Variation of entanglement entropy in scattering process

Shigenori Seki [b], I.Y. Park [a], Sang-Jin Sin [c],*

[a] *Department of Applied Mathematics, Philander Smith College, Little Rock, AR 72223, USA*
[b] *Research Institute for Natural Science, Hanyang University, Seoul 133-791, Republic of Korea*
[c] *Department of Physics, Hanyang University, Seoul 133-791, Republic of Korea*

ARTICLE INFO

ABSTRACT

In a scattering process, the final state is determined by an initial state and an S-matrix. We focus on two-particle scattering processes and consider the entanglement between these particles. For two types initial states, *i.e.*, an unentangled state and an entangled one, we calculate perturbatively the change of entanglement entropy from the initial state to the final one. Then we show a few examples in a field theory and in quantum mechanics.

© 2015 Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP[3].

1. Introduction

Entanglement is a characteristic feature in a quantum theory. The entanglement in quantum field theories has been studied extensively in the past decade. When one considers a sub-system A and its complement $\overline{A}$, the entanglement entropy between A and $\overline{A}$ is defined by the von Neumann entropy $S_E = -\operatorname{tr}_A \rho_A \log \rho_A$ with the reduced density matrix ρ_A. Calabrese and Cardy have systematically studied it in a conformal field theory with the use of a replica trick [1]. The other remarkable recent progress is the holographic derivation of entanglement entropy suggested by Ryu and Takayanagi [2,3]. Following it, one can obtain an entanglement entropy by calculating $S_E = \mathcal{A}/(4G_N)$, where $\mathcal{A}$ is the area of a minimal surface whose boundary is the boundary of the subsystem A. In other words, the holographic entanglement entropy provides us with a geometric understanding of entanglement.

Then there is the other geometric interpretation of entanglement entropy conjectured recently by Maldacena and Susskind [4]. Its original purpose was to resolve the firewall paradox [5].[1] This conjecture states that an Einstein–Rosen–Podolski pair, *i.e.*, a pair of entangled objects, is connected by an Einstein–Rosen bridge (or a wormhole). Therefore the conjecture is symbolically called the ER=EPR conjecture. From the point of view of the AdS/CFT correspondence, some examples supporting the ER=EPR conjecture have been shown. An entangled pair of accelerating quark and anti-quark was studied in Ref. [7]. Investigating the causal structure on the world-sheet minimal surface that is the holographic bulk dual of such a quark and anti-quark on the AdS boundary, Ref. [7] has found that there exists a wormhole on the minimal surface and that any open strings connecting the quark and anti-quark must go through the wormhole. Therefore the entanglement of the accelerating quark and anti-quark coincides with the existence of the wormhole. Furthermore, Ref. [8] considered Schwinger pair creation of a quark and an anti-quark and confirmed that there is a wormhole on the string world-sheet of their bulk dual. Ref. [9] focused on a pair of scattering gluons as an EPR pair. Since Ref. [10] had shown the minimal surface solution corresponding to the gluon scattering, Ref. [9] calculated the induced metric on the minimal surface and found a wormhole connecting the gluon pair. One can then naturally guess that, in a scattering process,[2] an interaction induces the variation of entanglement from an incoming state to an outgoing one. We know these states are associated with each other by an S-matrix. So the question is how the variation of entanglement entropy and the S-matrix are related. In this paper we attack this problem by a perturbative analysis in a weak coupling λ. In order to evaluate the entanglement entropy, it is useful to calculate Rényi entropy by the replica trick when one can calculate it exactly. For instance, Ref. [12] explicitly calculated the time evolution of the entanglement entropy between two free scalar field theories with a specific interaction. However, this method is often unavailable for a perturbative analysis. Therefore we apply the method developed by Refs. [13,14], in which the

[1] See Ref. [6] for an earlier work. It has predicted an energetic curtain, which is similar to the firewall, on the assumptions different from Ref. [5].

[2] Ref. [11] has studied the entanglement entropy in a decay process in terms of the Wigner–Weisskopf method.

entanglement between two divided momentum spaces was studied perturbatively.

In Section 2, we consider the variations of entanglement entropy from two kinds of initial states; one is an unentangled initial state and the other is an entangled one. In Section 3, we evaluate the variation of entanglement entropy in the field theory with a ϕ^4-like interaction. We also consider the time-dependent interaction in quantum mechanics. Section 4 is devoted to conclusion and discussion.

2. Perturbative calculation of entanglement entropy

Since we are interested in a scattering process of two particles, A and B, and their entanglement, let us consider the Hamiltonian with an interaction:

$$H = H_0 + \lambda H_{\text{int}}, \quad H_0 = H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_B. \tag{2.1}$$

It is usually difficult to divide the total Hilbert space $\mathcal{H}$ to $\mathcal{H}_A \otimes \mathcal{H}_B$ due to the interaction. However an initial state far in the past and a final state far in the future in a scattering process can be regarded as states generated by an asymptotically free Hamiltonian. Furthermore, although a field theory in general includes arbitrary multi-particle states in its Hilbert space, we concentrate only on an elastic scattering of two particles such as $A + B \to A + B$ with a weak coupling. That is to say, we restrict the Hilbert space to the $(1 + 1)$-particle Fock space, in which the initial and final states are. Since such a restriction usually violates unitarity for local interaction terms, we assume in this paper specific theories that do not produce states of more than $1 + 1$ particles at lower orders of perturbation (see an example in Section 3.1). Then the unitarity is approximately recovered at a weak coupling. Under this assumption, we can divide the Hilbert space of the initial and final states to $\mathcal{H}_A \otimes \mathcal{H}_B$, and these states are denoted by a $(1 + 1)$-particle state generated by the free Hamiltonian H_0, namely, a state of a particle A and B with momentum p and q:

$$|p, q\rangle := |p\rangle_A \otimes |q\rangle_B. \tag{2.2}$$

One can express the infinite time evolution from the initial state to the final one in terms of S-matrix by definition,

$$\lim_{t \to \infty} \langle \text{fin}|e^{-iHt}|\text{ini}\rangle = \langle \text{fin}|\mathbf{S}|\text{ini}\rangle, \quad \mathbf{S} := \mathbf{1} + i\mathbf{T}. \tag{2.3}$$

