ and $\tilde{f}_p = \frac{f_1(p)}{f_0(p)}$.

2.3   Relations between the schemes with ϕ and f

In this subsection, we will discuss the relation between holographic Wilsonian renormalization groups of the primitive field $\phi_p(r)$ and the rescaled field $f_p(r)$. As mentioned, the two fields are related by $\phi_p(r) = \Omega(r)f_p(r)$ and it turns out that Hamilton-Jacobi equations of the two fields, (2.6) and (2.28) are also clearly transformed from one to another[9]. In order to perform such a transformation, we have used the definition of canonical momenta (2.12), (2.32) in both scheme, and following useful relations,

$$\frac{\delta S'_B(f_p)}{\delta f_{-p}} = \Omega(\epsilon)\frac{\delta S_B(\phi_p)}{\delta \phi_{-p}} - (d-1)\frac{f_p}{2\epsilon}, \quad \frac{\delta S_B(\phi_p)}{\delta \phi_p} = \frac{1}{\Omega(\epsilon)}\frac{\delta S_B(f_p)}{\delta f_p}, \quad (2.38)$$

$$\text{and } \partial_\epsilon S_B(\phi_p) = \partial_\epsilon S_B(f_p) - \int_{r=\epsilon} d^d p \frac{\delta S_B(f_p)}{\delta f_p}\frac{\partial_r \Omega(r)}{\Omega(r)}f_{-p},$$

which are derived from (2.26). The first two relations in (2.38) are obvious. The last one uses the chain rule of differentiation. While the first term on the RHS is the usual change from ϕ to f, second term on the RHS depends on the rescaling involved in the field redefinition. In the second term we extracts the Ω dependent piece from S_B (which has explicit r dependence) and write its contribution as the cut-off is varied. Taking this factors in to account correctly, one gets the last relation in (2.38).

The main relation between the two schemes is manifestly the relation between each double trace deformation, namely (2.26). It can be easily proved that the double trace deformation parts (2.22) and (2.37) in each scheme are related to each other via (2.26) by the field redefinition $\phi_p = \Omega(r)f_p$.

3   Stochastic quantization

In this section, we will develop the Langevin dynamics and the Fokker-Planck approach respectively to reproduce the radial flows of double trace deformation in massive scalar field in AdS space.

3.1   Langevin equation with explicit time dependence

In this section, we will find the Langevin equation which allows us to derive the stochastic 2-point correlation function which, in turn, is in one to one correspondence with the boundary effective action obtained in the previous section via the relation obtained in [10]:

$$\langle \phi_p(t)\phi_q(t)\rangle_H^{-1} = \langle \phi_p(t)\phi_q(t)\rangle_S^{-1} - \frac{1}{2}\frac{\delta^2 S_c}{\delta \phi_p \delta \phi_{-p}}, \quad (3.1)$$

where $\langle \phi_p(t)\phi_q(t)\rangle_S$ is the stochastic two point correlation function, $\langle \phi_p(t)\phi_q(t)\rangle_H^{-1} = \frac{\delta^2 S_B}{\delta \phi_p(t)\delta \phi_q(t)}$ and S_c is called classical action, which will be defined soon.

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[9]To cancel the irrelevant terms in derivation, it is useful to substitute explicit form of $\Omega(r) = r^{\frac{d-1}{2}}$.

The Langevin equation that we want to solve has the following form:

$$\frac{1}{\Omega(t)}\frac{d\phi_p(t)}{dt} = -\frac{1}{\Omega(t)}\left(|p| - \frac{\partial_t\Omega(t)}{\Omega(t)}\right)\phi_p(t) + \eta(t,p), \tag{3.2}$$

where p_i are d-dimensional momenta and $\eta(t,p)$ is the stochastic noise satisfying [10]

$$\langle\eta(t,p)\eta(t',p')\rangle = \delta(t-t')\delta^d(p-p'). \tag{3.4}$$

Unlike the usual Langevin equation, the equation (3.2) has explicit time dependence appearing through $\Omega(t)$. Consistency with the Fokker-Planck approach requires $\Omega(t)$ to satisfy following condition,

$$\frac{d^2\Delta(t)}{dt^2} = \left(\frac{d^2-1}{4}\right)\Delta^{\frac{d+3}{d-1}}(t), \tag{3.5}$$

where $\Delta(t) \equiv \frac{1}{\Omega(t)}$(A related discussion will appear in the next subsection).