$\mathbf{T}$ is a transition matrix in $\mathcal{O}(\lambda)$ which is induced by the interaction. Then the final state is described as

$$|\text{fin}\rangle = \int dk\,dl\,|k, l\rangle \langle k, l|\mathbf{S}|\text{ini}\rangle, \tag{2.4}$$

in which we used the completeness relation of $(1+1)$-particles' states, i.e., $(\mathbf{1})_{(1+1)\text{-particles}} = \int dk\,dl\,|k, l\rangle\langle k, l|$, and an inner product of states, i.e., $\langle k, l|p, q\rangle = \delta(k - p)\delta(l - q)$. Although the norm $\langle p, q|p, q\rangle =: V$ has an infinite volume, we shall fix a normalization at the stage of a reduced density matrix. Here we comment that one can easily formulate the case of discrete spectra by replacing $\int dk\,dl$ with $\sum_{k,l}$. As an example we shall show in Section 3.2 the theory with a time-dependent interaction in non-relativistic quantum mechanics.

The total density matrix of the final state is $\rho^{(\text{fin})} = |\text{fin}\rangle\langle\text{fin}|$, and we obtain the reduced density matrix $\rho_A^{(\text{fin})}$ by taking trace of $\rho^{(\text{fin})}$ with respect to the particle B, i.e., $\rho_A^{(\text{fin})} = \text{tr}_B\,\rho^{(\text{fin})}$ up to normalization. In the case of (2.4) we can write down the reduced density matrix as

$$\rho_A^{(\text{fin})} = \frac{1}{\mathcal{N}} \int dk\,dk' \left(\int dl \langle k, l|\mathbf{S}|\text{ini}\rangle\langle\text{ini}|\mathbf{S}^\dagger|k', l\rangle \right) |k\rangle\langle k'|, \tag{2.5}$$

where $\mathcal{N}$ is a normalization constant determined by $\text{tr}_A\,\rho_A^{(\text{fin})} = 1$, namely,

$$\mathcal{N} = \int dk\,dl\,|\langle k, l|\mathbf{S}|\text{ini}\rangle|^2. \tag{2.6}$$

Then the entanglement entropy between A and B in the final state is

$$S_E^{(\text{fin})} = -\text{tr}\,\rho_A^{(\text{fin})} \log \rho_A^{(\text{fin})}, \tag{2.7}$$

and the variation of entanglement entropy from the initial state to the final one is

$$\Delta S_E = S_E^{(\text{fin})} - S_E^{(\text{ini})}, \tag{2.8}$$

where $S_E^{(\text{ini})}$ is the entanglement entropy of the initial state. We shall calculate these entanglement entropies perturbatively.

The replica trick allows us to calculate a Rényi entropy, $S(n) = \frac{1}{1-n} \log \text{tr}_A\,\rho_A^n$. The entanglement entropy is given by the $n \to 1$ limit of Rényi entropy, namely, $S_E = \lim_{n \to 1} S(n) = -\lim_{n \to 1} \frac{\partial}{\partial n} \text{tr}_A\,\rho_A^n$. Therefore the method to derive an entanglement entropy via a Rényi entropy is often useful. However, we are confronted with a problem when we analyze a quantum theory with a coupling λ in terms of perturbation. When one obtains a perturbative expansion of $\text{tr}_A\,\rho_A^n$, the term of order λ^n relevantly contributes to the entanglement entropy because the operation $\lim_{n \to 1} \frac{\partial}{\partial n}$ acts on λ^n and yields a term of $\lambda \log \lambda$ order. In other words, the higher order terms in the Rényi entropy are responsible for the convergence of the entanglement entropy under the $n \to 1$ limit. Hence any λ^n-order terms in $\text{tr}_A\,\rho_A^n$ are necessary in order to obtain a meaningful entanglement entropy. In this paper, instead of the replica trick, we apply the perturbative method developed by Ref. [13] for calculating an entanglement entropy.

2.1. Unentangled initial state

Let us consider the simplest single state with fixed momenta p_1 and q_1 for the initial state of particle A and B,

$$|\text{ini}\rangle \sim |p_1, q_1\rangle. \tag{2.9}$$

The normalization of states will be properly fixed later in normalizing a density matrix so that $\text{tr}_A\,\rho_A^{(\text{fin})} = 1$. This initial state is obviously unentangled, i.e., $S_E^{(\text{ini})} = 0$. Then we can describe the final state (2.4) as

$$|\text{fin}\rangle = \int dk\,dl\,|k, l\rangle \mathcal{S}_{kl;p_1q_1}$$
$$= \frac{\mathcal{S}_{p_1q_1;p_1q_1}}{V^2}|p_1, q_1\rangle + i\lambda \int_{k \neq p_1} dk \frac{\mathcal{T}_{kq_1;p_1q_1}}{V}|k, q_1\rangle$$
$$+ i\lambda \int_{l \neq q_1} dl \frac{\mathcal{T}_{p_1l;p_1q_1}}{V}|p_1, l\rangle$$
$$+ i\lambda \int_{\substack{k \neq p_1 \\ l \neq q_1}} dk\,dl\,\mathcal{T}_{kl;p_1q_1}|k, l\rangle, \tag{2.10}$$

where we introduced an infinite spacial volume $V := \int dx\,e^{ix\cdot 0} = \delta(0)$ due to the divergence of norms, i.e., $\langle p|p\rangle_A = \langle q|q\rangle_B = \delta(0)$. The integral $\int_{k \neq p} dk$ means $\int dk(1 - V^{-1}\delta(k - p))$. $\mathcal{S}_{kl;pq}$ and $\mathcal{T}_{kl;pq}$ denote S- and T-matrix elements,

$$\mathcal{S}_{kl;pq} := \langle k, l|\mathbf{S}|p, q\rangle, \quad \mathcal{T}_{kl;pq} := \frac{1}{\lambda}\langle k, l|\mathbf{T}|p, q\rangle. \tag{2.11}$$

S includes an identity **1**, while **T** is given by an interaction with coupling λ. Therefore the possible lowest orders of (2.11) with respect to λ are

$$\mathcal{S}_{pq;pq} \sim \mathcal{O}(\lambda^0), \qquad \mathcal{S}_{p'q';pq}|_{(p',q') \neq (p,q)} \sim \mathcal{O}(\lambda),$$
$$\mathcal{T}_{kl;pq} \sim \mathcal{O}(\lambda^0). \tag{2.12}$$