Since there is explicit time dependence in the Langevin equation, we cannot follow the usual method of stochastic quantization. We will therefore propose a more general concept for the classical action given by

$$S_c^\phi = \int_{\tilde{t}=t} d^dp\frac{1}{\Omega^2(\tilde{t})}\left(|p| - \frac{\partial_{\tilde{t}}\Omega(\tilde{t})}{\Omega(\tilde{t})}\right)\phi_p(\tilde{t})\phi_{-p}(\tilde{t}). \tag{3.6}$$

This definition is a bit strange when compared with the usual procedure of stochastic quantization. Normally the classical action in stochastic quantization has no explicit time dependence. We will interpret this classical action and the resulting Langevin equation in the following manner. We define the classical action at $\tilde{t} = t$ time slice. At that time slice, the time dependent factor $\Omega(\tilde{t})$ becomes a number as $\Omega(t)$. The Langevin equation satisfied by this classical action at any given time slice is

$$\frac{d\phi_p(t)}{dt} = -\frac{1}{2}\Omega^2(t)\frac{\delta S_c^\phi(\phi,t)}{\delta\phi_{-p}} + \Omega(t)\eta(t,p). \tag{3.7}$$

This is equivalent to the Langevin equation (3.2).

The most general form of the solution of Langevin equation (3.2) is

$$\phi_p(t) = \Omega(t)\int_{t_0}^t dt' e^{-|p|(t-t')}\eta(t',p). \tag{3.8}$$

Using the δ- function correlations of $\langle\eta(t,p)\eta(t,p')\rangle$ (3.4), 2-point equal time correlation function of $\phi_p(t)$ is obtained as

$$\langle\phi_p(t)\phi_{p'}(t)\rangle_S = \Omega^2(t)\frac{\delta^d(p-p')}{2|p|}\left(1 - e^{2|p|(t_0-t)}\right). \tag{3.9}$$

---

[10]As discussed in literature[2, 10], 1 and 2-point functions are given by

$$\langle\eta_p(t)\rangle = 0, \quad \langle\eta_p(t)\eta_{p'}(t')\rangle = \delta^d(p-p')\delta(t-t'). \tag{3.3}$$

Expectation values of odd number of insertions of η vanishes and any even number of insertions of it will be re-written as summation of all possible products of pairs of two point functions of η.

Relation between Langevin dynamics and massive scalar in AdS_{d+1} with non-zero momenta: Let us go back to the bulk theory in AdS_{d+1} of a scalar field with mass $m^2 = -\frac{d^2-1}{4}$. The most general form of the bulk solution with non-zero momentum is given in (2.19). This solution diverges in the interior[11] . To remove this divergence, we impose a regularity condition on the solution at the Poincare horizon. This gives a condition $\phi_0(p) + \phi_1(p) = 0$. This condition forces the solution to decay exponentially as it approaches $r = \infty$. The regular solution is then given by

$$\phi_p(r) = \phi_0(p) r^{\frac{d-1}{2}} e^{-|p|r}. \tag{3.10}$$

Using bulk equations of motion, boundary on-shell action at radial cut-off $r = \epsilon$ can be obtained as

$$S = \frac{1}{2} \int_{r=\epsilon} d^d p \sqrt{\bar{g}} g^{rr} \phi_p(r) \partial_r \phi_{-p}(r). \tag{3.11}$$