We employ the method developed by Ref. [13] in order to perturbatively calculate the entanglement entropy. Since Eq. (2.10) is rewritten as

$$|\text{fin}\rangle = \frac{\mathcal{S}_{p_1 q_1; p_1 q_1}}{V^2} |\tilde{p}_1\rangle_A \otimes |\tilde{q}_1\rangle_B$$
$$+ \int\limits_{\substack{k \neq p_1 \\ l \neq q_1}} dk\, dl \left(\lambda^2 \frac{\mathcal{T}_{kq_1; p_1 q_1} \mathcal{T}_{p_1 l; p_1 q_1}}{\mathcal{S}_{p_1 q_1; p_1 q_1}} + i\lambda \mathcal{T}_{kl; p_1 q_1} \right) |k, l\rangle, \tag{2.13}$$

with

$$|\tilde{p}_1\rangle_A = |p_1\rangle_A + i\lambda V \int\limits_{k \neq p_1} dk \frac{\mathcal{T}_{kq_1; p_1 q_1}}{\mathcal{S}_{p_1 q_1; p_1 q_1}} |k\rangle_A,$$
$$|\tilde{q}_1\rangle_B = |q_1\rangle_B + i\lambda V \int\limits_{l \neq q_1} dl \frac{\mathcal{T}_{p_1 l; p_1 q_1}}{\mathcal{S}_{p_1 q_1; p_1 q_1}} |l\rangle_B, \tag{2.14}$$

we can calculate the reduced density matrix (2.5) as

$$\rho_A^{(\text{fin})} = \frac{1}{\mathcal{N}_1} \left(\frac{|\mathcal{S}_{p_1 q_1; p_1 q_1}|^2}{V^3} |\tilde{p}_1\rangle\langle\tilde{p}_1| \right.$$
$$\left. + \lambda^2 V^2 \int\limits_{k, k' \neq p_1} dk\, dk'\, M_{kk'} |k\rangle\langle k'| \right),$$
$$M_{kk'} = \frac{1}{V^2} \int\limits_{l \neq q_1} dl \left(\lambda \frac{\mathcal{T}_{kq_1; p_1 q_1} \mathcal{T}_{p_1 l; p_1 q_1}}{\mathcal{S}_{p_1 q_1; p_1 q_1}} + i\mathcal{T}_{kl; p_1 q_1} \right)$$
$$\times \left(\lambda \frac{\mathcal{T}_{k'q_1; p_1 q_1} \mathcal{T}_{p_1 l; p_1 q_1}}{\mathcal{S}_{p_1 q_1; p_1 q_1}} + i\mathcal{T}_{k'l; p_1 q_1} \right)^*. \tag{2.15}$$

$\mathcal{N}_1$ is the normalization factor which is fixed by $\text{tr}_A \rho_A^{(\text{fin})} = 1$, namely,

$$\mathcal{N}_1 = \frac{|\mathcal{S}_{p_1 q_1; p_1 q_1}|^2}{V^2} + \lambda^2 V^2 \int\limits_{k \neq p_1} dk\, M_{kk}. \tag{2.16}$$

Here we recall (2.12) and it leads to $M_{kk'} \sim \mathcal{O}(1)$. After a perturbative expansion, the reduced density matrix (2.15) becomes

$$\rho_A^{(\text{fin})} = \left(1 - \lambda^2 \int\limits_{k \neq p_1} dk\, M_{kk} \right) \frac{1}{V} |\tilde{p}_1\rangle\langle\tilde{p}_1|$$
$$+ \lambda^2 \int\limits_{k, k' \neq p_1} dk\, dk'\, M_{kk'} |k\rangle\langle k'| + \mathcal{O}(\lambda^3), \tag{2.17}$$

$$M_{kk'} = \frac{1}{V^2} \int\limits_{l \neq q_1} dl\, \mathcal{T}_{kl; p_1 q_1} \mathcal{T}^*_{k'l; p_1 q_1} + \mathcal{O}(\lambda) \quad (k, k' \neq p_1) \tag{2.18}$$

When the eigenvalues of $M_{kk'}$ at leading order are denoted by m_k, we obtain

$$\int\limits_{k \neq p_1} dk\, M_{kk} = \text{tr}_A M_{kk'} = \int\limits_{k \neq p_1} dk\, m_k, \tag{2.19}$$

up to $\mathcal{O}(\lambda)$. Therefore the entanglement entropy of final state (2.7) becomes

$$S_E^{(\text{fin})} = -\left(1 - \lambda^2 \int\limits_{k \neq p_1} dk\, m_k \right) \log\left(1 - \lambda^2 \int\limits_{k \neq p_1} dk\, m_k \right)$$
$$- \int\limits_{k \neq p_1} dk\, (\lambda^2 m_k) \log(\lambda^2 m_k) + \mathcal{O}(\lambda^3)$$
$$= -\lambda^2 \log \lambda^2 \int\limits_{k \neq p_1} dk\, m_k + \lambda^2 \int\limits_{k \neq p_1} dk\, m_k (1 - \log m_k)$$
$$+ \mathcal{O}(\lambda^3). \tag{2.20}$$

Only the T-matrix elements $\mathcal{T}_{kl; p_1 q_1}$ with $k \neq p_1$ and $l \neq q_1$ contribute to the entanglement entropy of the final state at leading order. Of course, since the entanglement entropy of the unentangled initial state (2.9) vanishes, the variation of entanglement entropy (2.8), ΔS_E, is equal to $S_E^{(\text{fin})}$ itself in (2.20).