With substitution of regular solution (3.10) and using explicit expression of the background metric into (3.11), we get

$$\begin{aligned} I_{os}(r = \epsilon) &= -\frac{1}{2} \int_{r=\epsilon} d^d p\, e^{-2|p|r} \left(|p| - \frac{d-1}{2r} \right) \phi_0(p)\phi_0(-p), \\ &= -\frac{1}{2} \int_{r=\epsilon} \frac{d^d p}{r^{d-1}} \left(|p| - \frac{d-1}{2r} \right) \phi_p(r)\phi_{-p}(r) \end{aligned} \tag{3.12}$$

This boundary on-shell action is not yet regularized since there is a divergent term in it, namely the second term in the parenthesis. This divergence occurs as we take $r \to 0$ limit. However, it turns out that to capture the radial evolution of the corresponding double trace deformation, we can choose our classical action as

$$S_c(\epsilon) = -2I_{os}(\epsilon), \tag{3.13}$$

at the radial cut-off $r = \epsilon$. The prescription for stochastic quantization with such classical action is that (since we will identify the radial variable r to stochastic time t) $S_c(\epsilon)$ becomes classical action defined at $t = \epsilon$ time slice. In fact, the classical action (3.6) from bulk on-shell action (3.13) can be reproduced by substituting $\Omega(t) = t^{\frac{d-1}{2}}$.

The stochastic 2-point correlator is known from (3.9), which is given by

$$\langle \phi_p(t)\phi_{p'}(t) \rangle_S = t^{d-1} \frac{\delta^d(p - p')}{2|p|} \left(1 - e^{2|p|(t_0 - t)} \right). \tag{3.14}$$

It is clear that (3.14) precisely reproduce the radial flow of double trace deformation,

$$\langle \phi_p(r)\phi_{p'}(r) \rangle_H^{-1} \equiv \frac{\delta^2 S_B}{\delta \phi_p(r)\delta \phi_{p'}(r)} = \frac{1}{r^{d-1}} \left[\frac{d-1}{2} + |p|r \frac{\sinh(|p|r) + \tilde{\phi}(p)\cosh(|p|r)}{\cosh(|p|r) + \tilde{\phi}(p)\sinh(|p|r)} \right] \tag{3.15}$$

via the relation (3.1), when 'r' is identified to 't' and the initial time [12] in (3.14) is chosen as $t_0 = -\frac{1}{|p|} \coth^{-1}[\tilde{\phi}(p)]$. Here, the new constant $\tilde{\phi}(p)$ is $\tilde{\phi}(p) = \frac{\phi_1(p)}{\phi_0(p)}$.

---

[11]We note that one should impose regularity condition on the bulk solution to evaluate bulk on-shell action, I_{os}, however, when on compute HWRG by using the most general solution (2.19), there is no such regularity issue at all.

[12]The initial stochastic time is chosen so as to match 2-point stochastic correlations with the double trace deformation. For more detailed manipulations, see [10]

3.2 The Fokker-Planck approach

Fokker-Planck action is not precisely of the usual form in this case. In fact, it has deformation from its original form by time dependent factor $\Omega(t)$. In this section, we will derive the correct form of Fokker-Planck Lagrangian, show that it has the same form with bulk Lagrangian, and the double trace deformation will be correctly obtained via the relation proposed in [10] ,

$$S_B = \int_{t_0}^t dt' d^d p \mathcal{L}_{FP}(\phi(t'), \partial\phi(t'); t'). \tag{3.16}$$

To derive the Fokker-Planck action, the stochastic partition function is the best starting point:

$$Z = \int [D\eta] exp\left(-\frac{1}{2}\int_{t_0}^t \eta_p(t')\eta_{-p}(t') d^d p dt'\right). \tag{3.17}$$

We substitute the Langevin equation (3.2) into the partition function (3.17) to replace η by stochastic field $\phi_p(t)$. Functional integral measure part will transform by the Jacobian factor,

$$J\left(\frac{\delta\eta}{\delta\phi}\right) = \exp\left[\frac{1}{4}\int_{t_0}^t dt' d^d p\ \Omega^2(t')\frac{\delta^2 S_c(\phi, t')}{\delta\phi_p(t')\delta\phi_{-p}(t')}\right]. \tag{3.18}$$