2.2. Entangled initial state

Let us consider an entangled initial state,

$$|\text{ini}\rangle \sim u_1 |p_1, q_1\rangle + u_2 |p_2, q_2\rangle, \tag{2.21}$$

with $p_1 \neq q_1$, $p_2 \neq q_2$, $u_1^2 + u_2^2 = 1$, $u_{1,2} \neq 0$ and $u_1 \geq u_2$. The entanglement entropy of this state is

$$S_E^{(\text{ini})} = \sum_{j=1}^{2} |u_j|^2 \log |u_j|^2. \tag{2.22}$$

We can write down the final state in terms of the S-matrix (or T-matrix),

$$|\text{fin}\rangle = \frac{\mathcal{S}_{p_1 q_1}}{V^2} |p_1, q_1\rangle + \frac{\mathcal{S}_{p_2 q_2}}{V^2} |p_2, q_2\rangle$$
$$+ i\lambda \frac{\mathcal{T}_{p_1 q_2}}{V^2} |p_1, q_2\rangle + i\lambda \frac{\mathcal{T}_{p_2 q_1}}{V^2} |p_2, q_1\rangle$$
$$+ i\lambda \int\limits_{l \neq q_1, q_2} dl \sum_{j=1}^{2} \frac{\mathcal{T}_{p_j l}}{V} |p_j, l\rangle$$
$$+ i\lambda \int\limits_{k \neq p_1, p_2} dk \sum_{j=1}^{2} \frac{\mathcal{T}_{k q_j}}{V} |k, q_j\rangle$$
$$+ i\lambda \int\limits_{\substack{k \neq p_1, p_2 \\ l \neq q_1, q_2}} dk\, dl\, \mathcal{T}_{kl} |k, l\rangle, \tag{2.23}$$

where

$$\mathcal{S}_{kl} := u_1 \mathcal{S}_{kl; p_1 q_1} + u_2 \mathcal{S}_{kl; p_2 q_2},$$
$$\mathcal{T}_{kl} := u_1 \mathcal{T}_{kl; p_1 q_1} + u_2 \mathcal{T}_{kl; p_2 q_2}. \tag{2.24}$$

Note that $\mathcal{S}_{p_1 q_1} = u_1 V^2 + i\lambda \mathcal{T}_{p_1 q_1}$ and $\mathcal{S}_{p_2 q_2} = u_2 V^2 + i\lambda \mathcal{T}_{p_2 q_2}$. Firstly we diagonalize the first line in (2.23) by the use of

$$\mathbb{Q} = \begin{pmatrix} \mathcal{S}_{p_1 q_1} & i\lambda \mathcal{T}_{p_1 q_2} \\ i\lambda \mathcal{T}_{p_2 q_1} & \mathcal{S}_{p_2 q_2} \end{pmatrix},$$
$$W = \begin{pmatrix} i\lambda \mathcal{T}_{p_2 q_1} & \mathcal{S}_{p_2 q_2} - \zeta_2 \\ \mathcal{S}_{p_1 q_1} - \zeta_1 & i\lambda \mathcal{T}_{p_1 q_2} \end{pmatrix},$$
$$W \mathbb{Q} W^{-1} = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix}, \tag{2.25}$$

where

$$\zeta_1 + \zeta_2 = \mathcal{S}_{p_1 q_1} + \mathcal{S}_{p_2 q_2},$$

$$\zeta_1 - \zeta_2 = \sqrt{(\mathcal{S}_{p_1 q_1} - \mathcal{S}_{p_2 q_2})^2 - 4\lambda^2 \mathcal{T}_{p_1 q_2} \mathcal{T}_{p_2 q_1}}. \quad (2.26)$$

Following this diagonalization, the basis is transformed as

$$\begin{pmatrix} |p_1\rangle \\ |p_2\rangle \end{pmatrix} = W^t \begin{pmatrix} |\hat{p}_1\rangle \\ |\hat{p}_2\rangle \end{pmatrix},$$

$$\begin{pmatrix} |q_1\rangle \\ |q_2\rangle \end{pmatrix} = W^{-1} \begin{pmatrix} |\hat{q}_1\rangle \\ |\hat{q}_2\rangle \end{pmatrix}. \quad (2.27)$$

Then we can rewrite the final state (2.23) as

$$|\text{fin}\rangle = \sum_{j=1}^{2} \frac{\zeta_j}{V^2} |\hat{p}_j, \hat{q}_j\rangle + i\lambda \int_{l \neq q_1, q_2} dl \sum_{j=1}^{2} \frac{A_j(l)}{V} |\hat{p}_j, l\rangle$$

$$+ i\lambda \int_{k \neq p_1, p_2} dk \sum_{j=1}^{2} \frac{B_j(k)}{V} |k, \hat{q}_j\rangle$$

$$+ i\lambda \int_{\substack{k \neq p_1, p_2 \\ l \neq q_1, q_2}} dk\, dl\, \mathcal{T}_{kl} |k, l\rangle, \quad (2.28)$$

where

$$A_1(l) = i\lambda \mathcal{T}_{p_1 l} \mathcal{T}_{p_2 q_1} + \mathcal{T}_{p_2 l}(\mathcal{S}_{p_2 q_2} - \zeta_2),$$

$$A_2(l) = \mathcal{T}_{p_1 l}(\mathcal{S}_{p_1 q_1} - \zeta_1) + i\lambda \mathcal{T}_{p_2 l} \mathcal{T}_{p_1 q_2},$$

$$B_1(k) = \frac{i\lambda \mathcal{T}_{k q_1} \mathcal{T}_{p_1 q_2} - \mathcal{T}_{k q_2}(S_{p_1 q_1} - \zeta_1)}{\det W},$$

$$B_2(k) = \frac{-\mathcal{T}_{k q_1}(S_{p_2 q_2} - \zeta_2) + i\lambda \mathcal{T}_{k q_2} \mathcal{T}_{p_2 q_1}}{\det W}. \quad (2.29)$$

Furthermore we can rearrange the basis so that

$$|\text{fin}\rangle = \sum_{j=1}^{2} \frac{\zeta_j}{V^2} |\tilde{p}_j\rangle_A \otimes |\tilde{q}_j\rangle_B$$

$$+ \int_{\substack{k \neq p_1, p_2 \\ l \neq q_1, q_2}} dk\, dl \left(\lambda^2 \sum_{j=1}^{2} \frac{A_j(l) B_j(k)}{\zeta_j} + i\lambda \mathcal{T}_{kl} \right) |k, l\rangle, \quad (2.30)$$

where

$$|\tilde{p}_j\rangle_A = |\hat{p}_j\rangle_A + i\lambda \frac{V}{\zeta_j} \int_{k \neq p_1, p_2} dk\, B_j(k) |k\rangle_A,$$

$$|\tilde{q}_j\rangle_B = |\hat{q}_j\rangle_B + i\lambda \frac{V}{\zeta_j} \int_{l \neq q_1, q_2} dk\, A_j(k) |k\rangle_B \quad (j = 1, 2). \quad (2.31)$$