The stochastic partition function is given by

$$Z = \int [D\phi]e^{-S} = \int [D\phi] \exp\left[-\int_{t_0}^t dt' \int d^d p L(\phi, \partial\phi, t')\right], \tag{3.19}$$

where

$$
\begin{aligned}
L(\phi, \partial\phi, t) &= \frac{1}{2\Omega^2(t)}\left[\left(\frac{\partial\phi_p(t)}{\partial t}\right)\left(\frac{\partial\phi_{-p}(t)}{\partial t}\right) + \frac{1}{4}\left(\frac{\delta S_c(\phi, t)}{\delta\phi_p}\right)\left(\frac{\delta S_c(\phi, t)}{\delta\phi_{-p}}\right)\right. \\
&\quad \left. - \frac{1}{2}\Omega^4(t)\frac{\delta^2 S_c(\phi, t)}{\delta\phi_p(t)\delta\phi_{-p}(t)} + \Omega^2(t)\left(\frac{\delta S_c(\phi, t)}{\delta\phi_{-p}}\right)\left(\frac{\partial\phi_{-p}}{\partial t}\right)\right].
\end{aligned}
\tag{3.20}
$$

The first term on the second line in (3.20) does not depend on field ϕ and it becomes an overall constant in the partition function Z. The last term in L is not a total derivative since classical action contains explicit time dependence. To deal with L more clearly, we plug in the explicit form of the classical action (3.6). If we now assume that $\Omega(t)$ satisfies (3.5), then $L(\phi, \partial\phi, t)$ can be brought into the following form[13]

$$S = \int_{t_0}^t dt' \int_{-\infty}^{\infty} d^d p L(\phi, \partial\phi; t') = \int_{t_0}^t dt' \int_{-\infty}^{\infty} d^d p \mathcal{L}_{FP}(\phi, \partial\phi; t') + \frac{1}{2}\int_{t_0}^t dt' \partial_{t'} S_c^\phi(\phi, t'), \tag{3.21}$$

where $\mathcal{L}_{FP}$ is the Fokker-Planck Lagrangian density, which is given by

$$\mathcal{L}_{FP} = \frac{1}{2}\Omega(t)^{-\frac{2(d+1)}{d-1}}\left[\Omega(t)^{\frac{4}{d-1}}\left(\frac{\partial\phi_p}{\partial t}\right)\left(\frac{\partial\phi_{-p}}{\partial t}\right) + \Omega(t)^{\frac{4}{d-1}}|p|^2\phi_p\phi_{-p} - \frac{d^2-1}{4}\phi_p\phi_{-p}\right]. \tag{3.22}$$

---

[13]In the stochastic partition function, the usual form of the exponent is as (3.21). See equation (3.81) in [2].

We point out that $\mathcal{L}_{FP}$ has the same form as that of the bulk Lagrangian density [14] (2.2) with $m^2 = -\frac{d^2-1}{4}$ when $\Omega(t) = t^{\frac{d-1}{2}}$ and 't' is identified to 'r'. $\Omega(t) = t^{\frac{d-1}{2}}$ is the solution of equation (3.5). Therefore, there is no contradiction with the previous derivation of $\mathcal{L}_{FP}$.

Finally, we develop double trace part of the boundary effective action S_B using the prescription (3.16). Since (3.22) is a free theory on a certain time dependent background, it is enough to evaluate $\mathcal{L}_{FP}$ using its classical solutions if one does not consider back reaction. Equation of motion derived from (3.22) is given by

$$0 = \partial_t^2 \phi_p - \frac{d-1}{t} \partial_t \phi_p + \left(\frac{d^2-1}{4t^2} - |p|^2 \right) \phi_p, \tag{3.23}$$

and its most general form of solution is

$$\phi_p(t) = t^{\frac{d-1}{2}} [\Phi_0(p) \cosh(|p|t) + \Phi_1(p) \sinh(|p|t)], \tag{3.24}$$