As a result, we obtain the reduced density matrix (2.5) after a similarity transformation,

$$\rho_A^{(\text{fin})} = \frac{1}{\mathcal{N}_2} \left(\sum_{j=1}^{2} \frac{|\zeta_j|^2}{V^3} |\tilde{p}_j\rangle\langle\tilde{p}_j| \right.$$

$$\left. + \lambda^2 V^2 \int_{k,k' \neq p_1, p_2} dk\, dk'\, R_{kk'} |k\rangle\langle k'| \right), \quad (2.32)$$

$$R_{kk'} = \frac{1}{V^2} \int_{l \neq q_1, q_2} dl \left(\lambda \sum_{j=1}^{2} \frac{A_j(l) B_j(k)}{\zeta_j} + i\mathcal{T}_{kl} \right)$$

$$\times \left(\lambda \sum_{j=1}^{2} \frac{A_j(l) B_j(k')}{\zeta_j} + i\mathcal{T}_{k'l} \right)^*. \quad (2.33)$$

The leading term of $R_{kk'}$ does not depend on A_j and B_j but on $\mathcal{T}_{kl}$ ($k \neq p_1, p_2$, $l \neq q_1, q_2$). Using the normalization $\text{tr}_A \rho_A^{(\text{fin})} = 1$, $\mathcal{N}_2$ is computed as

$$\mathcal{N}_2 = \sum_{i=1}^{2} \frac{|\zeta_j|^2}{V^2} + \lambda^2 V^2 \int_{k \neq p_1, p_2} dk\, R_{kk}. \quad (2.34)$$

Then one can write down the reduced density matrix in perturbative expansion,

$$\rho_A^{(\text{fin})} = \left(u_1^2 + \lambda f + \lambda^2 g - \lambda^2 u_1^2 \int_{k \neq p_1, p_2} dk\, R_{kk} \right) \frac{1}{V} |\tilde{p}_1\rangle\langle\tilde{p}_1|$$

$$+ \left(u_2^2 - \lambda f - \lambda^2 g - \lambda^2 u_2^2 \int_{k \neq p_1, p_2} dk\, R_{kk} \right) \frac{1}{V} |\tilde{p}_2\rangle\langle\tilde{p}_2|$$

$$+ \lambda^2 \int_{k,k' \neq p_1, p_2} dk\, dk'\, R_{kk'} |k\rangle\langle k'| + \mathcal{O}(\lambda^3), \quad (2.35)$$

$$R_{kk'} = \frac{1}{V^2} \int_{l \neq q_1, q_2} dl\, \mathcal{T}_{kl} \mathcal{T}_{k'l}^* + \mathcal{O}(\lambda), \quad (2.36)$$

with

$$V^2 f = 2 u_1 u_2 (u_1 \,\text{Im}\, \mathcal{T}_{p_2 q_2} - u_2 \,\text{Im}\, \mathcal{T}_{p_1 q_1}), \quad (2.37)$$

$$V^4 g = 4 u_1 u_2 (u_1 \,\text{Im}\, \mathcal{T}_{p_1 q_1} + u_2 \,\text{Im}\, \mathcal{T}_{p_2 q_2})$$

$$\times (u_1 \,\text{Im}\, \mathcal{T}_{p_2 q_2} - u_2 \,\text{Im}\, \mathcal{T}_{p_1 q_1})$$

$$+ u_2^2 |\mathcal{T}_{p_1 q_1}|^2 - u_1^2 |\mathcal{T}_{p_2 q_2}|^2$$

$$- \frac{2 u_1 u_2 (u_1 + u_2)}{u_1 - u_2} \text{Re}(\mathcal{T}_{p_1 q_2} \mathcal{T}_{p_2 q_1}). \quad (2.38)$$

Note that f and g are anti-symmetric with respect to the indices 1 and 2. Since (2.35) implies a reduced density matrix after a similarity transformation, we can calculate $S_E^{(\text{fin})}$, the entanglement entropy of the final state. Here we introduce r_k ($k \neq p_1, p_2$) which denotes the eigenvalues of $R_{kk'}$. Subtracting the initial entanglement entropy (2.22) from $S_E^{(\text{fin})}$, we obtain the variation of entanglement entropy as

$$\Delta S_E = -\lambda^2 \log \lambda^2 \int_{k \neq p_1, p_2} dk\, r_k - \lambda f \log \frac{u_1^2}{u_2^2}$$

$$+ \lambda^2 \left(\int_{k \neq p_1, p_2} dk\, r_k (1 - S_E^{(\text{ini})} - \log r_k) \right.$$

$$\left. - \frac{f^2}{2 u_1^2 u_2^2} - g \log \frac{u_1^2}{u_2^2} \right) + \mathcal{O}(\lambda^3). \quad (2.39)$$

The leading term is of order $\lambda^2 \log \lambda^2$ and is similar to the case of the unentangled initial state (2.20). While the sub-leading term in the case of the unentangled initial state is of order λ^2, the sub-leading term in the case of the entangled initial state appears at order λ. This order λ contribution comes from the mutual transition between the states, $|p_1, q_1\rangle$ and $|p_2, q_2\rangle$. When the particles A and B at the initial state are maximally entangled, i.e., $u_1 = u_2 = 1/\sqrt{2}$, the term of order λ vanishes.

In the same way, one can consider an n coherent state as an initial state, namely, $|\text{ini}\rangle \sim \sum_{j=1}^{n} u_j |p_j, q_j\rangle$, $\sum_{j=1}^{n} u_j^2 = 1$.

Since the final state includes $S_{p_i q_j} V^{-2} |p_i q_j\rangle$, we firstly diagonalize the matrix $\mathbb{Q} = (S_{p_i q_j})$ $(i, j = 1, \ldots, n)$ so that $W \mathbb{Q} W^{-1} = \mathrm{diag}(\zeta_1, \ldots, \zeta_n)$, and replace $|p_i, q_i\rangle$ with $|\hat{p}_i, \hat{q}_i\rangle$ like (2.27). Then, by a procedure similar to (2.31), we can obtain a simplified reduced density matrix like (2.33). Therefore the leading contribution to the variation of entanglement entropy is $\lambda^2 \log \lambda^2 \times \int_{k \neq p_1, \ldots, p_n} dk \int_{l \neq q_1, \ldots, q_n} dl \, V^{-2} \mathcal{T}_{kl} \mathcal{T}_{kl}^*$, in which $\mathcal{T}_{kl} = \sum_{j=1}^{n} u_j \mathcal{T}_{kl; p_j q_j}$.

3. Examples

3.1. Field theory with ϕ^4-like interaction

We consider two real scalar fields, ϕ_A and ϕ_B, of which action with a ϕ^4-like interaction is

$$
S = -\int d^{d+1} x \left(\frac{1}{2} \partial_\mu \phi_A \partial^\mu \phi_A + \frac{1}{2} \partial_\mu \phi_B \partial^\mu \phi_B \right.
$$
$$
\left. + \frac{1}{2} m^2 (\phi_A^2 + \phi_B^2) + \frac{\lambda}{4} \phi_A^2 \phi_B^2 \right). \tag{3.1}
$$

We focus on a scattering process of two incoming particles and two outgoing particles such as $A + B \to A + B$. Since we can assume that the incoming and outgoing particles are free on-shell particles in the far past and future, one can describe a Fock space of such $(1 + 1)$-particle states as

$$
|\vec{p}, \vec{q}\rangle = a_{\vec{p}}^\dagger |0\rangle_A \otimes b_{\vec{q}}^\dagger |0\rangle_B. \tag{3.2}
$$

$a_{\vec{p}}^\dagger$ and $b_{\vec{q}}^\dagger$ are the creation operators of particles A and B and are defined by the following mode expansion for free scalar fields:

$$
\phi_A(x) = \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{1}{2E_{\vec{p}}} (a_{\vec{p}} e^{-ip \cdot x} + a_{\vec{p}}^\dagger e^{ip \cdot x}),
$$
$$
\phi_B(x) = \int \frac{d^d \vec{q}}{(2\pi)^d} \frac{1}{2E_{\vec{q}}} (b_{\vec{q}} e^{-iq \cdot x} + b_{\vec{q}}^\dagger e^{iq \cdot x}), \tag{3.3}
$$

where $p^0 = E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. The factor $d^d \vec{p} / ((2\pi)^d 2E_{\vec{p}})$ is a Lorentz invariant integration measure. The creation and annihilation operators obey the commutation relations:

$$
[a_{\vec{p}}, a_{\vec{k}}^\dagger] = 2E_{\vec{p}} (2\pi)^d \delta^{(d)}(\vec{p} - \vec{k}),
$$
$$
[b_{\vec{q}}, b_{\vec{l}}^\dagger] = 2E_{\vec{q}} (2\pi)^d \delta^{(d)}(\vec{q} - \vec{l}). \tag{3.4}
$$

Now let us study the case that the initial state is $|\mathrm{ini}\rangle = |\vec{p}_1, \vec{q}_1\rangle$. Since the identity operator on the $(1 + 1)$-particle Hilbert space is

$$
(\mathbf{1})_{(1+1)\text{-particle}} = \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{1}{2E_{\vec{p}}} \frac{d^d \vec{q}}{(2\pi)^d} \frac{1}{2E_{\vec{q}}} |\vec{p}, \vec{q}\rangle \langle \vec{p}, \vec{q}|, \tag{3.5}
$$

the final state (2.4) is described as

$$
|\mathrm{fin}\rangle = \mathbf{S} |\mathrm{ini}\rangle
$$
$$
= \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{2E_{\vec{k}}} \frac{d^d \vec{l}}{(2\pi)^d} \frac{1}{2E_{\vec{l}}} |\vec{k}, \vec{l}\rangle \langle \vec{k}, \vec{l} | \mathbf{S} | \vec{p}_1, \vec{q}_1\rangle
$$
$$
= \frac{1}{2E_{\vec{p}_1} 2E_{\vec{q}_1} L^{2d}} |\vec{p}_1, \vec{q}_1\rangle \langle \vec{p}_1, \vec{q}_1 | \mathbf{S} | \vec{p}_1, \vec{q}_1\rangle
$$
$$
+ \int_{\substack{k \neq \vec{p}_1 \\ l \neq \vec{q}_1}} \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{2E_{\vec{k}}} \frac{d^d \vec{l}}{(2\pi)^d} \frac{1}{2E_{\vec{l}}} |\vec{k}, \vec{l}\rangle \langle \vec{k}, \vec{l} | i\mathbf{T} | \vec{p}_1, \vec{q}_1\rangle, \tag{3.6}
$$

where L originates from the spacial volume of phase space, $L^d = (2\pi)^d \delta^{(d)}(0) = \int d^d \vec{x} e^{i\vec{x} \cdot \vec{0}}$. The final state (3.6) does not contain the

states proportional to $|\vec{k}(\neq \vec{p}_1), \vec{q}_1\rangle$ and $|\vec{p}_1, \vec{l}(\neq \vec{q}_1)\rangle$, which appear in the second line of (2.10), because such states vanish due to the factor of momentum conservation in the S-matrix element, namely, $\langle \vec{k}, \vec{l} | \mathbf{S} | \vec{p}_1, \vec{q}_1\rangle \sim \delta^{(d+1)}(k + l - p_1 - q_1)$.

As we have studied in Section 2, the variation of entanglement entropy in a scattering process is determined by the transition matrix $\mathbf{T}$. From the action (3.1), we perturbatively calculate the S-matrix element,

$$
\langle \vec{k}, \vec{l} | \mathbf{S} | \vec{p}_1, \vec{q}_1\rangle = 2E_{\vec{p}_1} 2E_{\vec{q}_1} (2\pi)^d \delta^{(d)}(\vec{k} - \vec{p}_1)(2\pi)^d \delta^{(d)}(\vec{l} - \vec{q}_1)
$$
$$
- i\lambda (2\pi)^{d+1} \delta^{(d+1)}(k + l - p_1 - q_1)
$$
$$
+ \mathcal{O}(\lambda^2). \tag{3.7}
$$