with arbitrary d-momenta, p_i dependent functions: $\Phi_0(p)$ and $\Phi_1(p)$. When we manipulate S_B, we can bring one term to be proportional to (3.23). The remaining term then is a total derivative and contributes only a boundary term. With this manipulation (3.16) becomes

$$S_B = \frac{1}{2} \int d^d p \frac{1}{\Omega^2(\tilde{t})} \phi_p(\tilde{t}) \partial_{\tilde{t}} \phi_{-p}(\tilde{t}) \Big|_{\tilde{t}=t_0}^{\tilde{t}=t} . \tag{3.25}$$

To evaluate the correct boundary effective action, we set two boundary conditions. (1)The initial time t_0 is set to be

$$t_0 = -\frac{1}{|p|} \coth^{-1} \tilde{\Phi}(p), \tag{3.26}$$

where $\tilde{\Phi}(p) = \frac{\Phi_1(p)}{\Phi_0(p)}$. At $\tilde{t} = t_0$, the solution (3.24) of the equation of motion (3.23) becomes zero, $\phi_p(t_0) = 0$.[15] (2) At $\tilde{t} = t$, we want $\phi_p(\tilde{t} = t) = \phi_p(t)$. Therefore, it is requested that

$$\phi_p(\tilde{t}) = \left(\frac{\tilde{t}^{\frac{d-1}{2}} [\cosh(|p|\tilde{t}) + \tilde{\Phi}(p) \sinh(|p|\tilde{t})]}{t^{\frac{d-1}{2}} [\cosh(|p|t) + \tilde{\Phi}(p) \sinh(|p|t)]} \right) \phi_p(t). \tag{3.27}$$

Substituting (3.27) into (3.25) and applying the initial boundary condition (3.26) on it, we get

$$S_B = \frac{1}{2} \int d^d p \frac{1}{t^d} \left(\frac{d-1}{2} + |p|t \frac{\sinh(|p|t) + \tilde{\Phi}(p) \cosh(|p|t)}{\cosh(|p|t) + \tilde{\Phi}(p) \sinh(|p|t)} \right). \tag{3.28}$$

It is easy to see that (3.28) is precisely the same with (2.22) once stochastic time 't' is identified to the radial variable 'r' in AdS space and $\tilde{\phi}(p) = \tilde{\Phi}(p)$.

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[14]Once t is identified to r, then $\sqrt{g} = \Omega(t)^{-\frac{2(d+1)}{d-1}}$, $g^{rr} = g^{ii} = \Omega(t)^{\frac{4}{d-1}}$ provided by $\Omega(t) = t^{\frac{d-1}{2}}$.

[15]For detailed discussion about such choice of the initial time, see [10].

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4 Toward a better-defined Langevin equation via field re-definition

Even though the Langevin equation (3.2) does not look like that of the usual form, it might be justified that (3.2) is the correct formulation by the fact that the usual form of Langevin equation can be derived from it by a field re-definition

$$\phi(t, p) = \Omega(t) f_p(t), \tag{4.1}$$

where $f_p(t)$ is a new stochastic field. It turns out that the new field $f_p(t)$ satisfies a new Langevin equation

$$\frac{df_p(t)}{dt} = -|p| f_p(t) + \eta(t, p), \tag{4.2}$$

which can be easily derived from (3.2) by using (4.1). The first term on the right hand side of (4.2) can be written as

$$|p| f_p(t) = \frac{1}{2} \frac{\delta S_c(\phi)}{\delta \phi_{-p}}, \tag{4.3}$$

which implies the classical action can be written as

$$S_c^f = \int d^d p |p| f_p f_{-p}. \tag{4.4}$$

This is precisely what the authors present in [10] for the theory of massless scalar field in 2-dimensional flat space. Langevin equation (4.2) has no explicit time dependent factors in it and nor does the classical action (4.4). Therefore, usual rules of stochastic quantization can be applied to this classical action without any modifications. We point out that this justification of the time dependent stochastic dynamics is very similar to that presented in [20].

4.1 Stochastic quantization of $f(x)$

It turns out that Langevin equation (4.2) together with the classical action (4.4) captures the radial evolution of double trace operator S_B' defined in (2.27) in the limit of free field theory. Euclidean action S_c will be identified to $-2I_{os}$ as demonstrated in (3.13). Using the bulk equation of motion(2.25) in momentum space using the Fourier transform as Eq.(2.5) with $\lambda = 0$, its on-shell action at $r = \epsilon$ cut-off is given by

$$I_{os} = \frac{1}{2} \int_{r=\epsilon} d^d p f_p(r) \partial_r f_{-p}(r). \tag{4.5}$$

The bulk equation of motion in the momentum space is

$$\partial_r^2 f_p(r) - p^2 f_p(r) = 0, \tag{4.6}$$

and the most general form of the solution is given by

$$f_p(r) = f_0(p) \cosh(|p|r) + f_1(p) \sinh(|p|r), \tag{4.7}$$

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where $f_0(p)$(the boundary value of the bulk field $f(x)$) and $f_1(p)$ are r-independent constants. This solution should be regular in the interior of AdS space as $r \to \infty$. To prevent divergent behavior of the solution, we impose a condition $f_0(p) + f_1(p) = 0$. Final form of the regular solution after imposing the regularity condition is

$$f_p(r) = f_0(p)e^{-|p|r}. \tag{4.8}$$

By using the explicit form of bulk solution (4.8), we get

$$I_{os} = -\frac{1}{2} \int_{r=\epsilon} d^d p |p| f_p(r) f_{-p}(r) \tag{4.9}$$

The Langevin dynamics & the Fokker-Planck approach: To evaluate stochastic 2-point correlator, we follow the prescription given in [10]. The Euclidean action is given by

$$S_c^f = -2I_{os} = \int d^d p |p| f_p(t) f_{-p}(t), \tag{4.10}$$

where we identify the radial cut-off ϵ with the time slice t. We plug the Euclidean action into Langevin equation

$$\frac{df_p(t)}{dt} = -\frac{1}{2} \frac{\delta S_c^f}{\delta f_{-p}(t)} + \eta(p, t) = -|p| f_p(t) + \eta(p, t), \tag{4.11}$$

where $\eta(p, t)$ is called the stochastic white noise which provides interactions with the surroundings and has its 2-point correlations as given in (3.4). The most general solution of the Langevin equation then becomes

$$f_p(t) = \int_{t_0}^{t} d\tilde{t} e^{-|p|(t-\tilde{t})} \eta(p, \tilde{t}), \tag{4.12}$$

Choice of the initial time t_0 is obtained by following the prescription given in [10],

$$t_0 = -\frac{1}{|p|} \coth^{-1}(\bar{f}_p), \tag{4.13}$$

where $\bar{f}_p$ an arbitrary momentum dependent function, which should be chosen as $\bar{f}_p = \tilde{f}_p$ to reproduce the correct double trace deformation, S'_B for the theory defined in (2.27).

For the final step, we evaluate 2-point correlator using correlation functions of the stochastic noise, which is given by

$$\langle f_0(p) f_0(p') \rangle_S = \frac{1}{2|p|} \delta^d(p - p') \left(1 - \frac{\bar{f}_p - 1}{\bar{f}_p + 1} e^{-2|p|t} \right) \tag{4.14}$$

It turns out that this stochastic 2-point function reproduces the kernel of S'_B, $\langle f_p f_{p'} \rangle_H^{-1} \equiv \frac{\delta^2 S'_B}{\delta f_p \delta f_{p'}}$ correctly through relation (3.1) using $\frac{1}{2} \frac{\delta^2 S_c}{\delta f_p \delta f_{p'}} = |p| \delta^d(p - p')$, when $r = t$ and $\bar{f}_p = \tilde{f}_p$. Fokker-Planck approach gives result which is consistent with the Langevin dynamics.