Substituting this S-matrix element into (3.6), we obtain the final state. Then the reduced density matrix automatically becomes block-diagonal,

$$
\rho_A^{(\mathrm{fin})} = \left(1 - \lambda^2 \int_{\vec{k} \neq \vec{p}_1} \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{2E_{\vec{k}}} M_{\vec{k}\vec{k}} \right) \frac{1}{2E_{\vec{p}_1} L^d} |\vec{p}_1\rangle \langle \vec{p}_1|
$$
$$
+ \lambda^2 \int_{\vec{k} \neq \vec{p}_1} \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{2E_{\vec{k}}} M_{\vec{k}\vec{k}} \frac{1}{2E_{\vec{k}} L^d} |\vec{k}\rangle \langle \vec{k}| + \mathcal{O}(\lambda^4), \tag{3.8}
$$

$$
M_{\vec{k}\vec{k}} = \frac{1}{2E_{\vec{p}_1} 2E_{\vec{q}_1} 2E_{\vec{p}_1 + \vec{q}_1 - \vec{k}} L^d}
$$
$$
\times \left\{ 2\pi \delta(E_{\vec{k}} + E_{\vec{p}_1 + \vec{q}_1 - \vec{k}} - E_{\vec{p}_1} - E_{\vec{q}_1}) \right\}^2. \tag{3.9}
$$

Notice that we have normalized this density matrix so that $\mathrm{tr}_A \rho_A^{(\mathrm{fin})} = 1$. Then the variation of entanglement entropy (2.20) is computed as

$$
\Delta S_E = -\lambda^2 \log \lambda^2 \int_{\vec{k} \neq \vec{p}_1} \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{2E_{\vec{k}}} M_{\vec{k}\vec{k}}
$$
$$
+ \lambda^2 \int_{\vec{k} \neq \vec{p}_1} \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{2E_{\vec{k}}} M_{\vec{k}\vec{k}} \left(1 - \log \frac{M_{\vec{k}\vec{k}}}{2E_{\vec{k}} L^d} \right)
$$
$$
+ \mathcal{O}(\lambda^4). \tag{3.10}
$$

We shall calculate it further by employing a center of mass frame, that is, $\vec{p}_1 = -\vec{q}_1 =: \vec{p}_{cm}$ and $\sqrt{\vec{p}_{cm}^2 + m^2} =: E_{cm}$. Of course the momenta of outgoing particles obey $\vec{k} = -\vec{l}$ due to the momentum conservation. Then the d-dimensional integration can be replaced with a spherical integration as $d^d \vec{k} = dk d\Omega_{d-1} k^{d-1}$, because the integration kernel in (3.10) depends only on the norm of $\vec{k}$. Therefore we finally obtain

$$
\Delta S_E = -\lambda^2 \log \lambda^2 \frac{\pi^{1 - \frac{d}{2}}}{2^{d+3} \Gamma(\frac{d}{2}) L^{d-1}} \frac{|\vec{p}_{cm}|^{d-2}}{E_{cm}^3}
$$
$$
+ \lambda^2 \frac{\pi^{1 - \frac{d}{2}}}{2^{d+3} \Gamma(\frac{d}{2}) L^{d-1}} \frac{|\vec{p}_{cm}|^{d-2}}{E_{cm}^3} \left(1 + \log(16 E_{cm}^4 L^{2d-2}) \right)
$$
$$
+ \mathcal{O}(\lambda^4). \tag{3.11}
$$

When the number of the spacial dimension d is equal to three, the leading term of the variation of entanglement entropy is proportional to $|\vec{p}_{cm}| / E_{cm}^3$. This is consistent with the cross section, which is $(d\sigma/d\Omega)_{cm} = \frac{\lambda^2}{64\pi^2} |\vec{p}_{cm}| / E_{cm}^3$, because both the variation of entanglement entropy and the cross section originate from a square of the absolute value of the scattering amplitude. Notice

that the remaining factor L in the entanglement entropy is an artifact caused by choosing the single-mode initial state whose norm has delta-functional divergence. The volume dependence of entanglement entropy in field theories was discussed also in Refs. [13, 14], where the momentum-space entanglement entropy is proportional to a spacial volume. The difference between the volume dependence of Ref. [13,14] and ours is mostly caused by the absence of integration with respect to the initial state momenta in our calculation.

3.2. Time-dependent interaction in quantum mechanics

In this subsection we turn to quantum mechanics with a time-dependent interaction, $\lambda H_{\mathrm{int}}(t)$. We set the initial state so that $|\mathrm{ini}\rangle = |p_1, q_1\rangle$ at $t = 0$. Then the time evolution of this initial state is described as

$$|\Psi(t)\rangle = |p_1, q_1\rangle + \lambda \sum_{k \neq p_1} C_{kq_1;p_1q_1}(t)e^{-iE_k t}|k, q_1\rangle$$
$$+ \lambda \sum_{l \neq q_2} C_{p_1 l;p_1 q_1}(t)e^{-iE_l t}|p_1, l\rangle$$
$$+ \lambda \sum_{k \neq p_1, l \neq q_1} C_{kl;p_1 q_1}(t)e^{-iE_{kl}t}|k, l\rangle, \qquad (3.12)$$

up to normalization. E_p, E_q and E_{pq} are energy eigenvalues which are defined in terms of the non-interacting part of the Hamiltonian (see (2.1)),

$$H_A|p\rangle_A = E_p|p\rangle_A, \qquad H_B|q\rangle_B = E_q|q\rangle_B,$$
$$H_0|p, q\rangle = E_{pq}|p, q\rangle. \qquad (3.13)$$

The interacting Hamiltonian $\lambda H_{\mathrm{int}}(t)$ yields $C_{kl;pq}(t)$. By the use of the well-known time-dependent perturbation theory, we can calculate

$$C_{kl;p_1 q_1}(t) = -i \int_0^t dt' e^{i\omega_{kl;p_1 q_1} t'} T_{kl;p_1 q_1}(t'),$$

$$T_{kl;p_1 q_1}(t) := \langle k, l|H_{\mathrm{int}}(t)|p_1, q_1\rangle, \qquad (3.14)$$

where $\omega_{kl;p_1 q_1} := E_{kl} - E_{pq}$. Since the time-dependent density matrix of $|\Psi(t)\rangle$ is given by $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$, we can calculate the reduced density matrix $\rho_A(t) = \mathcal{N}^{-1} \mathrm{tr}_B \rho(t)$ together with the normalization by $\mathrm{tr}_A \rho_A = 1$. After the same procedure as in Ref. [13] or Section 2.1, we obtain the entanglement entropy,

$$S_E(t) = -\lambda^2 \log \lambda^2 \sum_{k \neq p_1, l \neq q_1} \int_0^t dt' \int_0^t dt'' \, e^{i\omega_{kl;p_1 q_1}(t'-t'')}$$
$$\times T_{kl;p_1 q_1}(t') T_{kl;p_1 q_1}^*(t'') + \mathcal{O}(\lambda^2). \qquad (3.15)$$

Notice that one can regard $\lambda T_{kl;p_1 q_1}(t = \infty)$ as a kind of transition matrix.