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4.2 Transformation of the Fokker-Planck action with field re-scaling

It is rather trivial that the Langevin equation with the original field ϕ_p transforms into that with the rescaled field f_p using the field re-definition(4.1). The new Langevin equation gives the consistent relationship between the radial flow of the double trace deformation of massless scalar field theory in flat space-time and the corresponding stochastic quantization with the classical action(4.4) as demonstrated in the last section. In this section, to explain our framework more clearly, we will demonstrate that the scale transformation maps the time dependent Fokker-Planck action to the new one without explicit time dependence and usual flat space form. Let us start with the action S defined in (3.21). The action S is comprised of two pieces: Fokker-Planck action and the total derivative term with respect to t. The total derivative term has the form of $\int dt \partial_t S_c^\phi$, where S_c^ϕ is the classical action defined in (3.6). This is the usual form of the action S derived from the stochastic partition function (3.17) [16]. Now what we want to show is that using the field rescaling (4.1), the action S will transform into the form of

$$S = S_{FP}(f_p) + \frac{1}{2} \int_{t_0}^{t} dt' \partial_{t'} S_c^f, \tag{4.15}$$

where S_{FP} is the Fokker-Planck action in terms of the rescaled field f_p and S_c^f is the classical action given in (4.4).

Once the relation (4.1) is plugged into the action $S = \int dt \int d^d p L$ defined in (3.21), it becomes

$$
\begin{aligned}
L(f, \partial f; t) &= \left[\frac{1}{2} \partial_t f_p(t) \partial_t f_{-p}(t) + \frac{1}{2} |p|^2 f_p(t) f_{-p}(t) \right] \\
&+ \frac{1}{2} f_p(t) f_{-p}(t) \left[\frac{\Omega'^2(t)}{\Omega^2(t)} - \frac{d^2 - 1}{4} \Omega^{-\frac{4}{d-1}}(t) \right] + \frac{\Omega'(t)}{\Omega(t)} \partial_t [f_p(t) f_{-p}(t)] \\
&+ \partial_t \left[\frac{1}{2} |p| f_p(t) f_{-p}(t) - \frac{\Omega'(t)}{\Omega(t)} f_p(t) f_{-p}(t) \right].
\end{aligned}
\tag{4.16}
$$

We point out that the scale factor $\Omega(t)$ is not arbitrary but it is what satisfies the differential equation (3.5). In terms of $\Omega(t)$, it becomes

$$\frac{\Omega''(t)}{\Omega(t)} - 2\frac{\Omega'^2(t)}{\Omega^2(t)} = -\frac{d^2 - 1}{4} \Omega^{-\frac{4}{d-1}}(t). \tag{4.17}$$

Using (4.17), the term proportional to $-\frac{d^2-1}{4}\Omega^{-\frac{4}{d-1}}(t)$ in the second line in (4.16) can be replaced by the left hand side of (4.17). Then, the second line in (4.16) becomes total derivative and which precisely cancels the last term in (4.16). Finally, (4.16) becomes

$$L(f, \partial f; t) = \left[\frac{1}{2} \partial_t f_p(t) \partial_t f_{-p}(t) + \frac{1}{2} |p|^2 f_p(t) f_{-p}(t) \right] + \partial_t \left[\frac{1}{2} |p| f_p(t) f_{-p}(t) \right]. \tag{4.18}$$

The terms in the first square bracket are precisely the Fokker-Planck action and the term in total derivative is half of the classical action (4.4). Therefore, the Fokker-Planck actions in both schemes are clearly related by the scale transformation.

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[16]e.g. See [2, 3]

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4.3 Relations between two different schemes of stochastic quantization with ϕ and f

In both schemes with the original field ϕ and the new field f, they satisfy the relations between their 2-point stochastic correlation functions and double trace couplings in AdS/CFT respectively. Namely, the theories with field ϕ satisfies the relation (3.1) and for the new field f_p, the similar relation as

$$\langle f_p(t)f_q(t)\rangle_H^{-1} = \langle f_p(t)f_q(t)\rangle_S^{-1} - \frac{1}{2}\frac{\delta^2 S_c^f}{\delta f_p \delta f_{-p}} \tag{4.19}$$

is satisfied.