4. Conclusion and discussion

We have studied the variation of entanglement entropy from an initial state to a final state in a scattering process. We concentrated on the scattering of $2 \to 2$ particles and perturbatively calculated the entanglement entropy of final states for the two kinds of simple initial states: the unentangled state (2.9) and the entangled state (2.21). In both cases the leading terms of the variation of entanglement entropy, (2.20) and (2.39), are of order $\lambda^2 \log \lambda^2$ and are proportional to the trace of a square of the absolute value of

T-matrix elements, which are, in other words, the scattering amplitudes. The next leading term in the case of the unentangled initial state is of order λ^2. On the other hand the next leading term in the case of the entangled initial state appears at order λ, because there is a mutual transition between the states $|p_1, q_1\rangle$ and $|p_2, q_2\rangle$.

We have considered the model of two real scalar fields with the ϕ^4-like interaction as an example in a field theory. The variation of entanglement entropy has been computed perturbatively. If we employ the center of mass frame, the leading term (at order $\lambda^2 \log \lambda^2$) in the variation of entanglement entropy depends on the momenta of initial particles as $|\vec{p}_{cm}|^{d-2}/E_{cm}^3$. Notice that this factor becomes $|\vec{p}_{cm}|/E_{cm}^3$, when the space dimension is equal to three, i.e., the coupling λ is dimensionless. The same factor also appears in the cross section, because it originally comes from the scattering amplitude. Therefore, as we expected, the variation of entanglement entropy is proportional to the cross section.

We have also mentioned the time-dependent interaction as an example in quantum mechanics. The time-evolution of entanglement entropy from the simple initial state $|p_1, q_1\rangle$ can be written in terms of the transition matrix at the leading order $\lambda^2 \log \lambda^2$.

With the AdS/CFT correspondence, one can identify the scattering amplitude in a field theory of strong coupling with $\exp(-\mathcal{A})$, where $\mathcal{A}$ is an area of minimal surface in a bulk gravity theory, while the holographic entanglement entropy [2,3] is given by $\mathcal{A}'/(4G_N)$, where $\mathcal{A}'$ is an area of another minimal surface. That is to say, both of the scattering amplitude and entanglement entropy in a strongly coupled field theory are associated with minimal surfaces from the point of view of the AdS/CFT correspondence. In this paper we have shown the relation between the scattering and the variation of entanglement entropy by the perturbative calculations in a weak coupling. It is then in order to ask whether we can clarify such a relation from a field theory in a strong coupling. For this purpose we need to test it in an exactly calculable model. Moreover, the holographic understanding of such a relation, or a relation between those minimal surfaces, is another problem for the future.

References

[1] P. Calabrese, J.L. Cardy, Entanglement entropy and quantum field theory, J. Stat. Mech. 0406 (2004) P06002, arXiv:hep-th/0405152.
[2] S. Ryu, T. Takayanagi, Holographic derivation of entanglement entropy from AdS/CFT, Phys. Rev. Lett. 96 (2006) 181602, arXiv:hep-th/0603001.
[3] S. Ryu, T. Takayanagi, Aspects of holographic entanglement entropy, J. High Energy Phys. 0608 (2006) 045, arXiv:hep-th/0605073.
[4] J. Maldacena, L. Susskind, Cool horizons for entangled black holes, Fortschr. Phys. 61 (2013) 781, arXiv:1306.0533 [hep-th].
[5] A. Almheiri, D. Marolf, J. Polchinski, J. Sully, Black holes: complementarity or firewalls?, J. High Energy Phys. 1302 (2013) 062, arXiv:1207.3123 [hep-th].
[6] S.L. Braunstein, Black hole entropy as entropy of entanglement, or it's curtains for the equivalence principle, arXiv:0907.1190v1 [quant-ph];
S.L. Braunstein, S. Pirandola, K. Życzkowski, Better late than never: information retrieval from black holes, Phys. Rev. Lett. 110 (2013) 101301, arXiv:0907.1190 [quant-ph].
[7] K. Jensen, A. Karch, Phys. Rev. Lett. 111 (21) (2013) 211602, arXiv:1307.1132 [hep-th].

[8] J. Sonner, Holographic Schwinger effect and the geometry of entanglement, Phys. Rev. Lett. 111 (21) (2013) 211603, arXiv:1307.6850 [hep-th].

[9] S. Seki, S.-J. Sin, EPR = ER and scattering amplitude as entanglement entropy change, Phys. Lett. B 735 (2014) 272, arXiv:1404.0794 [hep-th].

[10] L.F. Alday, J.M. Maldacena, J. High Energy Phys. 0706 (2007) 064, arXiv:0705.0303 [hep-th].

[11] L. Lello, D. Boyanovsky, R. Holman, Entanglement entropy in particle decay, J. High Energy Phys. 1311 (2013) 116, arXiv:1304.6110 [hep-th].

[12] A. Mollabashi, N. Shiba, T. Takayanagi, Entanglement between two interacting CFTs and generalized holographic entanglement entropy, J. High Energy Phys. 1404 (2014) 185, arXiv:1403.1393 [hep-th].

[13] V. Balasubramanian, M.B. McDermott, M. Van Raamsdonk, Momentum-space entanglement and renormalization in quantum field theory, Phys. Rev. D 86 (2012) 045014, arXiv:1108.3568 [hep-th].

[14] T.C.L. Hsu, M.B. McDermott, M. Van Raamsdonk, Momentum-space entanglement for interacting fermions at finite density, J. High Energy Phys. 1311 (2013) 121, arXiv:1210.0054 [hep-th].