In fact, stochastic 2-point correlator in each scheme enjoy the relation as

$$\langle \phi_p(t)\phi_{-p}(t)\rangle_S = \Omega^2(t)\langle f_p(t)f_{-p}(t)\rangle_S. \tag{4.20}$$

This is clear from (3.14) and (4.14). The classical actions in both theories also have a relation as

$$S_c^\phi(\phi) = S_c^f(f) - \int d^d p \frac{\partial_t \Omega(t)}{\Omega(t)} f_p f_{-p}. \tag{4.21}$$

This relation is also well understood by looking at (3.5), (4.4) and (4.1). (4.21) leads to

$$\frac{\delta^2 S_c^f(f)}{\delta f_p \delta f_{-p}} = \Omega^2(t)\frac{\delta^2 S_c^\phi(\phi)}{\delta \phi_p \delta \phi_{-p}} + 2\frac{\partial_t \Omega(t)}{\Omega(t)}, \tag{4.22}$$

where we have used $\frac{\delta}{\delta f_p} = \Omega(t)\frac{\delta}{\delta \phi_p}$. Using (4.20) and (4.22), one can manipulate the right hand side of (4.19) and obtain the relation between double trace deformations in the two different schemes. Then, (4.19) becomes

$$\langle f_p(t)f_{-p}(t)\rangle_H^{-1} = \Omega^2(t)\left(\langle \phi_p(t)\phi_{-p}(t)\rangle_H^{-1} - \frac{\partial_t \Omega(t)}{\Omega^3(t)}\right), \tag{4.23}$$

where we have used (3.1) to switch the stochastic 2-point function with the double trace deformation in theory with the old field ϕ_p. This relation is precisely the same with the relation(2.26) between two different boundary effective actions, S_B and S_B' obtained as the solutions of their Hamilton-Jacobi equations. It is clear that one can derive (4.23) from (2.26) using definitions of the double trace couplings as $\langle f_p(t)f_{-p}(t)\rangle_H^{-1} = \frac{\delta^2 S_B'(f)}{\delta f_p \delta f_{-p}}$ and $\langle \phi_p(t)\phi_{-p}(t)\rangle_H^{-1} = \frac{\delta^2 S_B(\phi)}{\delta \phi_p \delta \phi_{-p}}$.

In summary,we have shown that all the rescaling arguments in the bulk theories with scalar field with the specific mass square $m^2 = -\frac{d^2-1}{4}$ are consistent with their description with stochastic quantization, in which one can also have scaling argument and all the quantities are in one to one correspondence with those quantities in the holographic description.

5 Conclusion

In this paper, we have constructed a precise one to one mapping between holographic Wilsonian renormalization group(HWRG) of conformally coupled scalar field in AdS$_{d+1}$ and stochastic

quantization(SQ) obtained from the classical action by identifying it with the on-shell action of the bulk scalar field theory evaluated at a certain radial cut-off of AdS space. Our Langevin equation and Fokker-Planck Hamiltonian dynamics present explicit stochastic time dependences in them and they cannot be dealt with the usual methodology of SQ. However, we have suggested more general definition of classical action and it turns out that SQ with such classical action reproduces the radial evolution of the boundary effective action of the conformally coupled scalar obtained from its HWRG computation correctly. Moreover, we have proved that SQ with such general definition of the classical action is consistent with the usual stochastic quantization method up to a field redefinition.

This field re-scaling argument continues to be valid even when the theory contains a certain class of interaction of the field ϕ of the type $L_{int} \sim \lambda \phi^{\frac{2(d+1)}{d-1}}$. Thus, it opens a new playground where one investigates HWRG and SQ of interacting theories and their mathematical relation. The scaling property seems to be very crucial ingredient to construct exact mapping between the two schemes.

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